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A Dual-Channel Supply Chain Model with Distribution-Free Approach using Customization

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Abstract:
In today's competitive scenario, many business organizations provide their customers with personalization options, thereby increasing customer satisfaction through a wide range of choices. This strategy boosts profitability for various technologically advanced businesses. In this study, a dual-channel supply chain model with customization is developed to enhance firm profits. Dual-channel retailing delivers personalized products online, while standard products are available via traditional retail channels.
This paper modifies the existing dual-channel model by considering customer switching behavior between the two channels. It also introduces a pre-assigned threshold that signifies demand decline when the price difference between the online and offline channels exceeds a specific limit. Furthermore, the model incorporates demand uncertainty and fluctuation using a distribution-free approach. Demand price-sensitivity is also considered in constructing the centralized dual-channel supply chain model. The study concludes that adopting a dual-channel policy leads to better predictability and performance compared to conventional single-channel systems. Additionally, customer switching depends on which channel offers a lower price, especially when the price gap surpasses the threshold.
Keywords: supply chain management, max-min distribution-free method, dual-channel retailing, demand uncertainty, customization, SDG-12

1. INTRODUCTION

The introduction of technology has made the retailing system more organized and competitive. E-commerce is increasing with the advancement of technology. The concept of smart manufacturing is evolving from simple digitization and utilizing the data collected from the

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customers to make decisions on the products.Sustainability in smart supply chain management also takes into account all three pillars of sustainability like environmental, social, and economic due to inflexible rules and regulations. In 2023, air pollution claimed approximately 7.1 million deaths (or approximately one in 10 deaths) globally, consequently, declaring it the fifth leading risk factor for mortality according to the State of global air/2019. Furthermore, the report discovered that air pollution claims more lives than traffic accidents or malaria. the ejection of $CO2$ from firms is manifested as a rate of manufacturing, speed of machinery, and consumption of energy function as all these contribute to the total greenhouse gases (Bazan et al., 2015). Moreover, government charges penalties to the firms once they cross the carbon emission ceiling. The European Union Emission Trading System (EU-ETS) works on a cap-and-trade policy where firms are penalized if they cross the carbon emission limit. If emissions are less than the limit firms can trade with another firm that surpasses the limit. Thus, many organizations are emphasizing on reducing greenhouse gas emissions by investing in green technologies (Reddy et al., 2020). By prioritizing sustainability, the company is committed to driving positive change in the industry and making substantial contributions to the global effort to combat climate change. Also, a sustainable model is exemplified by including out-of-order products and manageable $CO2$ emissions from the firm (Mashud et al., 2020). A supply chain model is all about handling the complete manufacturing of items from raw materials to the final product.

Table 1: Overview of Related Literature

Author	**DCSC**	**SP**	**CP**	**UCS**	**SSMD**	**MTO**	**DU**	**DFA**	**CLT**	**MM**
Takahashi et al.	✓	✓								
Zhang and Choi	✓	✓								
Chiang and Monahan	✓	✓								
Shao	✓	✓				✓				
Batarfi et al.	✓	✓				✓				✓
Yue and Liu	✓	✓					✓			
Yan and Pei	✓	✓							✓	
Sarkar et al.										✓
Chiang et al.	✓	✓								
Dan et al.	✓	✓								
Hua et al.	✓	✓						✓		
Majumder et al.										✓
Kaya et al.	✓	✓							✓	
Tsay and Agrawal	✓	✓								
Modak and Kelle	✓	✓								
Li et al.	✓	✓				✓				
Jing et al.	✓	✓		✓						
Zhou et al.	✓	✓								
Majumder et al.								✓		
Proposed paper	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

Abbreviations: DCSC: Dual-channel supply chain; SP: Standard product; CP: Customized product; UCS: Unequal customer shifting (online/offline); MTO: Make-to-order; DU: Demand uncertainty; DFA: Distribution-free approach; CLT: Controllable lead time; MM: Markup margin.

Table 2: Key Decision Variables in the Model

Decision Variable	Description
Q_d	Quantity of core products ordered for customization (units)
k	Safety factor (units) to account for demand variability
L	Retailer's lead time (days)
Q_r	Quantity of standard products ordered by the retailer (units)
n	Number of batches delivered from the manufacturer to the retailer in one production cycle (positive integer)

Table 3: Parameters of the model.

Parameters	Description
P_r	Production rate for the standard product ($P_r > a_1$) (positive number)
β_1	Price sensitivity in retail channel (customer/day)
β_2	Price sensitivity in online channel (customer/day)
δ_1	Number of customers switching from retail channel to online channel
a_2	Number of customers prefer online channel
σ	Standard deviation of demand per unit time
C_p	Production cost for standard product ($/unit)
δ_2	Number of customers switching from online channel to retail channel
D_r	Variable demand of retail channel (units/year)
π	Unit backlogging cost for the retailer ($/unit)
p_r	Retailer's selling price of the standard product ($p_r > C_p$) ($/order)
D_d	Variable demand of online channel (units/year)
ϕ_{di}	Percentage of core product stock used for customized product ($i = 1, 2, \ldots, N$)
h_1	Manufacturer's holding cost including financial and storage cost ($/unit)
A_r	Ordering cost of the retailer per order ($/order)
m	Markup margin (percentage)
C_{di}	Production cost for customized product ($i = 1, 2, \ldots, N$) ($/unit)
p_{di}	Manufacturer's selling price of customized product i ($/order)
r_v	Holding cost rate of manufacturer ($/unit/unit time)
r_b	Holding cost rate of retailer ($/unit/unit time)
S_d	Manufacturer's setup cost for core product customized product ($/setup)
S_r	Manufacturer's setup cost for standard product ($/setup)
C_b	Unit production cost paid by retailer ($/unit)
C_{vr}	Unit production cost paid by manufacturer ($/unit)
R	Reorder point of the retailer (units)
s	Safety factor of the retailer (units)
$E(\cdot)$	Mathematical expectation

2. MATHEMATICAL MODEL

This part explains the retailing system's total cost function, optimal decisions, and algorithm for evaluating the solution of this model.

2.1. *FORMULATION OF TOTAL COST*

The equations of the total cost of manufacturer and retailer for core and personalized items are derived in this segment.

2.1.1 *Cost parameter for Retailer*

The costs associated with the core item, bore by the retailer are given as follows. **Ordering cost**

The cost bear by the retailer for placing an order is given as

$$OC_1 = \frac{O_r d_1}{q_1} \tag{1}$$

Holding cost

Retailer's inventory follows an "Economic Order Quantity (EOQ)" model. The retailer places the order of q_1 quantity whenever the inventory measure sinks to R reorder point. Thus, inventory measure sinks to $R - d_1 l$ before the order is received. Moreover, $q_1 + R - d_1 l$ depicts the expected level of inventory after receiving quantity q_1. Over a cycle, the average inventory is represented as $\frac{q_1}{2} + R - d_1 l$. Therefore, the retailer's inventory cost of holding is

$$HC_3 = h_r C_b \left(\frac{q_1}{2} + R - d_1 l\right) \tag{2}$$

Shortage cost

If ρ is the stochastic demand of lead time and R is the reorder point then, the expected shortage at the end of the cycle is given as $E(\rho - R)^+$ and the shortage cost is

$$\text{Short}_{\text{cost}} = \frac{\nu d_1}{q_1} E(\rho - R)^+ \tag{3}$$

$$\leq \frac{\nu d_1}{q_1} \left[\frac{\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2 \sigma^2 l - 2 d_1 l \delta \sigma \sqrt{l}} + (d_1 l - \delta \sigma \sqrt{l})}{2} \right] \tag{4}$$

Lead time crashing cost

The lead time 'l' has 'm' collectively independent elements. For the 'jth' element, a_j =minimum time span, b_j = normal time span, and c_j = crashing cost per unit time. Practically, $c_1 < c_2 < \ldots < c_m$. Additionally, $l_0 = \sum_{j=1}^{n}$ and l_i be the duration of the lead time with elements 1, 2, ..., i sinks to their least span of time, then L_i can be represented as $l_i = l_0 - \sum_{j=1}^{n}(b_j - a_j)$, $i = 1, 2, \ldots, n$. The lead time crashing cost per cycle CL is manifested as $CL = c_i(l_{i-1} - l) + \sum_{j=1}^{i-1} c_j(b_j - a_j)$ (Sarkar et al., 2018). Therefore, the cost of lead time crashing is

$$CLL = \frac{d_1 CL}{q_1} \tag{5}$$

Total cost

The retailer's aggregate cost per unit of time by selling the core item is

TC_1 = Cost of ordering + Cost of holding + Cost of shortage + Cost of lead time crashing

$$TC_1 = \frac{O_r d_1}{q_1} + h_r C_b \left(\frac{q_1}{2} + R - d_1 l\right) + \frac{\nu d_1}{q_1} E(\rho - R)^+ + \frac{d_1 CL}{q_1} \tag{6}$$

2.1.2 *Manufacturer's cost parameters for standard product*

The costs associated with the standard product, bore by the manufacturer are given as follows.

Setup cost

The manufacturer's setup cost is equal to $\frac{S_c^m d1}{\eta q_1}$ that is derived from machines and cutting tools i.e., the setup of resources. Moreover, as q_1 is the quantity ordered by the retailer, $\eta q1$ quantity is manufactured by the producer where η is a positive integer. Therefore, the setup orders of the producer are $\frac{d1}{\eta q1}$ and the cost is $\frac{S_c^m d1}{\eta q_1}$. Henceforth, the setup cost for the manufacturer is given as

$$SC_1 = \frac{S_c^m d1}{\eta q_1} \tag{7}$$

Holding cost

Holding cost is induced when the producer stores the produced product for some time. The producer's average inventory is evaluated as follows. Whereas, the membership functions

$$\frac{d_1\left(\eta q_1\left(\frac{q_1}{P_1} + \frac{(\eta-1)q_1 d_1}{\eta}\right) - \frac{\eta^2 q_1^2}{2P_1} - \left(\frac{q_1^2}{d_1}\left(1 + 2 + \ldots + (\eta - 1)\right)\right)\right)}{\eta q_1} = \frac{q_1}{2}\left(\eta\left(1 - \frac{d_1}{P_1}\right) - 1 + \frac{2d_1}{P_1}\right)$$

The manufacturer's cost of holding is $\frac{h_m c_p q_1}{2}\left(\eta\left(1 - \frac{d_1}{P_1}\right) - 1 + \frac{2d_1}{P_1}\right)$. . Therefore, the expected cost of holding per unit time per unit item is

$$HC_1 = \frac{h_m c_p q_1}{2}\left(\eta\left(1 - \frac{d_1}{P_1}\right) - 1 + \frac{2d_1}{P_1}\right)$$

Manufacturer's production cost

For manufacturing the finished product, The manufacture has to invest in the form of capital, energy, and labors which is termed as cost of production. Thus, the cost of production is

$$PC_1 = C_{vr} d_1 \tag{8}$$

Cost of imperfect items

In the course of production of ηq_1 lost size, nearly $\frac{\eta q_1 \zeta}{2}$ defective items are believed to produce. Thus, the year-long cost of imperfect items is believed to be

$$IC_1 = \frac{s d_1 \eta q_1 \zeta}{2} \tag{9}$$

Investment in the quality improvement of the product

The capital expenditure I_ζ is presumed for reducing the probability of out-of-control ζ. Therefore, I_ζ can be exhibited as $I_\zeta = b \ln\left(\frac{\zeta_0}{\zeta}\right)$ for $0 < \zeta \leq \zeta_0$, i.e., $I_\zeta = b(\ln \zeta_0 - \ln \zeta)$, where as initial probability is given by ζ_0, for which the manufacturing can go out-of-control. And $b = \frac{1}{\Delta}$, where Δ depicts the shrink percentage in ζ per dollar rise in I_ζ. Therefore, the financing done for improving the quality of the product is given as

$$I = \alpha b(\ln \zeta_0 - \ln \zeta) \tag{10}$$

Where the capital investment's fraction of annual cost is α.

Total cost

The aggregate cost of selling the core item through the retail channel per unit of time is

TC_2 = Cost of setup + Cost of holding + Cost of production + Cost of imperfect items + Investment in the quality improvement of the product

$$TC_2 = \frac{S_c^m d_1}{\eta q_1} + \frac{h_m c_p q_1}{2}\left[\eta\left(1 - \frac{d_1}{P_1}\right) - 1 + \frac{2d_1}{P_1}\right] + C_{vr} d_1 + \frac{s d_1 \eta q_1 \zeta}{2} + \alpha b(\ln \zeta_0 - \ln \zeta) \tag{11}$$

2.1.3 *Manufacturer's cost parameters for customized product*

The costs associated with the customized product, bore by the manufacturer are given as follows.

Setup cost

Setup cost includes the charge of supplies and materials, i.e., cutting tools, materials, and machines. Thus the setup cost for the manufacturer is given as

$$SC_2 = \frac{S_p^m d_2}{q_2} \tag{12}$$

Holding cost

The manufacturer's average inventory is evaluated as

$$Average \quad inventory = \frac{q_2}{2}\left(1 - \frac{d_2}{P_2}\right)$$

The cost of holding item by the manufacturer is $\frac{hq_2}{2}\left(1 - \frac{d_2}{P_2}\right)$.

Therefore, the expected cost of holding per unit time per unit item is

$$HC_2 = \frac{hq_2}{2}\left(1 - \frac{d_2}{P_2}\right)$$

Manufacturing cost

For manufacturing the customized product, the manufacturer has to invest separately in the form of workers, energy, and capital. Thus, the production cost is given as

$$MC_2 = \sum_{j=1}^{K} c_j \psi_i d_2 \tag{13}$$

Total cost

In e-commerce, the aggregate cost of production per unit of time by selling the personalized item from the online route is

$TC_3 =$ Setup cost + Holding cost + Manufacturing cost

$$TC_3 = \frac{S_p^m d_2}{q_2} + \left(\frac{hq_2}{2}\right)\left(1 - \frac{d_2}{P_2}\right) + \sum_{j=1}^{K} c_j \psi_i d_2 \tag{14}$$

Total cost $TC_S$of the retailing system following a single route is derived by summing TC_1 and TC_2 which is

$$TC_S = TC_1 + TC_2$$

$$\begin{aligned} TC_S = \frac{S_c^m d_1}{\eta q_1} + \frac{h_m c_p q_1}{2}\left[\eta\left(1 - \frac{d_1}{P_1}\right) - 1 + \frac{2d_1}{P_1}\right] + C_{vr} d_1 + \frac{s d_1 \eta q_1 \zeta}{2} + \alpha b(\ln \zeta_0 - \ln \zeta) \\ + \frac{O_r d_1}{q_1} + h_r C_b\left(\frac{q_1}{2} + R - d_1 l\right) + \frac{\nu d_1}{q_1} E(\rho - R)^+ + \frac{d_1 CL}{q_1} \end{aligned} \tag{15}$$

Thus, for retailing system having centralized strategy with single-route, the expected aggregate cost TC_S is given by summing TC_1 and TC_2 that is

$$\begin{aligned} TC_S &= TC_1 + TC_2 \\ &= \frac{O_r d_1}{q_1} + h_r C_b\left(\frac{q_1}{2} + \delta\sigma\sqrt{l}\right) + \frac{d_1 CL}{q_1} \\ &\quad + \frac{\nu d_1}{2q_1}\left[\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2\sigma^2 l - 2d_1 l\delta\sigma\sqrt{l}} + (d_1 l - \delta\sigma\sqrt{l})\right] \\ &\quad + \frac{S_c^m d_1}{q_1 \eta} + \frac{h_m c_p q_1}{2}\left[\eta\left(1 - \frac{d_1}{P_1}\right) - 1 + \frac{2d_1}{P_1}\right] \\ &\quad + C_{vr} d_1 + \frac{s d_1 \eta q_1 \zeta}{2} + \alpha b(\ln \zeta_0 - \ln \zeta) \end{aligned} \tag{16}$$

And for the retailing system having a centralized strategy with dual-route, the expected aggregate cost TC_D is given by summing $TC1$, $TC2$, and $TC3$ is

$$\begin{aligned} TC_D &= TC_1 + TC_2 + TC_3 \\ &= \frac{O_r d_1}{q_1} + h_r C_b\left(\frac{q_1}{2} + \delta\sigma\sqrt{l}\right) + \frac{d_1 CL}{q_1} \\ &\quad + \frac{\nu d_1}{2q_1}\left[\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2\sigma^2 l - 2d_1 l\delta\sigma\sqrt{l}} + (d_1 l - \delta\sigma\sqrt{l})\right] + \frac{S_c^m d_1}{q_1\eta} \\ &\quad + \frac{h_m c_p q_1}{2}\left[\eta\left(1 - \frac{d_1}{P_1}\right) - 1 + \frac{2d_1}{P_1}\right] + C_{vr} d_1 + \frac{s d_1 \eta q_1 \zeta}{2} \\ &\quad + \alpha b(\ln\zeta_0 - \ln\zeta) + \frac{S_p^m d_2}{q_2} + \left(\frac{hq_2}{2}\right)\left(1 - \frac{d_2}{P_2}\right) + \sum_{j=1}^{K} c_j\psi_i d_2 \end{aligned} \quad (17)$$

2.2. *OUTCOME OF THE RETAILING SYSTEM*

Since, Equation (19) is non-linear in nature therefore for constant positive integer μ, partial derivative of the cost with respect to q_2, q_1, δ, & ζ is taken and then equated to zero for obtaining the optimal solution. Henceforth, we get

$$q_2^* = \sqrt{\frac{S_p^m d_2}{\frac{h}{2}\left(1 - \frac{d_2}{P_2}\right)}} \quad (18)$$

$$q_1^* = \sqrt{\frac{-d_1 CL + S_c^m d_1 + O_r d_1 + \frac{\nu d_1}{2}\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2\sigma^2 l - 2d_1 l\delta\sigma\sqrt{l}} + d_1 l - \delta\sigma\sqrt{l}}{h_r C_b + \frac{h_m}{2}\left(\eta\left(1 - \frac{d_1}{P_1}\right) - 1 + \frac{2d_1}{P_1}\right) + \frac{s d_1 \eta\zeta}{2}}} \quad (19)$$

$$\delta^* = \frac{\frac{(-\sigma\sqrt{q_1 C_b h_r(-q_1 C_b h_r + \nu d_1)})}{-q_1 C_b h_r + \nu d_1} + \frac{d_1 \nu\sigma\sqrt{q_1 C_b h_r(-q_1 C_b h_r + \nu d_1)l}}{2q_1 C_b h_r(\nu d_1 - q_1 C_b h_r)} + l d_1}{\sqrt{l}\sigma} \quad (20)$$

$$\zeta^* = \frac{\alpha b}{s d_1 \eta q_1} \quad (21)$$

Optimal solution for q_1^*, q_2^*, δ^*, ζ^* and so obtained dependents upon each other. Moreover, it is difficult to evaluate the closed-form expression for the centralized total cost function. Therefore, a numerical procedure is required for evaluating these optimal values. Along with the following algorithm an iteration method is utilized to find the managerial decisions.

2.3. *ANALYZING DIFFERENT PILLARS OF SUSTAINABILITY IN SINGLE CHANNEL*

2.3.1 *Environmental pillar*

1. CO_2 emissions throughout the production

The aggregate of CO_2 emitted (ton/unit) in the production process is:

$$E = E(P_1) = x_1 P_1^2 - x_2 P_1 + x_3 \tag{22}$$

where x_1, x_2, and x_3 can be experimentally verified from Bazan et al. (2015). The experiment gives a way to understand how operating a machine tool adds a carbon emission burden and gave a quadratic Equation (24) reflecting the equivalent CO_2 emissions. Therefore, the cost of the carbon exude because of production is given as

$$EC_1 = Ed_1 C^{ec} \tag{23}$$

2. Penalty because of excess of CO_2 emission

   When the carbon burden from the firm surpasses the predetermined ceiling, then the penalty cost is collected from it. Thus, the penalty cost as a consequence of carbon emission is given as

$$EC_2 = \sum_{i=1}^{l} Y_i C_{ep,i} \tag{24}$$

   where

$$Y_i = \begin{cases} 1 & Ed_1 > E_{li} \quad (i = 1, 2, ..., l) \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

2.3.2 *Social pillar*

Cumulative social cost as a result of hard labor, medical maintenance, welfare, and social consciousness (Khan et al., 2016) for single channel supply chain model, is

$$SC_1 = \eta S_1 q_1 \tag{26}$$

Therefore, for the supply chain having a centralized policy and following a single route, the expected aggregate cost TC_S is

$$\begin{aligned} TC_S = {} & \frac{O_r d_1}{q_1} + h_r C_b \left(\frac{q_1}{2} + \delta\sigma\sqrt{l}\right) + \frac{d_1 CL}{q_1} \\ & + \frac{\nu d_1}{2q_1}\left[\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2\sigma^2 l - 2d_1 l\delta\sigma\sqrt{l}} + (d_1 l - \delta\sigma\sqrt{l})\right] + \frac{S_c^m d_1}{q_1\eta} \\ & + \frac{h_m q_1}{2}\left[\eta\left(1 - \frac{d_1}{P_1}\right) - 1 + \frac{2d_1}{P_1}\right] + C_{vr} d_1 + \frac{s d_1 \eta q_1 \zeta}{2} \\ & + \alpha b(\ln\zeta_0 - \ln\zeta) + Ed_1 C_{ec} + \sum_{i=1}^{l} Y_i C_{ep,i} + \eta S_1 q_1 \end{aligned} \tag{27}$$

2.4. *ANALYZING DIFFERENT PILLARS OF SUSTAINABILITY IN DUAL CHANNEL*

2.4.1 *Environmental pillar*

1. CO_2 emissions throughout the production.

The aggregate of CO_2 emitted (ton/unit) in the production process is:

$$E' = E(P) = x_1 P^2 - x_2 P + x_3 \tag{28}$$

where x_1, x_2, and x_3 can be experimentally verified from Bazan et al. (2015). The experiment gives a way to understand how operating a machine tool adds a carbon emission burden and gave a quadratic Equation (30) reflecting the equivalent CO_2 emissions. Moreover, it also reflects that increased cutting speed converts tool wears into considerable lofty which shortens its life span and elevates CO_2 emissions. Further, there is also a trade relation with the cutting speed as carbon stress build-up by electricity utilization and the cooling liquid is comparable with time. The quadratic Equation (30) manifests the behavior of the corresponding carbon ejection. Therefore, the cost of the carbon exude because of production is given as

$$EC_1' = E'dC^{ec} \tag{29}$$

2. Penalty on firm because of excess of CO_2 emission

   When the carbon burden from the firm surpasses the predetermined ceiling then the penalty cost is collected from it. Thus, the penalty cost as a consequence of carbon emission is given as

$$EC_2' = \sum_{i=1}^{l} Y_i C_{ep,i} \tag{30}$$

where

$$Y_i = \begin{cases} 1 & E'd > E_{li} \quad (i = 1, 2, ..., l) \\ 0 & \text{otherwise} \end{cases} \tag{31}$$

2.4.2 *Social pillar*

Cumulative social cost as a result of hard labor, medical maintenance, welfare, and social consciousness (Khan et al., 2016) for supply chain model following dual route, is

$$SC_1 = \eta \left(S_1 q_1 + S_2 q_2 \right) \tag{32}$$

Therefore, for the supply chain having a centralized policy and following a dual route, the expected aggregate cost TC_D is

$$\begin{aligned}TC_D &= \frac{S_c^m d_1}{q_1\eta} + \frac{h_m q_1}{2}\left[\eta\left(1-\frac{d_1}{P_1}\right) - 1 + \frac{2d_1}{P_1}\right] + C_{vr}d_1 + \frac{sd_1\eta q_1\zeta}{2} \\ &+ \alpha b(\ln\zeta_0 - \ln\zeta) + \frac{O_r d_1}{q_1} + h_r C_b\left(\frac{q_1}{2} + \delta\sigma\sqrt{l}\right) + \frac{d_1 CL}{q_1} \\ &+ \frac{\nu d_1}{2q_1}\left[\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2\sigma^2 l - 2d_1 l\delta\sigma\sqrt{l}} + (d_1 l - \delta\sigma\sqrt{l})\right] + \frac{S_p^m d_2}{q_2} \\ &+ \left(\frac{hq_2}{2}\right)\left(1-\frac{d_2}{P_2}\right) + \sum_{j=1}^{K} c_j\psi_i d_2 + E'dC^{ec} + \sum_{i=1}^{l} Y_i C_{ep,i} + \eta(S_1 q_1 + S_2 q_2) \quad (33)\end{aligned}$$

2.5. *SOLUTION ALGORITHM*

To solve the current model succeeding ARV algorithm is applied.

1. Assign all parameters values as defined in Table 2.
2. Set $\eta = 1$.
3. Perform the underneath steps for all the values of $l_j; j = 1, 2, \ldots$
    (a) From Equation (20), evaluate q_2.
    (b) From Equation (21), evaluate q_1.
    (c) From Equation (22), evaluate δ.
    (d) From Equation (23), evaluate ζ.
    (e) Redo the steps from 3(a) to 3(d) unless there is no variation in the values of q_1, q_2, δ, and ζ until a specified accuracy level.
4. Evaluate E_1d and $E'd$ using the following steps:
    (a) If $E_1d < 220$ then $C_{ep,i} = 0$ and $Y_i = 0$ else go to step 4(b) and if $E'd < 220$ then $C_{ep,i} = 0$ and $Y_i = 0$ else go to step 4(b).
    (b) If $220 < E_1d < 330$ then $C_{ep,i} = 1000$ and $Y_i = 1$ else go to step 4(c) and if $220 < E'd < 330$ then $C_{ep,i} = 1000$ and $Y_i = 1$ else go to step 4(c).
    (c) If $330 < E_1d < 440$ then $C_{ep,i} = 2000$ and $Y_i = 1$ else go to step 4(d) and if $330 < E'd < 440$ then $C_{ep,i} = 2000$ and $Y_i = 1$ else go to step 4(d).
    (d) If $440 < E_1d < 550$ then $C_{ep,i} = 3000$ and $Y_i = 1$ else go to step 4(e) and if $440 < E'd < 550$ then $C_{ep,i} = 3000$ and $Y_i = 1$ else go to step 4(e).
    (e) If $550 < E_1d < 660$ then $C_{ep,i} = 4000$ and $Y_i = 1$ else go to step 4(f) and if $550 < E'd < 660$ then $C_{ep,i} = 4000$ and $Y_i = 1$ else go to step 4(f).
    (f) If $E_1d > 660$ then $C_{ep,i} = 4000$ and $Y_i = 1$ and if $E'd > 660$ then $C_{ep,i} = 4000$ and $Y_i = 1$.
5. Obtain the value of EC_1 and EC_1' from Equations (25) and (31).
6. Obtain the value of EC_2 and EC_2' from Equations (26) and (32).
7. Obtain the value of SC_1 and SC_2 from Equations (28) and (34).

8. Use the values of q_1, ζ, δ, Ed^r, EC_1, EC_2, and SC_1 to obtain TC_S from Equation (29).
9. Use the values of q_2, q_1, ζ, δ, E'_d, EC'_1, EC'_2, and SC_2 to obtain TC_D from Equation (35).
10. Set $\eta = \eta + 1$ and repeat steps 3 to 11.
11. If $TC_D(\eta) > TC_D(\eta + 1)$ then redo the steps from 2 to 6 or else end the algorithm.

3. MATHEMATICAL SOLUTION OF FIELD EQUATION

$$E(\rho - R)^+ = \frac{|\rho - R| + \rho - R}{2}$$
$$E(\rho - R)^+ \leq \frac{\sqrt{E(\rho - R)^2} + E(\rho - R)}{2}$$

Considering, $\rho = d_1\sqrt{L} + X$ summation of variability and randomness.

$$E(\rho - R)^+ \leq \frac{\sqrt{E(d_1\sqrt{l} + E(X - R))^2} + E(d_1\sqrt{l} + X - R)}{2}$$

where, $R = d_1 l + \delta\sigma\sqrt{l}$ is a safety factor

$$E(\rho - R)^+ \leq \frac{\sqrt{E(d_1 l + X - d_1 l + \delta\sigma\sqrt{l})^2} + E(d_1 l + X - d_1 l + \delta\sigma\sqrt{l})}{2}$$

In this current model, the worst practicable distribution of random variable d_1 with mean $d_1 l$ and standard deviation $\sigma\sqrt{l}$. Thus we obtain,

$$\begin{aligned}\mathbb{E}(\rho - R)^+ &= \frac{\sqrt{\mathbb{E}(X + \delta\sigma\sqrt{l})^2} + \mathbb{E}((X + \delta\sigma\sqrt{l}))}{2} \\ &= \frac{\sqrt{\mathbb{E}(X^2 + \delta^2\sigma^2 l + 2X\delta\sigma\sqrt{l})} + \mathbb{E}(X + \delta\sigma\sqrt{l})}{2} \\ &= \left[\frac{\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2\sigma^2 l - 2d_1 l\delta\sigma\sqrt{l}} + (d_1 l - \delta\sigma\sqrt{l})}{2}\right]\end{aligned}$$

COMPUTATION OF PROPOSITION 1

Underneath mentioned is the Hessian matrix H for the retailing system with dual-route. The Hessian matrix H for retailing system having dual-route is given as

$$(H_{TC_D})_1 = \begin{bmatrix} \frac{\partial^2 TC_D}{\partial q_1^2} & \frac{\partial^2 TC_D}{\partial q_1 \partial q_2} & \frac{\partial^2 TC_D}{\partial q_1 \partial \delta} & \frac{\partial^2 TC_D}{\partial q_1 \partial \zeta} \\ \frac{\partial^2 TC_D}{\partial q_2 \partial q_1} & \frac{\partial^2 TC_D}{\partial q_2^2} & \frac{\partial^2 TC_D}{\partial q_2 \partial \delta} & \frac{\partial^2 TC_D}{\partial q_2 \partial \zeta} \\ \frac{\partial^2 TC_D}{\partial q_1 \partial \delta} & \frac{\partial^2 TC_D}{\partial q_2 \partial \delta} & \frac{\partial^2 TC_D}{\partial \delta^2} & \frac{\partial^2 TC_D}{\partial \delta \partial \zeta} \\ \frac{\partial^2 TC_D}{\partial q_1 \partial \zeta} & \frac{\partial^2 TC_D}{\partial q_2 \partial \zeta} & \frac{\partial^2 TC_D}{\partial \delta \partial \zeta} & \frac{\partial^2 TC_D}{\partial \zeta^2} \end{bmatrix}$$

where,

$$\frac{\partial^2 TC_D}{\partial q_2^2} = \frac{2}{q_1^3}\left[O_r d_1 + \frac{S_c^m d_1}{\eta} + d_1 CL + \nu d_1 \left(\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2 \sigma^2 l - 2 d_1 l \delta \sigma \sqrt{l}}\right)\right],$$
$$\frac{\partial^2 TC_D}{\partial q_1 \partial q_2} = \frac{\partial^2 TC_D}{\partial q_2 \partial \delta} = \frac{\partial^2 TC_D}{\partial q_2 \partial \zeta} = \frac{\partial^2 TC_D}{\partial \delta \partial \zeta} = 0.$$

$$\frac{\partial^2 TC_D}{\partial q_1 \partial \delta} = \frac{-\nu d_1}{2 q_1^2}\left(\frac{\delta \sigma^2 l - d_1 l \sigma \sqrt{l}}{\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2 \sigma^2 l - 2 d_1 l \delta \sigma \sqrt{l}}} - \sigma \sqrt{l}\right),$$
$$\frac{\partial^2 TC_D}{\partial q_1 \partial \zeta} = s d_1 \eta^2,$$
$$\frac{\partial^2 TC_D}{\partial q_2^2} = \frac{2 S_p^m d_2}{q_2^3}.$$

$$\frac{\partial^2 TC_D}{\partial \delta^2} = \frac{\nu d_1}{2 q_1}\left(\frac{\sigma^4 l^2}{\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2 \sigma^2 l - 2 d_1 l \delta \sigma \sqrt{l}}}\right),$$
$$\frac{\partial^2 TC_D}{\partial \zeta^2} = \frac{\alpha b}{\zeta^2}.$$

The Hessian matrix's $|(H_{TC_D})_1|$, principal minor of the order 1×1 is

$$|(H_{TC_D})_{1,1}| = \left|\frac{\partial^2 TC_D}{\partial q_1^2}\right|_{(q_1^*, q_2^*, \delta^*)}$$
$$= \frac{2}{q_1^3}\left[O_r d_1 + \frac{S_c^m d_1}{\eta} + d_1 CL\right.$$
$$\left. + \nu d_1 \left(\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2 \sigma^2 l - 2 d_1 l \delta \sigma \sqrt{l}}\right)\right] > 0.$$

The Hessian matrix's $|(H_{TC_D})_1|$, principal minor of the order 2×2 is

$$|((H_{TC_D})_1)_{2,2}|_{(q_1^*, q_2^*, \delta^*, \zeta^*)} = \begin{vmatrix} \frac{\partial^2 TC_D}{\partial q_1^2} & \frac{\partial^2 TC_D}{\partial q_1 \partial q_2} \\ \frac{\partial^2 TC_D}{\partial q_2 \partial q_1} & \frac{\partial^2 TC_D}{\partial q_2^2} \end{vmatrix}_{(q_1^*, q_2^*, \delta^*, \zeta^*)}$$

$$|((H_{TC_D})_1)_{2,2}|_{(q_1^*, q_2^*, \delta^*, \zeta^*)} = \left(\frac{\partial^2 TC_D}{\partial q_1^2}\right)\left(\frac{\partial^2 TC_D}{\partial q_2^2}\right)$$

$$= \left(\frac{2}{q_1^3} \left[O_r d_1 + \frac{S_c^m d_1}{\eta} + d_1 CL + \nu d_1 \left(\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2 \sigma^2 l - 2 d_1 l \delta \sigma \sqrt{l}} \right) \right] \right) \left(\frac{2 S_p^m d_2}{q_2^3} \right) > 0$$

The Hessian matrix's $|(H_{TC_D})_1|$, principal minor of the order 3×3 is

$$|((H_{TC_D})_1)_{3,3}|_{(q_1^*, q_2^*, \delta^*, \zeta^*)} = \begin{vmatrix} \frac{\partial^2 TC_D}{\partial q_2^2} & \frac{\partial^2 TC_D}{\partial q_2 \partial \delta} & \frac{\partial^2 TC_D}{\partial q_2 \partial \zeta} \\ \frac{\partial^2 TC_D}{\partial \delta \partial q_2} & \frac{\partial^2 TC_D}{\partial \delta^2} & \frac{\partial^2 TC_D}{\partial \delta \partial \zeta} \\ \frac{\partial^2 TC_D}{\partial \zeta \partial q_2} & \frac{\partial^2 TC_D}{\partial \zeta \partial \delta} & \frac{\partial^2 TC_D}{\partial \zeta^2} \end{vmatrix}_{(q_1^*, q_2^*, \delta^*, \zeta^*)}$$

$$|((H_{TC_D})_1)_{3,3}|_{(q_1^*, q_2^*, \delta^*, \zeta^*)} = \left(\frac{\partial^2 TC_D}{\partial q_2^2} \right) \left(\frac{\partial^2 TC_D}{\partial \delta^2} \right) \left(\frac{\partial^2 TC_D}{\partial \zeta^2} \right)$$

$$= \left(\frac{2 S_p^m d_2}{q_2^3} \right) \left(\frac{\nu d_1}{2 q_1} \left(\frac{\sigma^4 l^2}{\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2 \sigma^2 l - 2 d_1 \delta l \sigma \sqrt{l}}} \right) \right) \left(\frac{\alpha b}{\zeta^2} \right) > 0$$

The Hessian matrix's $|(H_{TC_D})_1|$, principal minor of the order 4×4 is

$$|((H_{TC_D})_1)_{4,4}|_{(q_1^*, q_2^*, \delta^*, \zeta^*)} = \begin{vmatrix} \frac{\partial^2 TC_D}{\partial q_1^2} & \frac{\partial^2 TC_D}{\partial q_1 \partial q_2} & \frac{\partial^2 TC_D}{\partial q_1 \partial \delta} & \frac{\partial^2 TC_D}{\partial q_1 \partial \zeta} \\ \frac{\partial^2 TC_D}{\partial q_2 \partial q_1} & \frac{\partial^2 TC_D}{\partial q_2^2} & \frac{\partial^2 TC_D}{\partial q_2 \partial \delta} & \frac{\partial^2 TC_D}{\partial q_2 \partial \zeta} \\ \frac{\partial^2 TC_D}{\partial \delta \partial q_1} & \frac{\partial^2 TC_D}{\partial \delta \partial q_2} & \frac{\partial^2 TC_D}{\partial \delta^2} & \frac{\partial^2 TC_D}{\partial \delta \partial \zeta} \\ \frac{\partial^2 TC_D}{\partial \zeta \partial q_1} & \frac{\partial^2 TC_D}{\partial \zeta \partial q_2} & \frac{\partial^2 TC_D}{\partial \zeta \partial \delta} & \frac{\partial^2 TC_D}{\partial \zeta^2} \end{vmatrix}_{(q_1^*, q_2^*, \delta^*, \zeta^*)}$$

$$= \left(\frac{\partial^2 TC_D}{\partial q_1^2} \right) \left(\frac{\partial^2 TC_D}{\partial q_2^2} \right) \left(\frac{\partial^2 TC_D}{\partial \delta^2} \right) \left(\frac{\partial^2 TC_D}{\partial \zeta^2} \right) - \left(\frac{\partial^2 TC_D}{\partial q_2^2} \right) \left(\frac{\partial^2 TC_D}{\partial \zeta^2} \right) \left(\frac{\partial^2 TC_D}{\partial \delta \partial q_1} \right)^2$$

$$\begin{aligned} |(H_{TC_S})_{1,1}| (q_1^*, \delta^*, \zeta^*) &= \left| \frac{\partial^2 TC_S}{\partial q_1^2} \right|_{(q_1^*, \delta^*, \zeta^*)} \\ &= \frac{2}{q_1^3} \Big[Ord_1 + \frac{S_c^m d_1}{\eta} + d_1 CL \\ &\quad + \nu d_1 \left(\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2 \sigma^2 l - 2 d_1 l \delta \sigma \sqrt{l}} \right) \Big] > 0 \end{aligned}$$

$$\left(\frac{\nu d_1}{2 q_1} \left(\frac{\sigma^4 l^2}{\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2 \sigma^2 l - 2 d_1 \delta l \sigma \sqrt{l}}} \right) \right) - \left(\frac{2 S_p^m d_2}{q_2^3} \right) \left(\frac{\alpha b}{\zeta^2} \right)$$

$$\left(-\frac{\nu d_1}{2 q_1^2} \left(\frac{\delta \sigma^2 l - d_1 l \sigma \sqrt{l}}{\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2 \sigma^2 l - 2 d_1 l \delta \sigma \sqrt{l}}} - \sigma \sqrt{l} \right) \right)^2 > 0$$

$$\implies \left(\frac{2}{q_1^3} \left[O_r d_1 + \frac{S_c^m d_1}{\eta} + d_1 CL + \nu d_1 \left(\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2 \sigma^2 l - 2 d_1 l \delta \sigma \sqrt{l}} \right) \right] \right)$$

$$\left(\frac{\nu d_1}{2q_1}\left(\frac{\sigma^4 l^2}{\sqrt{\sigma^2 l+(d_1 l)^2+\delta^2\sigma^2 l-2d_1\delta l\sigma\sqrt{l}}}\right)\right) >$$

$$\left(-\frac{\nu d_1}{2q_1^2}\left(\frac{\delta\sigma^2 l-d_1 l\sigma\sqrt{l}}{\sqrt{\sigma^2 l+(d_1 l)^2+\delta^2\sigma^2 l-2d_1 l\delta\sigma\sqrt{l}}}-\sigma\sqrt{l}\right)\right)^2$$

Since the Hessian matrix's every single principal minor is positive, therefore, at $(q_1^*, q_2^*, \delta^*, \zeta^*)$, $(H_{TC_D})_1$ is positive definite. Moreover, at the same point, the total cost of the firm offering a product online along with a traditional platform obtains its global minimum.

COMPUTATION OF PROPOSITION 2

Underneath mentioned is the Hessian matrix H for the retailing system with single-route. The Hessian matrix H for retailing system having single-channel is given as

$$(H_{TC_S})_2 = \begin{bmatrix} \frac{\partial^2 TC_S}{\partial q_1^2} & \frac{\partial^2 TC_S}{\partial q_1\partial\delta} & \frac{\partial^2 TC_S}{\partial q_1\partial\zeta} \\ \frac{\partial^2 TC_S}{\partial\delta\partial q_1} & \frac{\partial^2 TC_S}{\partial\delta^2} & \frac{\partial^2 TC_S}{\partial\delta\partial\zeta} \\ \frac{\partial^2 TC_S}{\partial\zeta\partial q_1} & \frac{\partial^2 TC_S}{\partial\zeta\partial\delta} & \frac{\partial^2 TC_S}{\partial\zeta^2} \end{bmatrix}$$

where,

$$\frac{\partial^2 TC_S}{\partial q_1^2} = \frac{2}{q_1^3}\left[O_r d_1 + \frac{S_c^m d_1}{\eta} + d_1 CL + \nu d_1\left(\sqrt{\sigma^2 l+(d_1 l)^2+\delta^2\sigma^2 l-2d_1 l\delta\sigma\sqrt{l}}\right)\right],$$

$$\frac{\partial^2 TC_S}{\partial q_1\partial\delta} = \frac{-\nu d_1}{2q_1^2}\left[\frac{\delta\sigma^2 l-d_1 l\sigma\sqrt{l}}{\sqrt{\sigma^2 l+(d_1 l)^2+\delta^2\sigma^2 l-2d_1 l\delta\sigma\sqrt{l}}}-\sigma\sqrt{l}\right],$$

$$\frac{\partial^2 TC_S}{\partial q_1\partial\zeta} = \frac{s d_1\eta}{2},$$

$$\frac{\partial^2 TC_S}{\partial\delta^2} = \frac{\nu d_1}{2q_1}\left[\frac{\sigma^4 l^2}{\sqrt{\sigma^2 l+(d_1 l)^2+\delta^2\sigma^2 l-2d_1 l\delta\sigma\sqrt{l}}}\right],$$

$$\frac{\partial^2 TC_S}{\partial\delta\partial\zeta} = \frac{\partial^2 TC_S}{\partial\zeta\partial\delta} = 0,$$

$$\frac{\partial^2 TC_S}{\partial\zeta^2} = \frac{\alpha b}{\zeta^2}.$$

The Hessian matrix's $|(H_{TC_S})_2|$, principal minor of the order 1×1 is

$$\begin{aligned}|(H_{TC_S})_{1,1}|\,(q_1^*,\delta^*,\zeta^*) &= \left|\frac{\partial^2 TC_S}{\partial q_1^2}\right|_{(q_1^*,\delta^*,\zeta^*)} \\ &= \frac{2}{q_1^3}\left[Ord_1 + \frac{S_c^m d_1}{\eta} + d_1 CL\right. \\ &\quad + \nu d_1\left(\sqrt{\sigma^2 l+(d_1 l)^2+\delta^2\sigma^2 l-2d_1 l\delta\sigma\sqrt{l}}\right) > 0\end{aligned}$$

The Hessian matrix's $|(H_{TC_S})_2|$, principal minor of the order 2×2 is

$$|(H_{TC_S})_{2,2}|\,(q_1^*,\delta^*,\zeta^*) = \begin{vmatrix} \frac{\partial^2 TC_S}{\partial \delta^2} & \frac{\partial^2 TC_S}{\partial \delta \partial \zeta} \\ \frac{\partial^2 TC_S}{\partial \zeta \partial \delta} & \frac{\partial^2 TC_S}{\partial \zeta^2} \end{vmatrix}_{(q_1^*,\delta^*,\zeta^*)} = \left(\frac{\partial^2 TC_S}{\partial \delta^2}\right)\left(\frac{\partial^2 TC_S}{\partial \zeta^2}\right) - \left(\frac{\partial^2 TC_S}{\partial \delta \partial \zeta}\right)\left(\frac{\partial^2 TC_S}{\partial \zeta \partial \delta}\right)$$

$$= \frac{\nu d_1 \alpha b}{2 q_1 \zeta^2}\left(\frac{\sigma^4 l^2}{\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2\sigma^2 l - 2 d_1 l \delta \sigma \sqrt{l}}}\right) > 0$$

The Hessian matrix's $\left|(H^T CS)_2\right|$, principal minor of the order 3×3 is

$$|(H_{TC_S})_{3,3}|_{(q_1^*,\delta^*,\zeta^*)} = \begin{vmatrix} \frac{\partial^2 TC}{\partial q_1^2} & \frac{\partial^2 TC}{\partial q_1 \partial \delta} & \frac{\partial^2 TC}{\partial q_1 \partial \zeta} \\ \frac{\partial^2 TC}{\partial \delta \partial q_1} & \frac{\partial^2 TC}{\partial \delta^2} & \frac{\partial^2 TC}{\partial \delta \partial \zeta} \\ \frac{\partial^2 TC}{\partial \zeta \partial q_1} & \frac{\partial^2 TC}{\partial \zeta \partial \delta} & \frac{\partial^2 TC}{\partial \zeta^2} \end{vmatrix}$$

$$= \left(\frac{\partial^2 TC}{\partial q_1^2}\right)\left(\frac{\partial^2 TC}{\partial \delta^2}\right)\left(\frac{\partial^2 TC}{\partial \zeta^2}\right) - \left(\frac{\partial^2 TC}{\partial q_1 \partial \delta}\right)\left(\frac{\partial^2 TC}{\partial \delta \partial q_1}\right)\left(\frac{\partial^2 TC}{\partial \zeta^2}\right) + \left(\frac{\partial^2 TC}{\partial q_1 \partial \zeta}\right)\left(-\frac{\partial^2 TC}{\partial \zeta \partial q_1}\right)\left(\frac{\partial^2 TC}{\partial \delta^2}\right)$$

$$\frac{2}{q_1^3}\left[O_r d_1 + \frac{S_c^m d_1}{\eta} + d_1 CL + \nu d_1\left(\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2\sigma^2 l - 2 d_1 l \delta \sigma \sqrt{l}}\right)\right]\left(\frac{\nu d_1}{2 q_1}\right)\left(\frac{\alpha b}{\zeta^2}\right)$$

$$\left[\frac{\sigma^4 l^2}{\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2\sigma^2 l - 2 d_1 \delta l \sigma \sqrt{l}}}\right]$$

$$-\left(\frac{\nu d_1}{2 q_1^2}\right)^2\left[\frac{\delta\sigma^2 l - d_1 l \sigma \sqrt{l}}{\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2\sigma^2 l - 2 d_1 \delta l \sigma \sqrt{l}}} - \sigma\sqrt{l}\right]^2\left(\frac{\alpha b}{\zeta^2}\right)$$

$$+\left(\frac{s d_1 \eta}{2}\right)^2 \frac{\nu d_1}{2 q_1}\left[\frac{\sigma^4 l^2}{\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2\sigma^2 l - 2 d_1 \delta l \sigma \sqrt{l}}}\right] > 0$$

$$\frac{2}{q_1^3}\left[O_r d_1 + \frac{S_c^m d_1}{\eta} + d_1 CL + \nu d_1\left(\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2\sigma^2 l - 2 d_1 l \delta \sigma \sqrt{l}}\right)\right]\left(\frac{\nu d_1}{2 q_1}\right)\left(\frac{\alpha b}{\zeta^2}\right)$$

$$\left[\frac{\sigma^4 l^2}{\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2\sigma^2 l - 2d_1\delta l\sigma\sqrt{l}}}\right]$$
$$+\left(\frac{sd_1\eta}{2}\right)^2 \frac{\nu d_1}{2q_1}\left[\frac{\sigma^4 l^2}{\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2\sigma^2 l - 2d_1\delta l\sigma\sqrt{l}}}\right]$$
$$> \left(\frac{\nu d_1}{2q_1^2}\right)^2\left[\frac{\delta\sigma^2 l - d_1 l\sigma\sqrt{l}}{\sqrt{\sigma^2 l + (d_1 l)^2 + \delta^2\sigma^2 l - 2d_1\delta l\sigma\sqrt{l}}} - \sigma\sqrt{l}\right]^2\left(\frac{\alpha b}{\zeta^2}\right)$$

Since the Hessian matrix's every single principal minor is positive, therefore, at $(q_1^*, q_2^*, \delta^*, \zeta^*)$, $(H_{TC_S})_1$ is positive definite. Moreover, at the same point, the total cost of the firm offering a product on a traditional platform obtains its global minimum.

4. CONCLUSION

Recently, many industries accept integrating dual-channel retailing systems into their business model. This is evident from the results of this study that traditional-online retailing increases the profitability of the company. Customized products have given the provision to the customer of choosing merchandise of their own choice. Moreover, this paper assumed a threshold amount and unequal shifting of customers between the channels. These two assumptions turned the model into a more practical scenario. The study also incorporated the policy of extensive quality checks in the production process as the customized product needs to be analyzed properly before delivery. The considerations of sustainable pillars have shown how the company could improve its profit while achieving sustainable goals.

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