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SCP Identities of Multiplicative (Generalized)-Derivations on One-Sided Ideals in Semiprime Rings

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Abstract: Let $\mathcal{B}$ be a ring. A map $\Gamma: \mathcal{B} \to \mathcal{B}$ is termed a multiplicative (generalized)-derivation (abbreviated as $Mult.(G) - D$ throughout this paper) if $\Gamma(\nu\omega) = \Gamma(\nu)\omega + \nu\delta(\omega)$ holds $\forall$ $\nu, \omega \in \mathcal{B}$ where $\delta: \mathcal{B} \to \mathcal{B}$ is any map (not necessarily a derivation). If Γ, Δ are $Mult.(G) - D$ associated with maps δ, ξ respectively. This paper aims to investigate the following algebric identities: (i) $[\Gamma(\nu), \Gamma(\omega)] = \pm[\nu, \omega]$ (SCP map Γ), (ii)$[\Gamma(\nu), \omega] = \pm[\nu, \Delta(\omega)]$, (iii) $[\xi(\nu), \Gamma(\omega)] = \pm[\nu, \omega]$ and (iv) $[\xi(\nu), \omega] = \pm[\nu, \Gamma(\omega)]$ for all ν, ω in a left ideal of a semiprime ring $\mathcal{B}$. Additionally, we present an example illustrating that the semiprimeness condition in our theorems cannot be omitted.

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1- Introduction

The concept of multiplicative derivation was first established in 1991 by Daif [1] who defined it as: A function $\delta: \mathcal{B} \to \mathcal{B}$ (not necessarily additive) is termed a *multiplicative derivation* of $\mathcal{B}$ if $\delta(\nu\omega) = \delta(\nu)\omega + \nu\delta(\omega)$ for all $\nu, \omega \in \mathcal{B}$. The research by Daif [1] drew inspiration from Martindale's work [2]. Subsequently, Goldmann and Ŝemrl provided a thorough characterization of these functions in [3].

This initial definition was later broadened to multiplicative generalized derivation by Daif and Tammam in [4], who characterized it as: A function $\Gamma: R \to \mathcal{B}$ (not necessarily additive) qualifies as a multiplicative generalized derivation if there exists a multiplicative derivation $\delta: \mathcal{B} \to \mathcal{B}$ such that $\Gamma(\nu\omega) = \Gamma(\nu)\omega + \nu\delta(\omega)$ $\forall$ $\nu, \omega \in \mathcal{B}$. Later, Dhara and Ali [5] expanded this definition of multiplicative generalized derivation by allowing δ to be any function on $\mathcal{B}$. Therefore, a function $\Gamma: R \to \mathcal{B}$ (not necessarily additive) is designated a $Mult.(G) - D$ $(Mult.(G) - D)$ if $\Gamma(\nu\omega) = \Gamma(\nu)\omega + \nu\delta(\omega)$ is satisfied $\forall$ $\nu, \omega \in \mathcal{B}$, where $\delta: R \to \mathcal{B}$ represents any function (not necessarily a derivation nor additive). Consequently, the framework of $Mult.(G) - D$ encompasses the framework of multiplicative derivation. Additionally, $Mult.(G) - D$ with $\delta = 0$ encompasses the notion of *multiplicative centralizer* (not necessarily additive).

It is evident that every generalized derivation constitutes a $Mult.(G) - D$ on $\mathcal{B}$. Nonetheless, the converse does not necessarily hold in all instances, as illustrated by the subsequent examples:

Example 1.1 *Consider* $\mathcal{B} = C[0,1]$, *the ring of all continuous real functions and define maps* $\delta: \mathcal{B} \to \mathcal{B}$, *as follows:*

$$\delta(f)(\nu) = \begin{cases} f(\nu) log|f(\nu)| & \forall\, f(\nu) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

And $\Gamma: \mathcal{B} \to \mathcal{B}$, as follows,

$$\Gamma(f)(\nu) = \begin{cases} f(\nu)(1 + log|f(\nu)|) & \forall\, f(\nu) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

It is evident that δ and Γ do not exhibit additive properties, while D functions as a multiplicative derivation, and Γ is classified as a $Mult.(G) - D$ associated with D.

Example 1.2 *Consider the ring* $\mathcal{B} = \{, \begin{pmatrix} 0 & 0 & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} | b, d, c \in \mathbb{R}\}$. *Define maps* $\Gamma: \mathcal{B} \to \mathcal{B}$ *and* $\delta: \mathcal{B} \to \mathcal{B}$ *as follows:*

$$\Gamma(\begin{pmatrix} 0 & 0 & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & 0 & bd \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and}$$

$$\delta(\begin{pmatrix} 0 & 0 & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & 0 & c^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is evident that δ and Γ do not exhibit additive properties, while D functions as a multiplicative derivation, and Γ is classified as a $Mult.(G) - D$ associated with D.

In the last thirty years, some researchers have demonstrated commutativity theorems for certain prime or semiprime rings having automorphisms or derivations which are centralizing or commuting on certain ordered subsets of $\mathcal{B}$. (see [6], [7], [8], [9], [10] and [11], where further references can be found).

Let S be a non-empty subset of $\mathcal{B}$. We define a function $\Gamma: \mathcal{B} \to \mathcal{B}$ as commutativity preserving on a subset S of $\mathcal{B}$ when the following condition holds: for any $\nu, \omega \in S$, if $[\nu, \omega] = 0$, then $[\Gamma(\nu), \Gamma(\omega)] = 0$. Furthermore, the function Γ is termed strong commutativity preserving (abbreviated as SCP) on S if $\forall$ elements $\nu, \omega \in S$, the equality $[\nu, \omega] = [\Gamma(\nu), \Gamma(\omega)]$ is satisfied. An expanding body of research exists concerning strong commutativity preserving (SCP) functions and derivations (see works by [12], [13], [14], among others)

In [15], Ali demonstrated that when $\mathcal{B}$ represents a semiprime ring and f constitutes an endomorphism that functions as a SCP map on a nonzero ideal U of $\mathcal{B}$, then f necessarily operates as a commuting map on U. Additionally, in [16], Samman established that an epimorphism of a semiprime ring exhibits strong commutativity preserving properties if and only if it is centralizing. Both derivations and SCP maps have been thoroughly investigated by numerous scholars within the domains of operator algebras, prime rings, and semiprime rings as well.

This article aims to establish several theorems concerning $Mult.(G)-D$ on semiprime rings that hold significant independent value. Specifically, our investigations extend previously established results by generalizing in two directions: first, by replacing a two-sided ideal with a left-sided ideal L (abbreviated as l-ideal), and second, by substituting a generalized derivation with a $Mult.(G)-D$ within the framework of semiprime rings. For our analysis, we will examine Γ and Δ as $Mult.(G)-D$ associated with the maps δ and ξ respectively. Our research will focus on investigating the following algebraic conditions: $[\Gamma(\nu),\Gamma(\omega)]=\pm[\nu,\omega]$ (SCP map Γ), $[\Gamma(\nu),\omega]=\pm[\nu,\Delta(\omega)]$, $[\xi(\nu),\Gamma(\omega)]=\pm[\nu,\omega]$ and $[\xi(\nu),\omega]=\pm[\nu,\Gamma(\omega)]$ $\forall \nu,\omega\in L$.

The following lemmas will be essential for establishing our results.

Lemma 1.3 *[17] . Let $\mathcal{B}$ be a 2-torsion free semiprime ring and L a l-ideal of $\mathcal{B}$. If elements $a,b\in\mathcal{B}$ satisfy the condition $a\nu b+b\nu a=0$ $\forall\nu\in L$, then $a\nu b=0$ and $b\nu a=0$ $\forall\nu\in L$.*

Lemma 1.4 *[11, Lemma 2.1] . Let $\mathcal{B}$ be a semiprime ring, $\mathcal{L}$ a nonzero two-sided ideal of $\mathcal{B}$. If an element $a\in\mathcal{L}$ satisfies the condition $a\nu a=0$ $\forall\nu\in\mathcal{L}$, then $a=0$.*

2- SCP condition on $Mult.(G)-D$ on l-ideal

Theorem 2.1 *Consider a 2-torsion free semiprime ring $\mathcal{B}$, L a nonzero l-ideal of $\mathcal{B}$ and $\Gamma:\mathcal{B}\to\mathcal{B}$ a $Mult.(G)-D$ linked to the map ξ. If $\Gamma(\nu\omega)=\nu\Gamma(\omega)+\xi(\nu)\omega$ $\nu,\omega\in L$ and Γ is SCP on L, then $L[\xi(\nu),\nu]=0$ and $L[\Gamma(\nu),\nu]=0$, $\forall\nu\in L$.*

Proof. Given that Γ exhibits the SCP property on L, it follows that

$$[\Gamma(\nu),\Gamma(\omega)]=[\nu,\omega]\quad\forall\,\nu,\omega\in L.\quad(1)$$

By substituting ω with $\omega\nu$ in (1), we obtain

$$[\Gamma(\nu),\Gamma(\omega)]\nu+\Gamma(\omega)[\Gamma(\nu),\nu]+[\Gamma(\nu),\omega]\xi(\nu)+\omega[\Gamma(\nu),\xi(\nu)]=[\nu,\omega]\nu\ \forall\,\nu,\omega\in L.\quad(2)$$

Multiplying (1) to the right by ν, we have

$$[\Gamma(\nu),\Gamma(\omega)]\nu=[\nu,\omega]\nu\quad\forall\,\nu,\omega\in L.\qquad(3)$$

Combining (2) and (3), we obtain

$$\Gamma(\omega)[\Gamma(\nu),\nu]+[\Gamma(\nu),\omega]\xi(\nu)+\omega[\Gamma(\nu),\xi(\nu)]=0\quad\forall\,\nu,\omega\in L.\ (4)$$

Now, if we replace ω with $z\omega$ in (4) and apply (4), we obtain

$$\xi(z)\omega[\Gamma(\nu),\nu]+[\Gamma(\nu),z]\omega\xi(\nu)=0\quad\forall\,\nu,\omega,z\in L.\qquad(5)$$

Take $z=\nu$, we have $\xi(\nu)\omega[\Gamma(\nu),\nu]+[\Gamma(\nu),\nu]\omega\xi(\nu)=0$ $\forall\nu,\omega\in L$. Lemma 1.3, gives

$$[\Gamma(\nu),\nu]\omega\xi(\nu)=0\quad\text{for all }\nu,\omega\in L.\qquad(6)$$

Since $\Gamma(\nu^2)=\Gamma(\nu)\nu+\nu\xi(\nu)=\nu\Gamma(\nu)+\xi(\nu)\nu$ $\forall\nu\in L$, this gives

$$[\Gamma(\nu),\nu]=[\xi(\nu),\nu]\quad\forall\,\nu\in L.\qquad(7)$$

Using (7) in (6) we have $[\xi(\nu),\nu]\omega\xi(\nu)=0\ \forall\nu\in L$, that is $L[\xi(\nu),\nu]\mathcal{B}L[\xi(\nu),\nu]=0\ \forall\nu\in L$. Using the semiprimeness of $\mathcal{B}$, we deduce that $L[\xi(\nu),\nu]=0\ \forall\ \nu\in L$. Substituting into (7), we obtain $L[\Gamma(\nu),\nu]=0\ \forall\nu\in L$. The case $[\Gamma(\nu),\Gamma(\omega)]=-[\nu,\omega]\ \forall\nu,\omega\in L$ is similar.

Remark 2.2 *The result can be established in a similar manner for the case where* $[\Gamma(\nu),\Gamma(\omega)]=-[\nu,\omega]$ *holds* $\forall\nu,\omega\in L$.

Corollary 2.3 *Let* $\mathcal{B}$ *be a 2-torsion free semiprime ring, and let* $\Gamma:\mathcal{B}\to\mathcal{B}$ *be a* $Mult.(G)-D$ *associated with the map* ξ. *If* $\Gamma(\nu\omega)=\nu\Gamma(\omega)+\xi(\nu)\omega\ \forall\nu,\omega\in\mathcal{B}$, *and if* Γ *is SCP on* L *or satisfies* $[\Gamma(\nu),\Gamma(\omega)]=-[\nu,\omega]\ \forall\nu,\omega\in\mathcal{B}$, *then* ξ *and* Γ *commute on* $\mathcal{B}$.

Theorem 2.4 *Let* $\mathcal{B}$ *be a semiprime ring,* L *a nonzero left ideal of* $\mathcal{B}$, *and* Γ,Δ *two* $Mult.(G)-D$ *associated with the maps* δ *and* ξ, *respectively. If* $[\Gamma(\nu),\omega]=[\nu,\Delta(\omega)]$ *or* $[\Gamma(\nu),\omega]=-[\nu,\Delta(\omega)],\forall\nu,\omega\in L$, *then* $L[\delta(\nu),\nu]=0$ *and* $L[\xi(\nu),\nu]=0,\forall\nu\in L$.

Proof. Assume

$$[\Gamma(\nu),\omega]=[\nu,\Delta(\omega)]\quad\forall\ \nu,\omega\in L.\quad(8)$$

If we replace in (8) ν by $\nu\omega$, we get

$$[\Gamma(\nu),\omega]\omega+[\nu,\omega]\delta(\omega)+\nu[\delta(\omega),\omega]=[\nu,\Delta(\omega)]\omega+\nu[\omega,\Delta(\omega)]\quad\forall\ \nu,\omega\in L.\ (9)$$

On the other hand, if we multiply (8) on the right by ω, we obtain

$$[\Gamma(\nu),\omega]\omega=[\nu,\Delta(\omega)]\omega\quad\forall\ \nu,\omega\in L.\qquad(10)$$

Subtracting (10) from (9), we get

$$[\nu,\omega]\delta(\omega)+\nu[\delta(\omega),\omega]=\nu[\omega,\Delta(\omega)],\quad\forall\nu,\omega\in L.\qquad(11)$$

Now if we substitute ν by $r\nu$ in (11), we have

$$r[\nu,\omega]\delta(\omega)+[r,\omega]\nu\delta(\omega)+r\nu[\delta(\omega),\omega]=r\nu[\omega,\Delta(\omega)]\quad\forall\ \nu,\omega\in L,r\in\mathcal{B}.\quad(12)$$

If we multiply (11) on the left by r, we get

$$r[\nu,\omega]\delta(\omega)+r\nu[\delta(\omega),\omega]=r\nu[\omega,\Delta(\omega)]\quad\forall\ \nu,\omega\in L,r\in\mathcal{B}.\quad(13)$$

From (12) and (13), we obtain $[r,\omega]\nu\delta(\omega)=0$. If we substitute r with $\delta(\omega)$, we have

$$[\delta(\omega),\omega]\nu\delta(\omega)=0\quad\forall\ \nu,\omega\in L.\quad(14)$$

If we multiply, (14) to the right by ω, we get $[\delta(\omega),\omega]\nu\delta(\omega)\omega=0$. Replacing ν by $\nu\omega$ in (14), to get $[\delta(\omega),\omega]\nu\omega\delta(\omega)=0$. By subtracting the last two identities, we arrive at $[\delta(\omega),\omega]\nu[\delta(\omega),\omega]=0$. Since L is a left ideal, it follows that $L[\delta(\omega),\omega]\mathcal{B}L[\delta(\omega),\omega]=0\ \forall\ \omega\in L$. Using the semiprimeness of $\mathcal{B}$, we conclude that $L[\delta(\omega),\omega]=0,\forall\omega\in L$. On the other hand, if we replace ω with $\omega\nu$ in (8), we obtain

$$[\Gamma(\nu),\omega]\nu+\omega[\Gamma(\nu),\nu]=[\nu,\Delta(\omega)]\nu+\omega[\nu,\xi(\nu)]+[\nu,\omega]\xi(\nu)\quad\forall\ \nu,\omega\in L.\quad(15)$$

Using (8) in (15) we get

$$\omega[\Gamma(\nu),\nu]=\omega[\nu,\xi(\nu)]+[\nu,\omega]\xi(\nu)\quad\forall\ \nu,\omega\in L.\ (16)$$

We can proceed in a similar manner as above and deduce that $L[\xi(\omega),\omega]=0,\forall\omega\in L$. The case where $[\Gamma(\nu),\omega]=-[\nu,\Delta(\omega)],\forall\nu,\omega\in L$ follows analogously.

Corollary 2.5 *Let $\mathcal{B}$ be a semiprime ring, $\mathcal{L}$ a nonzero two-sided ideal, and Γ,Δ two $Mult.(G)-D$ associated with the maps δ and ξ, respectively. If $[\Gamma(\nu),\omega]=[\nu,\Delta(\omega)]$ or $[\Gamma(\nu),\omega]=-[\nu,\Delta(\omega)]\forall\nu,\omega\in\mathcal{L}$, then δ and ξ commute on $\mathcal{L}$.*

Proof. By Theorem 2.4, we have $\mathcal{L}[\delta(\omega),\omega]=0\forall\omega\in\mathcal{L}$. Multiplying on the left by $[\delta(\omega),\omega]$, we obtain $[\delta(\omega),\omega]\mathcal{L}[\delta(\omega),\omega]=0$. By Lemma 1.4, it follows that $[\delta(\omega),\omega]=0\forall\omega\in\mathcal{L}$, which implies that δ commutes on $\mathcal{L}$. Similarly, we can show that ξ also commutes on $\mathcal{L}$.

Theorem 2.6 *Let $\mathcal{B}$ represent a semiprime ring that is free of 2-torsion, L a nonzero l-ideal within $\mathcal{B}$, and $\Gamma:\mathcal{B}\to\mathcal{B}$ a $Mult.(G)-D$ mapping linked to ξ. Suppose that $\forall\nu,\omega\in L$, the relation $\Gamma(\nu\omega)=\nu\Gamma(\omega)+\xi(\nu)\omega$ holds, along with either $[\xi(\nu),\Gamma(\omega)]=[\nu,\omega]$ or $[\xi(\nu),\Gamma(\omega)]=-[\nu,\omega]$. Then $L[\xi(\nu),\nu]=0$ and $L[\Gamma(\nu),\nu]=0,\forall\nu\in L$.*

Proof. Assume that

$$[\xi(\nu),\Gamma(\omega)]=[\nu,\omega]\quad\forall\,\nu,\omega\in L.\quad(17)$$

Substituting ω with $\omega\nu$ in (17), we obtain

$$[\xi(\nu),\Gamma(\omega)]\nu+\Gamma(\omega)[\xi(\nu),\nu]+[\xi(\nu),\omega]\xi(\nu)=[\nu,\omega]\nu\quad\forall\,\nu,\omega\in L.\quad(18)$$

Multiplying (17) to the right by ν, we obtain

$$[\xi(\nu),\Gamma(\omega)]\nu=[\nu,\omega]\nu\quad\forall\,\nu,\omega\in L.\quad(19)$$

Combining (18) and (19) we obtain

$$\Gamma(\omega)[\xi(\nu),\nu]+[\xi(\nu),\omega]\xi(\nu)=0\quad\forall\,\nu,\omega\in L.\quad(20)$$

If we substitute $z\omega$ for ω in (20), the result is

$$z\Gamma(\omega)[\xi(\nu),\nu]+\xi(z)\omega[\xi(\nu),\nu]+z[\xi(\nu),\omega]\xi(\nu)+[\xi(\nu),z]\omega\xi(\nu)=0\quad\forall\,\nu,\omega,z\in L.\quad(21)$$

Using (20) in (21) and take $z=\nu$, we get

$$\xi(\nu)\omega[\xi(\nu),\nu]+[\xi(\nu),\nu]\omega\xi(\nu)=0\quad\forall\,\nu,\omega\in L.\ (22)$$

Lemma 1.3, gives $[\xi(\nu),\nu]\omega\xi(\nu)=0,\forall\nu,\omega\in L$, this gives

$$L[\xi(\nu),\nu]\mathcal{B}L[\xi(\nu),\nu]=0,\forall\nu\in L.$$

Given that $\mathcal{B}$ is semiprime, it follows that $L[\xi(\nu),\nu]=0,\forall\nu\in L$. Since $\Gamma(\nu^2)=\Gamma(\nu)\nu+\nu\xi(\nu)=\nu\Gamma(\nu)+\xi(\nu)\nu\ \forall\ \nu\in L$, this gives $[\Gamma(\nu),\nu]=[\xi(\nu),\nu]$, thus $L[\Gamma(\nu),\nu]=L[\xi(\nu),\nu]=0,\forall\nu\in L$.

The case $[\xi(\nu),\Gamma(\omega)]=-[\nu,\omega],\forall\nu,\omega\in L$ is similar.

Corollary 2.7 *Let $\mathcal{B}$ be a semiprime ring that is free of 2-torsion, and let $\Gamma:\mathcal{B}\to\mathcal{B}$ be a $Mult.(G)-D$ mapping associated with ξ. If $\Gamma(\nu\omega)=\nu\Gamma(\omega)+\xi(\nu)\omega$ and $[\xi(\nu),\Gamma(\omega)]=[\nu,\omega]$ or $[\xi(\nu),\Gamma(\omega)]=-[\nu,\omega]$ holds $\forall\nu,\omega\in\mathcal{B}$, then ξ and Γ commute on $\mathcal{B}$.*

Theorem 2.8 *Let $\mathcal{B}$ represent a semiprime ring, L a nonzero l-ideal of $\mathcal{B}$, and Γ a $Mult.(G)-D$ linked to a mapping ξ. If it holds that $[\xi(\nu),\omega]=[\nu,\Gamma(\omega)]$ or $[\xi(\nu),\omega]=-[\nu,\Gamma(\omega)]$ $\forall$ elements $\nu,\omega\in L$, then it follows that $L[\delta(\nu),\nu]=0$ and $L[\Gamma(\nu),\nu]=0, \forall\nu\in L$.*

Proof. Assume that

$$[\xi(\nu),\omega]=[\nu,\Gamma(\omega)] \quad \forall\, \nu,\omega\in L. \quad (23)$$

By substituting ω with $\omega\nu$ in (23), we derive

$$[\xi(\nu),\omega]\nu+\omega[\xi(\nu),\nu]=[\nu,\Gamma(\omega)]\nu+\omega[\nu,\xi(\nu)]+[\nu,\omega]\xi(\nu) \quad \forall\, \nu,\omega\in L. \quad (24)$$

Multiplying (23) to the right by ν, we obtain

$$[\xi(\nu),\omega]\nu=[\nu,\Gamma(\omega)]\nu \quad \forall\, \nu,\omega\in L. \quad (25)$$

Combining (24) and (25), we have

$$2\omega[\xi(\nu),\nu]=[\nu,\omega]\xi(\nu) \quad \forall\, \nu,\omega\in L. \quad (26)$$

Now, by substituting ω with $z\omega$ in (26) and applying (26), we obtain

$$[\nu,z]\omega\xi(\nu)=0 \quad \forall\, \nu,\omega,z\in L. \quad (27)$$

By replacing z with rz in (27), we obtain

$$[\nu,r]z\omega\xi(\nu)=0 \quad \forall\, \nu,\omega,z\in Lr\in\mathcal{B}. \quad (28)$$

Replacing z by $\xi(\nu)z$ in (28), to get $0=[\nu,r]\xi(\nu)z\omega\xi(\nu)$, that is

$$[\nu,r]\xi(\nu)\mathcal{B}z\omega\xi(\nu)=(0) \quad \forall \quad \nu,\omega,z\in L, r\in R.$$

Interchanging z and ω and then subtracting one from the other, we have

$$[\nu,r]\xi(\nu)\mathcal{B}[z,\omega]\xi(\nu)=(0) \quad \forall \quad \nu,\omega,z\in L.$$

In particular, by setting $r=z$ and $\omega=\nu$, we have $[\nu,z]\xi(\nu)\mathcal{B}[\nu,z]\xi(\nu)=(0), \forall\nu,z\in L$. Since $\mathcal{B}$ is semiprime, it follows that

$$[\nu,z]\xi(\nu)=0 \quad \forall\, \nu,z\in L. \quad (29)$$

By multiplying (29) on the right by ν, we deduce that $[\nu,z]\xi(\nu)\nu=0, \forall\nu,z\in L$. Substituting z with $z\nu$, we find that $[\nu,z]\nu\xi(\nu)=0, \forall\nu,z\in L$. Subtracting these two equations yields $[\nu,z][\xi(\nu),\nu]=0, \forall\nu,z\in L$. Replacing z with $\xi(\nu)z$ in this result gives $[\nu,\xi(\nu)]z[\nu,\xi(\nu)]=0, \forall\nu,z\in L$, which can be rewritten as $L[\xi(\nu),\nu]\mathcal{B}L[\xi(\nu),\nu]=(0), \forall\nu\in L$. The semiprimeness of $\mathcal{B}$ implies that $L[\xi(\nu),\nu]=(0), \forall\nu\in L$. From our assumption, we have $[\nu,\Gamma(\nu)]=[\xi(\nu),\nu]$, and thus $L[\nu,\Gamma(\nu)]=L[\xi(\nu),\nu]=0, \forall\nu\in L$.

The case where $[\xi(\nu),\omega]=-[\nu,\Gamma(\omega)]$ $\forall\, \nu,\omega\in L$ follows similarly.

Corollary 2.9 *Let $\mathcal{B}$ be a semiprime ring, and let Γ be a $mult.(G)-D$ associated with a map ξ. If $[\xi(\nu),\omega]=[\nu,\Gamma(\omega)]$ or $[\xi(\nu),\omega]=-[\nu,\Gamma(\omega)], \forall\nu,\omega\in\mathcal{B}$, then Γ and ξ are commuting maps on $\mathcal{B}$.*

3- Example

The following example demonstrates that the semiprimeness condition in the preceding theorems is essential and cannot be omitted.

Example 3.1 $Let\ \mathcal{B} = \{\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in \mathbb{Z}\ (the\ set\ of\ integers)\}.$ *For any* $0 \neq b \in \mathbb{Z}$, $\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathcal{B} \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0)$, *then* $\mathcal{B}$ *is not a semiprime ring. Define* $\Gamma: \mathcal{B} \to \mathcal{B}$ *and* $\xi: \mathcal{B} \to \mathcal{B}$ *is given by:*

$\Gamma(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $\xi(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & a^2 & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$. It is easy to verify that Γ is a $Mult.(G) - D$ associated with the map ξ. Furthermore, it can be directly confirmed that: $\Gamma(\nu)\omega = \omega\Gamma(\nu) = 0,\ \ \forall \nu, \omega \in \mathcal{B}$. Consequently, $[\Gamma(\nu), \omega] = \pm[\nu, \Gamma(\omega)]$, is hold for all $\nu, \omega \in \mathcal{B}$. However, despite these properties, $\xi \neq 0$, ξ is not commuting on $\mathcal{B}$, and $\Gamma(\nu\omega) \neq \Gamma(\nu)\omega$.

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