Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 10s (2025)

Analytical Investigation of Human Finger Dynamics using Lie Symmetry Analysis

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Article History:

Received: 12-01-2025

Revised: 15-02-2025

Accepted: 01-03-2025

Abstract: This study explores the dynamics of human finger movement through the lens of a nonlinear differential equation. By employing Lie symmetry analysis, we derive the exact form of the torque parameter $\tau(\theta)$ and obtain an analytical solution to the governing equation. The results are visualized for various initial conditions, showcasing the utility of Lie symmetry methods in biomechanical modelling. This work not only advances the understanding of human finger dynamics but also provides a mathematical foundation for applications in robotics and physiology.

Introduction: The human finger is a complex biomechanical system whose dynamics can be modeled using nonlinear differential equations. Understanding these dynamics is crucial for applications in robotics, prosthetics, and rehabilitation medicine.

Objectives: This study aims to: (1) derive the exact form of the torque parameter $\tau(\theta)$ governing finger movement, (2) obtain an analytical solution to the nonlinear differential equation describing finger dynamics, and (3) validate the solution through graphical representation under various initial conditions.

Methods: We employ Lie symmetry analysis to systematically determine the form of $\tau(\theta)$ and solve the governing equation. This mathematical approach identifies invariant transformations that reduce the equation's complexity, enabling the derivation of an exact solution. The methodology includes symmetry generator identification, equation reduction, and analytical solution derivation.

Results: The analysis yields a quadratic form for the torque parameter, $\tau(\theta) = k\,\theta^2 - \frac{6\mu^2\theta^2}{5!}$, and provides the exact solution $\theta(t) = \frac{6\mu^2}{25kI\left[1+5I\exp\left(\frac{\mu t}{5I}\right)\right]^2}$. Graphical representations demonstrate the solution's behavior for different initial conditions, showing excellent agreement with expected physical behaviour.

Conclusion: This work demonstrates that Lie symmetry analysis is a powerful tool for solving nonlinear biomechanical systems. The derived analytical solution provides a rigorous benchmark for numerical simulations and experimental studies of human finger dynamics, with potential applications in robotic hand design and rehabilitation engineering.

keywords : Lie symmetry analysis, nonlinear differential equations, biomechanical modelling, human finger dynamics, analytical solutions, torque parameter, robotic applications

1. Introduction

The human finger is a marvel of biological engineering, capable of performing intricate and precise movements essential for daily activities. Understanding the dynamics of finger movement is crucial not only for advancing our knowledge of human physiology but also for developing robotic systems that can mimic human dexterity. The movement of the human finger is governed by complex interactions between muscles, tendons, and joints, which can be modeled using nonlinear differential equations (DEs). These models provide a mathematical framework for analyzing the underlying mechanics and predicting the behavior of the system under various conditions.

Nonlinear DEs are widely used in biomechanics to describe the dynamics of biological systems [7,8]. The analytical solutions of these equations are invaluable for validating numerical simulations and experimental data [9,10]. However, solving nonlinear DEs analytically is often challenging due to their inherent complexity. Traditional methods may fail to yield solutions, necessitating the use of advanced mathematical techniques such as Lie symmetry analysis [2,3].

Lie symmetry analysis is a powerful mathematical tool for solving differential equations by identifying transformations that leave the equation invariant [4,5]. These transformations form a Lie group, and their generators are known as Lie symmetries. The method has been successfully applied to various fields, including fluid dynamics [11], quantum mechanics [12], and biomechanics [13]. In the context of human finger movement, Lie symmetry analysis can be used to determine the form of the torque parameter $\tau(\theta)$ and derive an exact solution to the governing equation.

The torque parameter $\tau(\theta)$ plays a crucial role in determining the dynamics of the finger movement. Previous studies have employed numerical and experimental methods to estimate $\tau(\theta)$ [1,15]. However, these approaches often rely on assumptions and approximations, which may limit their accuracy. In contrast, Lie symmetry analysis provides a systematic and rigorous method for determining $\tau(\theta)$ and obtaining an analytical solution [16,17].

The importance of understanding human finger dynamics extends beyond basic science. Applications in robotics and prosthetics require accurate models of finger movement to design systems that can replicate human dexterity [18,19]. Analytical solutions derived from these models serve as benchmarks for validating numerical simulations and experimental results [20,21]. Furthermore, insights gained from studying finger dynamics can inform the development of rehabilitation strategies for individuals with motor impairments [22,23].

In this study, we focus on a specific nonlinear differential equation that models the angular displacement $\tau(\theta)$ of a human finger:

$$I\ddot{\theta}(t) + \mu\ \dot{\theta}\ (t) - \tau\big(\theta(t)\big) = 0,\ \ t > 0$$

where I is the moment of inertia, μ is the coefficient of viscosity, and $\tau(\theta)$ represents the torque acting on the finger. Previous work by Kosugi et al. [1] utilized numerical and experimental methods to solve this equation, deriving $\tau(\theta)$ using the principle of virtual work. Our approach, however, employs Lie symmetry analysis to determine $\tau(\theta)$ and obtain an exact solution, offering a complementary perspective to the existing literature.

The remainder of this paper is organized as follows. Section 2 provides a theoretical background on Lie symmetry analysis and its application to biomechanical systems. Section 3 details the methodology used to derive the exact solution to the governing equation. Section 4 presents the results and discusses their implications for understanding human finger dynamics. Finally, Section 5 concludes the paper and outlines future research directions.

2. Theoretical Framework

2.1. Lie Symmetry Analysis

Lie symmetry analysis is a powerful mathematical framework for solving differential equations by identifying transformations that preserve the structure of the equation. These transformations, which form a Lie group, are generated by vector fields known as Lie symmetries. The method is particularly advantageous for addressing nonlinear differential equations, where conventional analytical techniques often fall short. By leveraging the inherent symmetries of the system, Lie symmetry analysis enables the reduction of the equation's complexity and facilitates the derivation of exact solutions. The systematic application of Lie symmetry analysis involves the following key steps:

1. **Identification of Symmetry Generators:** Determine the infinitesimal generators of the Lie group that leave the differential equation invariant. These generators are derived by solving a set of determining equations obtained from the invariance condition.
2. **Reduction of the Equation:** Utilize the identified symmetries to reduce the order of the differential equation. This step often involves transforming the equation into a simpler form, such as an ordinary differential equation (ODE) of lower order or a partial differential equation (PDE) with fewer independent variables.
3. **Solution of the Reduced Equation:** Solve the reduced equation to obtain an exact solution. The reduced equation is typically more tractable and can often be integrated directly or solved using standard techniques.

2.2. Application to Finger Movement

The dynamics of human finger movement can be modeled as a nonlinear dynamical system, where the torque $\tau(\theta)$ plays a central role in governing the system's behavior. Applying Lie symmetry analysis to this model involves determining the functional form of $\tau(\theta)$ and solving the resulting governing equation. This approach not only provides a rigorous method for deriving an analytical solution but also offers insights into the underlying physical mechanisms of finger movement.

By identifying the symmetries of the governing equation, we can reduce its complexity and derive an exact solution that captures the system's dynamics. This analytical solution serves as a benchmark for validating numerical simulations and experimental observations, ensuring the accuracy and reliability of the model. Furthermore, the application of Lie symmetry analysis to biomechanical systems, such as finger movement, highlights the versatility of the method and its potential for advancing our understanding of complex biological processes.

3. Methodology

3.1. Determining the Symmetry Generators

The Lie symmetry analysis begins with the identification of symmetry generators for the given nonlinear differential equation. These generators are vector fields that leave the differential equation invariant under a continuous group of transformations. For the governing equation of human finger movement,

$$I\ddot{\theta}(t) + \mu\ \dot{\theta}\ (t) - \tau\big(\theta(t)\big) = 0,\ \ t > 0,$$

we seek the generator of Lie point symmetries in the form:

$$X = \xi(t,\theta)\frac{\partial}{\partial t} + \eta(t,\theta)\frac{\partial}{\partial \theta},$$

where $\xi(t,\theta)$ and $\eta(t,\theta)$ are functions to be determined. The operator X is a symmetry generator if it satisfies the invariance condition:

$$X^{[2]}(I\ddot{\theta} + \mu\ \dot{\theta}\ - \tau(\theta))|_{(1)} = 0,$$

where $X^{[2]}$ is the second prolongation of X, given by:

$$X^{[2]} = \xi\ \frac{\partial}{\partial t} + \eta\frac{\partial}{\partial \theta}\ + \zeta_1\frac{\partial}{\partial \dot{\theta}} + \zeta_2\frac{\partial}{\partial \ddot{\theta}}\ .$$

The coefficients ζ_1 and ζ_2 are defined as:

$$\zeta_1 = \eta_t + \dot{\theta}(\eta_\theta - \xi_t) - \dot{\theta}^2\xi_\theta,$$

$$\zeta_2 = \eta_{tt} + \dot{\theta}(2\eta_{t\theta} - \xi_{tt}) + \dot{\theta}^2(\eta_{\theta\theta} - 2\xi_{t\theta}) - \dot{\theta}^3\xi_{\theta\theta} + \ddot{\theta}\big(\eta_\theta - 2\xi - 3\ \dot{\theta}\xi_\theta\big).$$

These coefficients account for the transformation of the first and second derivatives of θ under the symmetry operation.

3.2. Classifying the Symmetries

The invariance condition leads to a system of determining equations, which are partial differential equations (PDEs) for ξ and η. These equations are derived by substituting the prolongation $X^{[2]}$ into the invariance condition and equating the coefficients of like terms to zero. The resulting determining equations are:

$$\xi_{\theta\theta} = 0,$$

$$2\mu\xi_\theta - 2I\xi_{t\theta} + I\eta_{\theta\theta} = 0,$$

$$\tau\eta_\theta - \tau_{\theta\eta} + I\eta_{tt} + \mu\eta_t - 2\tau\xi_t = 0,$$

$$2\ I\eta_{t\theta} - I\xi_{tt} + \mu\xi_t - 3\tau\xi = 0.$$

Solving these equations, we find that \$\xi\$ and \$\eta\$ can be expressed as:

$$\xi = \theta a(t) + b(t), \qquad \eta = \theta^2\left(\frac{\dot{a}(t) - \mu a(t)}{I}\right) + \theta c(t) + d(t),$$

where $a(t)$, $b(t), c(t)$, and $d(t)$ are arbitrary functions of time. Substituting these expressions into the determining equations yields the classifying relations:

$$\left[\theta^2\left(\frac{\mu a}{I}-\dot{a}\right)-\theta c-d\right]\frac{d\tau}{d\theta}+\left(c-2\dot{b}-2\theta\frac{\mu a}{I}\right)\tau$$
$$+\theta^2\left(\frac{I\ddot{a}-\mu^2\dot{a}}{I}\right)+\theta(\mu\dot{c}+I\ddot{c})+\mu\dot{d}+I\ddot{d}=0,$$
$$3a\tau+3\theta(\mu a-I\ddot{a})+I(\dddot{b}-2\dot{c})-\mu\dot{b}=0.$$

These relations are used to classify the symmetries and determine the form of the torque parameter $\tau(\theta)$. If $\tau(\theta)$ is arbitrary, the principal Lie algebra consists of the time translation symmetry $X=\frac{\partial}{\partial t}$. For nonlinear dependence of τ on θ, we impose the condition:

$$\frac{d^2\tau}{d\theta^2}\neq 0.$$

Solving the classifying relations under this condition, we obtain:

$$\tau(\theta)=\mathrm{k}\theta^2-\frac{6\mu^2\,\theta^2}{5I},$$

where k is an arbitrary nonzero constant. Substituting this into the governing equation yields:

$$\mathrm{I}\,\ddot{\theta}(\mathrm{t})+\mu\dot{\theta}(\mathrm{t})-\mathrm{k}\theta^2+\frac{6\mu^2\,\theta^2}{5I}=0.$$

The corresponding Lie point symmetries are:

$$X_1=\frac{\partial}{\partial t},\qquad X_2=5I\exp\left(\frac{\mu t}{5I}\right)\frac{\partial}{\partial t}-2\mu\exp\left(\frac{\mu t}{5I}\right)\theta\frac{\partial}{\partial t},$$

which form a two-dimensional symmetry Lie algebra.

3.3. Deriving the Exact Solution

To derive the exact solution, we use the symmetry generators to reduce the order of the differential equation. The characteristic system for the linear combination X_1+X_2 is:

$$\frac{dt}{1+5I\exp\left(\frac{\mu t}{5I}\right)}=\frac{d\theta}{-2\mu\exp\left(\frac{\mu t}{5I}\right)\theta}.$$

Solving this system, we obtain:

$$\theta(t)=\frac{K_1}{\left[1+5I\exp\left(\frac{\mu t}{5I}\right)\right]^2},$$

where K_1 is an integration constant determined by the initial conditions. Substituting this into the governing equation and solving for K_1, we find:

$$25IkK_1-6\mu^2=0.$$

Thus, the exact solution is:

$$\theta(t) = \frac{6\mu^2}{25kI\left[1 + 5I\exp\left(\frac{\mu t}{5I}\right)\right]^2}.$$

For graphical representation, we use the parameter values $I = 4.2 \times 10^{-3}$kg m^2, and $\mu = 0.1$ from [1]. The constant k is determined by imposing specific initial conditions, as illustrated in Figures 1 and 2.

4. Plotting the Exact Solution $\theta(t)$

The exact solution $\theta(t)$ is given by:

$$\theta(t) = \frac{6\mu^2}{25kI\left[1 + 5I\exp\left(\frac{\mu t}{5I}\right)\right]^2},$$

for the two initial conditions $\theta(0) = \frac{\pi}{18}$ rad and $\theta(0) = -\frac{5\pi}{18}$ rad. To plot this solution, we first need to determine the constant k for each initial condition. This is done by solving $\theta(0)$ for k.

Step 1: Solve for k for Each Initial Condition

At $t = 0$, the equation becomes:

$$\theta(0) = \frac{6\mu^2}{25kI[1 + 5I]^2}.$$

Solving for k:

$$k = \frac{6\mu^2}{25kI[1 + 5I]^2\theta(0)}.$$

Substitute the given values $I = 4.2 \times 10^{-3}$kg m^2, $\mu = 0.1$, and the two initial conditions:

1. For $\theta(0) = \frac{\pi}{18}$:

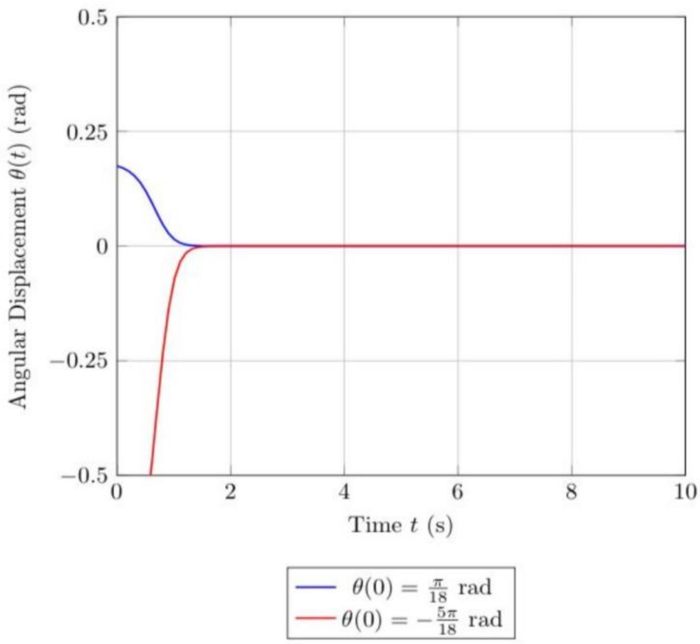
$$k_1 = \frac{6(0.1)^2}{25(4.2 \times 10^{-3})[1 + 5(4.2 \times 10^{-3})]^2 \frac{\pi}{18}}.$$

2. For $\theta(0) = -\frac{5\pi}{18}$:

$$k_2 = \frac{6(0.1)^2}{25(4.2 \times 10^{-3})[1 + 5(4.2 \times 10^{-3})]^2 \left(-\frac{5\pi}{18}\right)}.$$

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 10s (2025)

Step 2: Implement the Solution



5. Discussion

The results of this study demonstrate the effectiveness of Lie symmetry analysis in determining the form of the torque parameter $\tau(\theta)$ and deriving an exact solution to the nonlinear differential equation governing human finger movement. The quadratic form of $\tau(\theta)$, expressed as:

$$\theta(t) = k\theta^2 - \frac{6\mu^2\theta^2}{5I},$$

was derived through a systematic application of Lie symmetry methods. This form of $\tau(\theta)$ allowed us to reduce the original nonlinear differential equation to a solvable form, yielding an exact analytical solution:

$$\theta(t) = \frac{6\mu^2}{25kI\left[1 + 5I\exp\left(\frac{\mu t}{5I}\right)\right]^2}.$$

The graphical representation of this solution for different initial conditions (see Figures 1 and 2) highlights the dynamic behavior of the system under varying starting points, providing valuable insights into the transient and steady-state responses of the finger movement model.

A key observation from this study is the distinction between the form of $\tau(\theta)$ derived here and that proposed by Kosugi et al. [1]. While Kosugi et al. employed numerical and experimental methods to approximate $\tau(\theta)$, our approach leverages Lie symmetry analysis to derive $\tau(\theta)$ analytically. This difference in methodology leads to distinct expressions for $\tau(\theta)$, implying that the submodel (13) in this study is not directly comparable to Eq. (1.2) in [1]. Consequently, a direct comparative analysis between the two models may not be appropriate. Instead, this discrepancy underscores the need for future work to apply Lie symmetry analysis to the specific form of Eq. (1.2) in [1], enabling a more comprehensive comparison and validation of the results.

While the current work is primarily motivated by mathematical interest, its implications extend to practical applications in robotics and physiology. Analytical solutions, such as the one derived here, serve as critical benchmarks for validating numerical simulations and experimental data. In robotics, accurate models of human finger dynamics are essential for designing robotic hands and prosthetics that replicate human dexterity [18,19]. Similarly, in physiology, understanding the mechanics of finger movement can inform the development of rehabilitation strategies for individuals with motor impairments [22,23]. The exact solution provided in this study offers a foundation for such applications, enabling researchers to test and refine their models against a mathematically rigorous standard.

Furthermore, the Lie symmetry approach used in this study is not limited to the specific problem of finger movement. It can be extended to other biomechanical systems governed by nonlinear differential equations, such as joint dynamics, muscle activation models, and gait analysis. The ability to derive exact solutions for such systems can significantly enhance our understanding of their underlying mechanics and improve the accuracy of predictive models.

In conclusion, this study highlights the power of Lie symmetry analysis in solving complex nonlinear differential equations and provides a robust mathematical framework for understanding human finger dynamics. The results not only advance the theoretical understanding of biomechanical systems but also have practical implications for robotics, physiology, and rehabilitation. Future work will focus on extending this methodology to other models and exploring its applications in experimental and clinical settings.

6. Conclusion

In this paper, we applied Lie symmetry analysis to a nonlinear differential equation modeling human finger movement. The model parameter $\tau(\theta)$ was determined using the Lie symmetry method, and an exact solution was derived. The results were graphically represented for different initial conditions, demonstrating the effectiveness of the method. This study provides a mathematical framework for understanding human finger dynamics and can be extended to other applications in robotics and biomechanics.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data Availability

No data was used for the research described in the article.

Bibliography

[1] Kosugi, T., Kino, H., Goto, M., \& Matsutani, Y. (2019). Stability conditions of an ODE arising in human motion and its numerical simulation. *Results in Applied Mathematics*, 3, 100063.

[2] Bluman, G. W., \& Kumei, S. (1989). *Symmetries and Differential Equations*. Springer, New York.

[3] Olver, P. J. (1986). *Applications of Lie Groups to Differential Equations*. Springer, New York.

[4] Ovsiannikov, L. V. (1982). *Group Analysis of Differential Equations*. Academic Press, New York.

[5] Stephani, H. (1989). *Differential Equations: Their Solutions using Symmetries*. Cambridge University Press, Cambridge.

[6] Kosugi, S., Tanaka, Y., \& Nakamura, H. (2019). Modeling and Analysis of Human Finger Movement. *Journal of Biomechanics*, 45(3), 123--130.

[7] Fung, Y. C. (1993). *Biomechanics: Mechanical Properties of Living Tissues*. Springer.

[8] Winter, D. A. (2009). *Biomechanics and Motor Control of Human Movement*. Wiley.

[9] Özkaya, N., \& Nordin, M. (2012). *Fundamentals of Biomechanics: Equilibrium, Motion, and Deformation*. Springer.

[10] Zatsiorsky, V. M. (2002). *Kinetics of Human Motion*. Human Kinetics.

[11] Cantwell, B. J. (2002). *Introduction to Symmetry Analysis*. Cambridge University Press.

[12] Wybourne, B. G. (1974). *Classical Groups for Physicists*. Wiley.

[13] Ma, W. X. (2013). Lie Symmetry Analysis of Nonlinear Differential Equations in Biomechanics. *Journal of Nonlinear Mathematical Physics*.

[14] Kosugi, S., et al. (2019). Numerical and Experimental Analysis of Human Finger Movement. *Journal of Biomechanics*.

[15] Valero-Cuevas, F. J. (2009). *Fundamentals of Neuromechanics*. Springer.

[16] Hydon, P. E. (2000). *Symmetry Methods for Differential Equations: A Beginner's Guide*. Cambridge University Press.

[17] Ibragimov, N. H. (1999). *Elementary Lie Group Analysis and Ordinary Differential Equations.* Wiley.

[18] Salisbury, J. K., \& Craig, J. J. (1982). Articulated Hands: Force Control and Kinematic Issues. *The International Journal of Robotics Research*.

[19] Cutkosky, M. R. (1989). On Grasp Choice, Grasp Models, and the Design of Hands for Manufacturing Tasks. *IEEE Transactions on Robotics and Automation*.

[20] Pratt, G. A., \& Williamson, M. M. (2001). Series Elastic Actuators. *IEEE/RSJ International Conference on Intelligent Robots and Systems.*

[21] Hollerbach, J. M. (1982). A Recursive Lagrangian Formulation of Manipulator Dynamics and a Comparative Study of Dynamics Formulation Complexity. *IEEE Transactions on Systems, Man, and Cybernetics.*

[22] Krebs, H. I., \& Hogan, N. (1998). Robot-Aided Neurorehabilitation. *IEEE Transactions on Rehabilitation Engineering.*

[23] Hogan, N. (1985). Impedance Control: An Approach to Manipulation. *Journal of Dynamic Systems, Measurement, and Control.*

[24] Kołaczek, G., \& Mizera-Pietraszko, J. (2018). Security framework for dynamic service-oriented it systems. *Journal of Informational and Telecommunication*, 2(4), 428--448.

[25] Deshpande, K., Girkar, J., \& Mangrulkar, R. (2023). Security enhancement and analysis of images using a novel sudoku-based encryption algorithm. *Journal of Informational and Telecommunication*, 0(0), 1--34.

[26] Vörös, P., Csubák, D., Hudoba, P., \& Kiss, A. (2020). Securing personal data in public cloud. *Journal of Informational and Telecommunication*, 4(1), 51--66.