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Method of a Priori Estimate in the Case of Fractional Order Differential Equations with Integral Conditions

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Abstract:

This paper concentrate on exploring the existence and uniqueness of a solution for a non-linear boundary value problem that has integral conditions in the case of fractional partial differential equations. For this we split the proof into two sections: linear and non-linear problem; for the associated linear problem, we derive the a priori bound and demonstrate the density of the operator generated by the problem posed; we solve the non-linear problem by introducing a iterative process. The results show the efficiency of energy inequality method in the case of time fractional order differential equations with integral conditions our results illustrate the existence and uniqueness of the continuous dependence of solution on fractional order.

Keywords: fractional partial differential equations; integral conditions; priori estimate; density of the operator; non-linear problem.

1.Introduction

In mathematics, fractional calculus is a field of analysis that investigates the extension of differentiation and integration from integers to non-integers, commonly referred to as fractional orders. Fractional differentiation, in particular, has been a subject of interest for almost as long as the classical calculus that we know today.

In addition, many problems in physics and modern technology are formulated using non-local conditions for partial differential equations, which are described by integral conditions. These conditions have gained significant attention due to their applications in a variety of fields, such as population dynamics, blood flow models, chemical engineering, and cellular systems (A. Bouziani, 2002; A.Bouziani,2003; A.Bouziani, N.Merazga, A.Bouziani,2003; N.Merazga, A. Bouziani,2005). Numerous authors have studied the existence and uniqueness of solutions to problems involving fractional differential equations, including initial and boundary value problems (A. Anguraj, P.Karthikeyan, 2010; B. Ahmad, J. Nieto,2009; M. Benchohra, J. R. Graef, S. Hamani,2008; M. Belmekki, M. Benchohra,2008; R. P. Agarwal, M. Benchohra, S. Hamani,2005; R. W. Ibrahim, S. Momani,2007; X. J. Li, C. J. Xu,2010).

For this purpose, we employed the energy inequality method, which is a useful tool for studying fractional and non-local classical problems. Compared to other techniques, this method plays an essential role in proving the existence and uniqueness of the solution.

It depends on density arguments and certain a priori bounds.

2. Preliminary

Definition 2.1 (I.Podlubny, 1999)(Gamma function)

For any complex number $\mathbf{z}$ such that $\mathrm{Re}(\mathbf{z}) > 0$, we define the following function called Gamma and denoted by the Greek letter "$\boldsymbol{\Gamma}$".

$$\boldsymbol{\Gamma} : \mathbf{R}^{*+} \rightarrow \mathbf{R}$$

$$\mathbf{z} \rightarrow \boldsymbol{\Gamma}(\mathbf{z}) = \int_{\mathbf{0}}^{+\infty} \mathbf{t}^{\mathbf{z}-\mathbf{1}} \exp^{-\mathbf{t}} \mathbf{dt}. \quad (1)$$

Definition 2.2 (I.Podlubny, 1999) (The Riemann-Liouville integral)

The Riemann-Liouville integral of order $\alpha > 0$, for an integral function S, is defined by

$$D_t^{-\alpha} S(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{S(\tau)}{(t-\tau)^{1-\alpha}} d\tau. \quad (2)$$

Definition 2.3 (Haim Brezis, 1983)

Let R be a subspace vector of the Hilbert space H, then $\mathbf{R}^{\perp}$ the orthogonal complement of R is defined as:

$$\mathbf{R}^{\perp} = \{\mathbf{f} \in \mathbf{H}, (\mathbf{f}, \mathbf{g})_{\mathbf{H}} = \mathbf{0}, \forall \mathbf{g} \in \mathbf{R}\}.$$

Proposition 2.1

Let R be a subspace vector of the Hilbert space H. R is dense in H if and only if:

$$\mathbf{R}^{\perp} = \{\mathbf{0}\}.$$

Definition 2.4 (Bertram Ross, 2006) (Left Caputo derivative)

$${}_0^c\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\tau)}{\partial \tau} \frac{1}{(t-\tau)^\alpha} d\tau. \quad (3)$$

Definition 2.5 (Stefan G Samko, Anatoly A Kilbas, Oleg I Marichev, 1993) (Mittag-Leffler function)

For $\mathbf{z} \in \mathbb{C}$, Mittag-Leffler function $\mathbf{E}_{\boldsymbol{\alpha}}(\mathbf{z})$ is defined as follows:

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+1)}, \ \alpha > 0. \tag{4}$$

The two-term Mittag-Leffler function is vital in the fractional calculus theory and defined as:

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\beta)}, \quad (\alpha > 0, \beta > 0). \tag{5}$$

Cauchy-Schwarz inequality:

$$\forall(f, g) \in \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega),$$

we have:

$$\int_{\Omega}|f(t)g(t)|\, dt \leq \left(\int_{\Omega}(f(t))^2\, dt\right)^{\frac{1}{2}}\left(\int_{\Omega}(g(t))^2\, dt\right)^{\frac{1}{2}}, \tag{6}$$

Cauchy inequality:

$\forall(a, b) \in \mathbb{R}^2$:

$$|ab| \leq \frac{1}{2}a^2 + \frac{1}{2}b^2. \tag{7}$$

Cauchy inequality with ϵ:

Let ϵ be a strictly positive number, then$\forall(a, b) \in \mathbb{R}^2$:

$$|ab| \leq \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon}. \tag{8}$$

Poincaré inequality

Lemma 2.1(A.A. Alikhanov, 2010) For any function S(t) that is absolutely continuous on the interval [0,T], the following inequality holds:

$$S(t)\, \partial_t^{\beta}S(t) \geq \frac{1}{2}\partial_t^{\beta}S^2(t), \qquad 0 < \alpha < 1. \tag{9}$$

Lemma 2.2(A.A. Alikhanov, 2010) (Gnonwall Lemma) Let a non-negative, absolutely continuous function y(t) satisfy the inequality:

$$\partial_t^{\alpha}y(t) \leq k_1u(t) + k_2(t), \qquad 0 < \alpha < 1. \tag{10}$$

For all $t \in [0, T]$, where k_1 is a positive constant and $k_2(t)$ a non-negative integrale function over [0,T]. Then,

$$y(t) \leq y(0)E_{\alpha}(k_1t^{\alpha}) + \Gamma(\alpha)E_{\alpha,\alpha}(k_1t^{\alpha})D_t^{-\alpha}k_2(t), \tag{11}$$

where: E_α et $E_{\alpha,\alpha}$are Mittag-Leffler functions.

Lemma 2.3 (Ladyzhenskaya, 1985) Let u(t) be a non-negative, absolutely continuous function on [0,T], and for all $t \in [0, T]$, satisfies the inequality:

$$\frac{d\varphi}{dt} \le C(t)\varphi(t) + B(t). \tag{12}$$

Such that the functions C(t) and B(t) are summable and non-negative on [0,T]. Here:

$$\varphi(t) \le e^{\int_0^t C(\tau)d\tau}\varphi(0) + \int_0^t B(\xi)e^{\int_0^\xi C(\tau)d\tau}\, d\xi \tag{13}$$

Lemma 2.4 (Mesloub S, Mezhoudi R, Medjeden M, 2002) For any $n \in \mathbb{N}$, we have

$$\left\|\Im_x{}^{2n}u\right\|^2_{\mathbb{L}^2(0,1)} \le \left(\frac{1}{2}\right)^{2n} \|u\|^2_{\mathbb{L}^2(0,1)}. \tag{14}$$

Where:

$$\Im_x{}^{2n}u = \int_0^x \int_0^{\xi_1} \dots \int_0^{\xi_{2n-1}} u(\eta, t)\, d\eta d\xi_{2n-1} \dots d\xi_1 = \int_0^x \frac{(x-\xi)^{2n-1}}{(2n-1)!} u(\xi, t) d\xi$$

3. Problem Statement

In a rectangular domain: $\Omega = (0,1) \times (0, T)$, $0 \le T \le \infty$, consider the following fractional partial differential equation:

$$\mathcal{L}v = {}_0^c\partial_t^\delta v(x,t) - \alpha\frac{\partial^2 v}{\partial x^2} - \beta\frac{\partial^3 v}{\partial t\, \partial x^2} + \gamma v - \int_0^t a(t-s)v(x,s)ds = g(x,t,v,r),$$
$$0 < t < T, \quad 0 < \delta < 1 \tag{15}$$

where: $r(x,t) = \int_0^t g(x,s,v(x,s),r(x,s))ds.$

Where $a(t)$ is a function of t and satisfies the condition $0 < a_0 < a(t) < a_1$ and α, β and γ are strictly positive constants.

With Initial conditions,

$$\ell v = v(x,0) = \phi(x), \qquad qv = \frac{\partial v(x,0)}{\partial t} = \Psi(x), \quad 0 < x < 1, \tag{16}$$

and integral conditions,

$$\int_0^1 v(x,t)dx = m(t), \qquad \int_0^1 xv(x,t)dx = n(t), \qquad 0 < t \le T, \tag{17}$$

where $\Phi, \Psi, \mathrm{m}, \mathrm{n}$ and g are known functions.

Since the boundary conditions are non-homogeneous, we construct the function

$$\mathrm{U}(\mathrm{x},\mathrm{t}) = 6\big(2\mathrm{n}(\mathrm{t}) - \mathrm{m}(\mathrm{t})\big)\mathrm{x} - 2(3\mathrm{n}(\mathrm{t}) - 2\mathrm{m}(\mathrm{t})).$$

And we introduce a new function: $\mathrm{u}(\mathrm{x},\mathrm{t}) = \mathrm{v}(\mathrm{x},\mathrm{t}) - \mathrm{U}(\mathrm{x},\mathrm{t})$. Then, the problem (15)-(17) can be reformulated as follows:

$$\mathcal{L}\mathrm{u} = {}_0^c\partial_t^\delta \mathrm{u}(\mathrm{x},\mathrm{t}) - \alpha\frac{\partial^2\mathrm{u}}{\partial\mathrm{x}^2} - \beta\frac{\partial^3\mathrm{u}}{\partial\mathrm{t}\,\partial\mathrm{x}^2} + \gamma\mathrm{u} - \int_0^t \mathrm{a}(\mathrm{t}-\mathrm{s})\mathrm{u}(\mathrm{x},\mathrm{s})\mathrm{ds} = \mathrm{f}(\mathrm{x},\mathrm{t},\mathrm{u},\mathrm{r}), \tag{18}$$

where: $$\mathrm{f}\big(\mathrm{x},\mathrm{t},\mathrm{u}(\mathrm{x},\mathrm{t}),\mathrm{r}(\mathrm{x},\mathrm{t})\big) = \mathrm{g}(\mathrm{x},\mathrm{t},\mathrm{v},\mathrm{r}) - \mathcal{L}\mathrm{v} + \int_0^t \mathrm{a}(\mathrm{t}-\mathrm{s})\mathrm{U}(\mathrm{x},\mathrm{s})\mathrm{ds}.$$

The initial conditions

$$\ell\mathrm{u} = \mathrm{u}(\mathrm{x},0) = \phi(\mathrm{x}) - \ell\mathrm{U} = \varphi(\mathrm{x}), \quad \mathrm{qu} = \frac{\partial\mathrm{u}(\mathrm{x},0)}{\partial\mathrm{t}} = \Psi(\mathrm{x}) - \mathrm{qU} = \psi(\mathrm{x}), 0 < x < 1. \tag{19}$$

The integral conditions:

$$\int_0^t \mathrm{u}(\mathrm{x},\mathrm{t})\mathrm{dx} = 0, \qquad \int_0^t \mathrm{xu}(\mathrm{x},\mathrm{t})\mathrm{dx} = 0, \qquad 0 < t \le T\,. \tag{20}$$

4.Technical tools and associated linear problem

We define some function spaces and tools required to investigate the following linear problem associated with problems (18)-(20)

$$\mathcal{L}\mathrm{u} = {}_0^c\partial_t^\delta \mathrm{u}(\mathrm{x},\mathrm{t}) - \alpha\frac{\partial^2\mathrm{u}}{\partial\mathrm{x}^2} - \beta\frac{\partial^3\mathrm{u}}{\partial\mathrm{t}\,\partial\mathrm{x}^2} + \gamma\mathrm{u} - \int_0^t \mathrm{a}(\mathrm{t}-\mathrm{s})\mathrm{u}(\mathrm{x},\mathrm{s})\mathrm{ds} = \mathrm{f}(\mathrm{x},\mathrm{t}), \tag{21}$$

The initial conditions

$$\ell\mathrm{u} = \mathrm{u}(\mathrm{x},0) = \phi(\mathrm{x}) - \ell\mathrm{U} = \varphi(\mathrm{x}), \quad \mathrm{qu} = \frac{\partial\mathrm{u}(\mathrm{x},0)}{\partial\mathrm{t}} = \Psi(\mathrm{x}) - \mathrm{qU} = \psi(\mathrm{x}), \qquad 0 < x < 1. \tag{22}$$

The integrals conditions:

$$\int_0^t \mathrm{u}(\mathrm{x},\mathrm{t})\mathrm{dx} = 0, \qquad \int_0^t \mathrm{xu}(\mathrm{x},\mathrm{t})\mathrm{dx} = 0, \qquad 0 < t \le T\,. \tag{23}$$

We will show the existence and uniqueness of the solution of problem (21) – (23) , the proof will be based on a priori estimates and on the density of the set of values of the operator generated by problem (21) – (23).

For this, we must first convert problem (21) – (23) into an equivalent operational form:

$$\mathrm{Lu} = \mathcal{F} = (\mathrm{f},\varphi,\psi). \tag{24}$$

Where the operator $L = (\mathcal{L}, \ell, q)$ with $L: E \to F$ is defined in $D(L)$ suchaway that:

$$D(L) = \begin{Bmatrix} u \in \mathbb{L}^2(D),\ {}_0^c\partial_t^\delta u,\ \frac{\partial u}{\partial t},\ \frac{\partial u}{\partial x},\ \frac{\partial^2 u}{\partial x^2},\ \frac{\partial^3 u}{\partial t\,\partial x^2} \in \mathbb{L}^2(D) \\ \int_0^t u(x,t)dx = 0, \quad \int_0^t xu(x,t)dx = 0, \quad 0 < t \le T \end{Bmatrix} \tag{25}$$

and u satisfies the initial condition (4.2). E is the Banach space equipped with the following norm:

$$\|u\|^2{}_E = \sup\left(D_t^{\delta-1}\left\|\mathfrak{I}_x \frac{\partial u}{\partial \tau}\right\|^2{}_{\mathbb{L}^2(\Omega)} + \int_0^t \left\|\frac{\partial u}{\partial \tau}\right\|^2{}_{\mathbb{L}^2(\Omega)} d\tau + \int_0^t u^2\,dx\right), \tag{26}$$

and F is the Hilbert space composed of functions with the norm:

$$\|Lu\|^2{}_F = \|\varphi\|^2{}_{\mathbb{L}^2(\Omega)} + \|\mathfrak{I}_x\psi\|^2{}_{\mathbb{L}^2(\Omega)} + \|\mathfrak{I}_x f\|^2{}_{\mathbb{L}^2(\Omega)}. \tag{27}$$

5. Prior Estimation and Uniqueness of the Solution:

The priori estimation method, also known as the energy integral method, is one of the most effective functional analysis methods for solving partial differential equations with integral conditions, and is an important technique for proving the existence, uniqueness, and continuous dependence of solutions to PDE.

Theorem 5.1 For any function $u \in D(L)$, we have the a priori estimation

$$\|u\|_E \le C\,\|Lu\|_F, \tag{28}$$

where C is a constant that is independent of u.

Proof: We multiply (21) by $Mu = -\mathfrak{I}_x{}^2 \frac{\partial u}{\partial t} = -\int_0^t \int_0^\eta \frac{\partial u}{\partial t}(\xi,t)d\xi d\eta$, and integrate over the subdomain to obtain $\Omega = (0,1) \times (0,\tau)$, we obtain:

$$(\mathcal{L}u, Mu)_{\mathbb{L}^2(\Omega)} = -\left({}_0^c\partial_t^\delta u, \mathfrak{I}_x{}^2 \frac{\partial u}{\partial t}\right)_{\mathbb{L}^2(\Omega)} + \alpha\left(\frac{\partial^2 u}{\partial x^2}, \mathfrak{I}_x{}^2 \frac{\partial u}{\partial t}\right)_{\mathbb{L}^2(\Omega)}$$

$$+\beta\left(\frac{\partial^3 u}{\partial t\,\partial x^2}, \mathfrak{I}_x{}^2 \frac{\partial u}{\partial t}\right)_{\mathbb{L}^2(\Omega)} - \gamma\left(u, \mathfrak{I}_x{}^2 \frac{\partial u}{\partial t}\right)_{\mathbb{L}^2(\Omega)}$$

$$= -\left(\int_0^t a(t-s)u(x,s)ds, \mathfrak{I}_x{}^2 \frac{\partial u}{\partial t}\right)_{\mathbb{L}^2(\Omega)} - \left(f(x,t), \mathfrak{I}_x{}^2 \frac{\partial u}{\partial t}\right)_{\mathbb{L}^2(\Omega)} \tag{29}$$

By integrating by parts for each term on the left-hand side of (22), and using the conditions (20), we obtain

$$-\left({}_0^c\partial_t^\delta u, \Im_x{}^2 \frac{\partial u}{\partial t}\right)_{\mathbb{L}^2(\Omega)} = -\int_0^\tau \int_0^1 \left({}_0^c\partial_t^\delta u\, \Im_x{}^2 \frac{\partial u}{\partial t}\right) dxdt$$

$$= \int_0^\tau \int_0^1 \left({}_0^c\partial_t^\delta \Im_x \frac{\partial u}{\partial t}\right)\left(\Im_x \frac{\partial u}{\partial t}\right) dxdt. \quad (30)$$

$$\alpha\left(\frac{\partial^2 u}{\partial x^2}, \Im_x{}^2 \frac{\partial u}{\partial t}\right)_{\mathbb{L}^2(\Omega)} = \alpha \int_0^\tau \int_0^1 \left(\frac{\partial^2 u}{\partial x^2} \Im_x{}^2 \frac{\partial u}{\partial t}\right) dxdt$$

$$= \alpha \int_0^\tau \int_0^1 u \frac{\partial u}{\partial t} dxdt\,. \quad (31)$$

$$\beta\left(\frac{\partial^3 u}{\partial t\, \partial x^2}, \Im_x{}^2 \frac{\partial u}{\partial t}\right)_{\mathbb{L}^2(\Omega)} = \beta \int_0^\tau \int_0^1 \left(\frac{\partial^3 u}{\partial t\, \partial x^2} \Im_x{}^2 \frac{\partial u}{\partial t}\right) dxdt$$

$$= \beta \int_0^\tau \int_0^1 \left(\frac{\partial u}{\partial t}\right)^2 dxdt$$

$$= \beta \int_0^\tau \left\|\frac{\partial u}{\partial t}\right\|^2_{\mathbb{L}^2(0,1)} dt\,. \quad (32)$$

$$-\gamma\left(u, \Im_x{}^2 \frac{\partial u}{\partial t}\right)_{\mathbb{L}^2(\Omega)} = -\gamma \int_0^\tau \int_0^1 \left(u\, \Im_x{}^2 \frac{\partial u}{\partial t}\right) dxdt$$

$$= -\frac{\gamma}{2} \int_0^1 \left(\Im_x{}^2 u(x,\tau)\right)^2 dx + \frac{\gamma}{2} \int_0^1 (\varphi(x))^2 dx. \quad (33)$$

Applying the Cauchy inequalities (7) and (8), and integrating by parts for the two terms on the right-hand side of (29), we obtain:

$$-\left(f, \Im_x{}^2 \frac{\partial u}{\partial t}\right)_{\mathbb{L}^2(\Omega)} \leq \frac{\varepsilon}{2} \int_0^\tau \int_0^1 (\Im_x f)^2 dxdt + \frac{1}{2\varepsilon} \int_0^\tau \int_0^1 \left(\Im_x \frac{\partial u}{\partial t}\right)^2 dxdt \quad (34)$$

$$\left(\int_0^t a(t-s)u(x,s)ds\,, \Im_x{}^2 \frac{\partial u}{\partial t}\right)_{\mathbb{L}^2(\Omega)} = \int_0^\tau \int_0^1 \left(\int_0^t a(t-s)u(x,s)ds\right) \Im_x{}^2 \frac{\partial u}{\partial t} dxdt$$

$$\leq a_1 T^2 \|u\|^2_{\mathbb{L}^2(0,1)} + \int_0^\tau \left\|\Im_x \frac{\partial u}{\partial t}\right\|^2_{\mathbb{L}^2(0,1)} dt, \quad (35)$$

By substituting (30)-(35) into (29), and applying lemma (2.1), we obtain:

$$\frac{1}{2}\int_0^\tau\int_0^1\left({}_0^c\partial_t^\delta \Im_x\frac{\partial u}{\partial t}\right)^2 dxdt+\alpha\int_0^\tau\int_0^1 u\frac{\partial u}{\partial t}dxdt+\beta\int_0^\tau\left\|\frac{\partial u}{\partial t}\right\|^2_{\mathbb{L}^2(0,1)}dt$$
$$-\frac{\gamma}{2}\int_0^1\left(\Im_x{}^2u(x,\tau)\right)^2dx+\frac{\gamma}{2}\int_0^1(\varphi(x))^2dx$$
$$\le\frac{1}{2\varepsilon}\int_0^\tau\int_0^1(\Im_x f)^2dxdt+\frac{\varepsilon}{2}\int_0^\tau\int_0^1\left(\Im_x\frac{\partial u}{\partial t}\right)^2dxdt$$
$$+T^2\|u\|^2_{\mathbb{L}^2(0,1)}+\int_0^\tau\left\|\Im_x\frac{\partial u}{\partial t}\right\|^2_{\mathbb{L}^2(0,1)}dt, \tag{36}$$

Evaluating the first and third terms on the left-hand side, we have:

$$\alpha\int_0^\tau\int_0^1 u\frac{\partial u}{\partial t}dxdt=\alpha\int_0^1\int_0^\tau u\frac{\partial u}{\partial t}dtdx$$
$$=\frac{\alpha}{2}\int_0^1u^2(x,\tau)dx-\frac{\alpha}{2}\int_0^1\varphi^2(x)\,dx-\frac{\alpha}{2}\int_0^1\int_0^\tau u^2\,(x,t)dtdx \tag{37}$$

$$\frac{1}{2}\int_0^\tau\int_0^1\left({}_0^c\partial_t^\delta\Im_x\frac{\partial u}{\partial t}\right)^2dxdt=D_t{}^{\delta-1}\left\|\Im_x\frac{\partial u}{\partial t}\right\|^2_{\mathbb{L}^2(0,1)}-\frac{t^{1-\delta}}{\Gamma(1-\delta)}\|\Im_x\psi\|^2_{\mathbb{L}^2(0,1)}. \tag{38}$$

Substituting (37) and (38) as well as the conditions (22) into inequality (36), we obtain:

$$D_t{}^{\delta-1}\left\|\Im_x\frac{\partial u}{\partial t}\right\|^2_{\mathbb{L}^2(0,1)}+\int_0^\tau\left\|\frac{\partial u}{\partial t}\right\|^2_{\mathbb{L}^2(0,1)}dt+\int_0^1u^2dx$$

$$\le\eta_1\left(\begin{array}{c}\int_0^1\varphi^2(x)\,dx+\int_0^1\int_0^\tau u^2\,dtdx+\int_0^\tau\int_0^1(\Im_xf)^2\,dxdt+\|\Im_x\psi\|^2_{\mathbb{L}^2(0,1)}\\+\int_0^\tau\int_0^1\left(\Im_x\frac{\partial u}{\partial t}\right)^2dxdt\end{array}\right), \tag{39}$$

where:

$$\eta_1=\frac{\max\left(\frac{\alpha+\gamma}{2},a_1T^2+\frac{\alpha}{2},\ \frac{1}{2\varepsilon},\ \frac{\varepsilon}{2},\ \frac{T^{1-\delta}}{\Gamma(1-\delta)}\right)}{\min\left(\frac{1}{2},\beta,\frac{\alpha+\gamma}{2}\right)}. \tag{40}$$

For the second term on the right-hand side of (39), we apply the lemma (2.3) by letting

$$\varphi(t)=\int_0^t\int_0^1u^2dxdt\quad;\ \frac{\partial\varphi}{\partial t}=\int_0^1u^2\,dx\ ;\ \varphi(0)=0, \tag{41}$$

this leads to

$$\int_0^1\int_0^\tau u^2\,dtdx \le T\eta_1 e^{\eta_1 T}\left(\int_0^1 \varphi^2(x)\,dx + \int_0^\tau\int_0^1 (\Im_x f)^2\,dxdt + \|\Im_x\psi\|^2_{\mathbb{L}^2(0,1)} + \int_0^\tau\int_0^1\left(\Im_x\frac{\partial u}{\partial t}\right)^2 dxdt\right). \tag{42}$$

Thus, the inequality (39) becomes

$$D_t^{\delta-1}\left\|\Im_x\frac{\partial u}{\partial t}\right\|^2_{\mathbb{L}^2(0,1)} + \int_0^\tau\left\|\frac{\partial u}{\partial t}\right\|^2_{\mathbb{L}^2(0,1)} dt + \int_0^1 u^2 dx$$
$$\le \eta_2\left(\begin{array}{c}\int_0^1 \varphi^2(x)\,dx + \int_0^\tau\int_0^1 (\Im_x f)^2\,dxdt + \|\Im_x\psi\|^2_{\mathbb{L}^2(0,1)} \\ + \int_0^\tau\int_0^1\left(\Im_x\frac{\partial u}{\partial t}\right)^2 dxdt\end{array}\right), \tag{43}$$

such that:

$$\eta_2 = \max(\eta_1, \eta_1^2 T\, e^{\eta_1 T}). \tag{44}$$

Finally, we apply lemma (2.2) to the last term on the right-hand side of (43) by letting:

$$y(t) = \int_0^\tau\int_0^1\left(\Im_x\frac{\partial u}{\partial t}\right)^2 dxdt \quad ; \partial_t^\delta y(t) = D^{\delta-1}\left\|\Im_x\frac{\partial u}{\partial t}\right\|^2_{\mathbb{L}^2(0,1)}. \tag{45}$$

Thus, we obtain:

$$\int_0^\tau\int_0^1\left(\Im_x{}^2\frac{\partial u}{\partial t}\right)^2 dxdt \le \eta_2\Gamma(\delta)E_{\delta,\delta}(\eta_2 t^\delta)\left(\frac{T}{\delta\,\Gamma(\delta)}\|\varphi\|^2_{\mathbb{L}^2(0,1)} + \frac{T}{\delta\,\Gamma(\delta)}\|\Im_x\psi\|^2_{\mathbb{L}^2(0,1)} + D_t^{-\delta-1}\|\Im_x f\|^2_{\mathbb{L}^2(\Omega)}\right)$$
$$\le \eta_2\Gamma(\delta)E_{\delta,\delta}(\eta_2 t^\delta)\max\left(1,\frac{T}{\delta\,\Gamma(\delta)}\right)\left(D_t^{-\delta-1}\|\Im_x f\|^2_{\mathbb{L}^2(\Omega)} + \|\varphi\|^2_{\mathbb{L}^2(0,1)} + \|\Im_x\psi\|^2_{\mathbb{L}^2(0,1)}\right). \tag{46}$$

Substituting equation (46) into equation (43), we obtain:

$$D_t^{\delta-1}\left\|\Im_x\frac{\partial u}{\partial t}\right\|^2_{\mathbb{L}^2(0,1)} + \int_0^\tau\left\|\frac{\partial u}{\partial t}\right\|^2_{\mathbb{L}^2(0,1)} dt + \int_0^1 u^2 dx$$

$$\le \eta_3\left(\int_0^1 \varphi^2(x)\,dx + \int_0^\tau\int_0^1 (\Im_x f)^2\,dxdt + \|\Im_x\psi\|^2_{\mathbb{L}^2(0,1)} + D_t^{-\delta-1}\|\Im_x f\|^2_{\mathbb{L}^2(\Omega)}\right), \tag{47}$$

Where:

$$\eta_3 = \eta_2 \max\left(\eta_2, \eta_2\Gamma(\delta), E_{\delta,\delta}(\eta_2 t^{\delta}) \max\left(1, \frac{T}{\delta\,\Gamma(\delta)}\right)\right). \tag{48}$$

On the other hand:

$$D_t^{-\delta-1}\|\Im_x f\|^2_{\mathbb{L}^2(\Omega)} \leq \frac{T^{\delta}}{\Gamma(1+\delta)} \int_0^T \|\Im_x f\|^2_{\mathbb{L}^2(0,1)} dt \ . \tag{49}$$

Hence, inequality (47) becomes

$$D_t^{\delta-1}\left\|\Im_x \frac{\partial u}{\partial t}\right\|^2_{\mathbb{L}^2(0,1)} + \int_0^{\tau}\left\|\frac{\partial u}{\partial t}\right\|^2_{\mathbb{L}^2(0,1)} dt + \int_0^1 u^2 dx$$
$$\leq C\left(\int_0^1 \varphi^2(x)\, dx + \int_0^{\tau}\int_0^1 (\Im_x f)^2\, dxdt + \|\Im_x \psi\|^2_{\mathbb{L}^2(0,1)}\right), \tag{50}$$

such that:

$$C = \eta_3\left(1 + \frac{T^{\delta}}{\Gamma(1+\delta)}\right). \tag{51}$$

We observe that the right-hand side of inequality (50) is independent of τ, by taking the supremum of the left-hand side with respect to $\tau \in [0, T]$. We obtain the desired inequality, which concludes the proof.

Proposition 5.1: The operator L which is defined from E to F has a closure.

Theorem 5.1 holds for strong solutions, and we have the inequality:

$$\|u\|_E \leq \acute{C}\|\bar{L}u\|_F\,, \tag{52}$$

thus, we obtain:

Corollary 5.1: The strong solution of (21)-(23) is unique if it exists, and depends continuously on $\mathcal{F} \in F$.

Corollary 5.2: The set of values $R(\bar{L})$ of the operator $\bar{L}$is closed in F. ∎

6. Existence of the Solution

To prove the existence of the solution, we must demonstrate that: $R(L)$ is dense in F, for all: $u \in E$, and $\mathcal{F} = (f, \varphi, \psi) \in F$.

Theorem 5.1

For $z \in L^2(\Omega)$ and all $u \in E$, we have:

$$\int_\Omega Lu.z\,dxdt = 0, \tag{53}$$

then: z disappears almost everywhere in Ω, this implies that the problem (21)-(23) has a unique solution.

Proof:

The proof of this theorem consists of choosing $z \in R(L)^\perp$, we demonstrate that:

$$R(L)^\perp = \{0\} \Leftrightarrow \overline{R(L)} = F.$$

The scalar product in F is defined by:

$$(Lu, z)_F = \int_\Omega Lu.z\,dxdt\,. \tag{54}$$

Then (53) can be written as:

$$\int_\Omega \left({}_0^c\partial_t^\delta u(x,t) - \alpha \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial^3 u}{\partial t\,\partial x^2} + \gamma u, z \right) dxdt = 0. \tag{55}$$

If letting:

$$u(x,t) = \mathfrak{I}_t{}^2\mu = \int_0^t \int_0^s \mu(x,\xi)d\xi ds, \tag{56}$$

where ${}_0^c\partial_t^\delta\mu\,, \frac{\partial^2\mu}{\partial x^2}\,, \frac{\partial^3\mu}{\partial t\,\partial x^2}, \mu \in L^2(\Omega)$ and it also satisfies the initial boundary conditions (19) and (20), we can write equation (55) as:

$$\int_\Omega \left({}_0^c\partial_t^\delta \mathfrak{I}_t{}^2\mu - \alpha \frac{\partial^2 \mathfrak{I}_t{}^2\mu}{\partial x^2} - \beta \frac{\partial^3 \mathfrak{I}_t{}^2\mu}{\partial t\,\partial x^2} + \gamma \mathfrak{I}_t{}^2\mu\,, z \right) dxdt = 0. \tag{57}$$

We can express z as a function of μ as follows:

$$z(x,t) = \mathfrak{I}_t\mu - \mathfrak{I}_x{}^2\mathfrak{I}_t\mu. \tag{58}$$

We can then substitute (58) into (57) and do integrations by parts on each term

$$\int_\Omega \left({}_0^c\partial_t^\delta \mathfrak{I}_t{}^2\mu \,.\, \mathfrak{I}_t\mu\right) \mathrm{dxdt} = \int_\Omega \left({}_0^c\partial_t^\delta \mathfrak{I}_t\mu \,.\, \mathfrak{I}_t\mu\right) \mathrm{dxdt}$$

$$\geq \frac{1}{2}\int_0^\tau {}_0^c\partial_t^\delta \|\mathfrak{I}_t\mu\|_{\mathbb{L}^2(0,1)} \,\mathrm{dt}, \quad (59)$$

$$-\int_\Omega \left({}_0^c\partial_t^\delta \mathfrak{I}_t{}^2\mu \,.\, \mathfrak{I}_x{}^2\mathfrak{I}_t\mu\right) \mathrm{dxdt} = \int_0^\tau \int_0^1 {}_0^c\partial_t^\delta \mathfrak{I}_x(\mathfrak{I}_t\mu\,).\mathfrak{I}_x(\mathfrak{I}_t\mu\,)\,\mathrm{dxdt}$$

$$\geq \frac{1}{2}\int_0^\tau {}_0^c\partial_t^\delta \|\mathfrak{I}_x(\mathfrak{I}_t\mu)\|_{\mathbb{L}^2(0,1)}, \quad (60)$$

$$-\int_\Omega \left(\alpha \frac{\partial^2(\mathfrak{I}_t{}^2\mu)}{\partial x^2}.\mathfrak{I}_t\mu\right) \mathrm{dxdt} = -\alpha\int_0^\tau\int_0^1 \left(\frac{\partial}{\partial x}\left(\mathfrak{I}_t{}^2\mu\right)\right)^2 \mathrm{dx\,dt}, \quad (61)$$

$$\int_\Omega \left(\alpha \frac{\partial^2(\mathfrak{I}_t{}^2\mu)}{\partial x^2}.\mathfrak{I}_x{}^2\mathfrak{I}_t\mu\right) \mathrm{dxdt} = \alpha\int_0^\tau\int_0^1 \left(\mathfrak{I}_t{}^2\mu\right)^2 \mathrm{dxdt}, \quad (62)$$

$$-\int_\Omega \left(\beta \frac{\partial^3\mathfrak{I}_t{}^2\mu}{\partial t\,\partial x^2}.\mathfrak{I}_t\mu\right) \mathrm{dxdt} = \beta\int_0^\tau\int_0^1 \left(\frac{\partial}{\partial X}(\mathfrak{I}_t\mu)\right)^2 \mathrm{dxdt}, \quad (63)$$

$$-\int_\Omega \left(-\beta \frac{\partial^3\mathfrak{I}_t{}^2\mu}{\partial t\,\partial x^2}.\mathfrak{I}_x{}^2\mathfrak{I}_t\mu\right) \mathrm{dxdt} = \beta\int_0^\tau\int_0^1 (\mathfrak{I}_t\mu)^2 \,\mathrm{dxdt}, \quad (64)$$

$$\int_\Omega \left(\gamma\mathfrak{I}_t{}^2\mu.\,\mathfrak{I}_t\mu\right) \mathrm{dxdt} = \frac{\gamma}{2}\int_0^\tau\int_0^1 \left(\mathfrak{I}_t{}^2\mu(x,t)\right)^2 \mathrm{dxdt} - \frac{\gamma}{2}\int_0^\tau\int_0^1 \varphi^2\,\mathrm{dxdt}, \quad (65)$$

$$-\int_\Omega \left(\gamma\mathfrak{I}_t{}^2\mu.\,\mathfrak{I}_x{}^2\mathfrak{I}_t\mu\right) \mathrm{dxdt} = -\frac{\gamma}{2}\int_0^\tau\int_0^1 \left(\mathfrak{I}_x{}^2(\mathfrak{I}_t{}^2\mu(x,0))\right)^2 \mathrm{dxdt} +$$

$$\frac{\gamma}{2}\int_0^\tau\int_0^1 \left(\mathfrak{I}_x{}^2(\mathfrak{I}_t{}^2\mu(x,0))\right)^2 \mathrm{dxdt}$$

$$= -\frac{\gamma}{2}\int_0^\tau\int_0^1 \left(\mathfrak{I}_t{}^2\mu(x,t)\right)^2 \mathrm{dxdt} + \frac{\gamma}{2}\int_0^\tau\int_0^1 \varphi^2\,\mathrm{dxdt}. \quad (66)$$

By substituting (59)-(66) into (57), we obtain:

$$D_t^{\delta-1}\|\mathfrak{I}_t\mu\|^2_{\mathbb{L}^2(0,1)} + D_t^{\delta-1}\|\mathfrak{I}_x\mathfrak{I}_t\mu\|^2_{\mathbb{L}^2(0,1)} +$$

$$\int_0^\tau\int_0^1\left(\frac{\partial}{\partial X}(\mathfrak{I}_t\mu)\right)^2 dxdt + \int_0^\tau\int_0^1(\mathfrak{I}_t\mu)^2\, dxdt$$

$$\leq \rho_1\left(\int_0^\tau\int_0^1\left(\frac{\partial}{\partial x}\left(\mathfrak{I}_t{}^2\mu\right)\right)^2 dx\, dt + \int_0^\tau\int_0^1\left(\mathfrak{I}_t{}^2\mu\right)^2 dxdt\right), \tag{67}$$

where:

$$\rho_1 = \frac{1}{(1,2\beta,\alpha)}. \tag{68}$$

We apply lemma (2.3) and we obtain:

$$\varphi_1(t) = \int_0^\tau\int_0^1\left(\frac{\partial}{\partial x}\left(\mathfrak{I}_t{}^2\mu\right)\right)^2 dx\, dt \;\; ; \frac{\partial\varphi_1(t)}{\partial t} = \int_0^1\left(\frac{\partial}{\partial x}\left(\mathfrak{I}_t{}^2\mu\right)\right)^2 dx \;\; ; \;\; \varphi_1(0) = 0, \tag{69}$$

so

$$\varphi_1(t) \leq \rho_1 Te^{T\rho_1}\int_0^\tau\int_0^1\left(\left(\mathfrak{I}_t{}^2\mu\right)\right)^2 dx\, dt, \tag{70}$$

then, equation (67) can be transformed as follows:

$$D_t^{\delta-1}\|\mathfrak{I}_t\mu\|^2_{\mathbb{L}^2(0,1)} + D_t^{\delta-1}\|\mathfrak{I}_x\mathfrak{I}_t\mu\|^2_{\mathbb{L}^2(0,1)} +$$

$$\int_0^\tau\int_0^1\left(\frac{\partial}{\partial x}(\mathfrak{I}_t\mu)\right)^2 dxdt + \int_0^\tau\int_0^1(\mathfrak{I}_t\mu)^2\, dxdt$$

$$\leq \rho_2\left(\int_0^\tau\int_0^1\left(\left(\mathfrak{I}_t{}^2\mu\right)\right)^2 dx\, dt\right), \tag{71}$$

such that:

$$\rho_2 = \max(\rho_1{}^2 Te^{T\rho_1},\ \rho_1). \tag{72}$$

Applying lemma (2.3), we arrive at equation

$$\varphi_2(t) = \int_0^\tau\int_0^1\left(\mathfrak{I}_t{}^2\mu\right)^2 dx\, dt \;\; ; \frac{\partial\varphi_2(t)}{\partial t} = \int_0^1\left(\mathfrak{I}_t{}^2\mu\right)^2 dx \;\; ; \;\; \varphi_2(0) = 0, \tag{73}$$

so:

$$\varphi_2(t) = \int_0^\tau\int_0^1\left(\mathfrak{I}_t{}^2\mu\right)^2 dx\, dt \leq 0. \tag{74}$$

Hence, (71) is transformed into

$$D_t^{\delta-1}\|\mathfrak{I}_t\mu\|^2_{\mathbb{L}^2(0,1)} + D_t^{\delta-1}\|\mathfrak{I}_x\mathfrak{I}_t\mu\|^2_{\mathbb{L}^2(0,1)} +$$

$$\int_0^\tau\int_0^1\left(\frac{\partial}{\partial x}(\mathfrak{I}_t\mu)\right)^2 dxdt+\int_0^\tau\int_0^1(\mathfrak{I}_t\mu)^2\,dxdt\le 0. \tag{75}$$

Setting: μ=0 in (58) we conclude that: $z=0$ in$L^2(\Omega)$. Since $z\in R(L)^{\perp}$, so $R(L)^{\perp}=\{0\}$.

This proves that $\overline{R(L)}$=F.

This completes the proof. ■

7. The study of the nonlinear problem

This section is devoted to solving the main problems (15)–(17). Consider now the auxiliary problem with the homogenous equation:

$$\begin{cases}\mathcal{L}U={}_0^c\partial_t^\delta U(x,t)-\alpha\frac{\partial^2U}{\partial x^2}-\beta\frac{\partial^3U}{\partial t\,\partial x^2}+\gamma U-\int_0^t a(t-s)U(x,s)ds=0, & (76)\\ \ell U=U(x,0)=\varphi(x),\quad qU=\frac{\partial U(x,0)}{\partial t}=\psi(x),\qquad 0<x<1. & (77)\\ \int_0^t U(x,t)dx=0,\qquad \int_0^t xU(x,t)dx=0,\qquad 0<t\le T\,. & (78)\end{cases}$$

If V and u are solutions of problems (18)-(20),(21)-(23), respectively, then $h=u-V$ satisfies

$$\begin{cases}\mathcal{L}w={}_0^c\partial_t^\delta w(x,t)-\alpha\frac{\partial^2w}{\partial x^2}-\beta\frac{\partial^3w}{\partial t\,\partial x^2}+\gamma w-\int_0^t a(t-s)w(x,s)ds=\chi(x,t,w,\frac{\partial w}{\partial x}), & (79)\\ \ell w=w(x,0)=\varphi(x),\quad qw=\frac{\partial w(x,0)}{\partial t}=\psi(x),\qquad 0<x<1. & (80)\\ \int_0^t w(x,t)dx=0,\qquad \int_0^t xw(x,t)dx=0,\qquad 0<t\le T\,. & (81)\end{cases}$$

Such that the function $\chi\left(x,t,w,\frac{\partial w}{\partial x}\right)=\chi\left(x,t,w+U,\frac{\partial w}{\partial x}+\frac{\partial U}{\partial x}\right)$, verifies the following condition:

$$|\chi(x,t,w_1,y_1)-\chi(x,t,w_2,y_2)|\le M(|w_1-w_2|+|y_1-y_2|),\quad \forall\,(x,t)\in\mathbb{Q} \tag{82}$$

Now we will show that the solution of problems (79)-(81) is unique. We will establish a similar proof for problems (21)-(23).

First we introduce the following space:

$${}'_{C^1}(\mathbb{Q})=\left\{w\in C^1(\mathbb{Q})\ ,\frac{\partial w^2}{\partial t\,\partial x^2}\in C(\mathbb{Q})\right\} \tag{83}$$

We suppose that: $w,u\in {}'_{C^1}(\mathbb{Q})$ verify homogenous initiale and boundary conditions, we have:

$$(\mathcal{L}w, \mathfrak{J}_x u) = \left({}_0^c\partial_t^\delta w, \mathfrak{J}_x u\right)_{\mathbb{L}^2(\Omega)} - \alpha\left(\frac{\partial^2 w}{\partial x^2}, \mathfrak{J}_x u\right)_{\mathbb{L}^2(\Omega)} - \beta\left(\frac{\partial^3 w}{\partial t\,\partial x^2}, \mathfrak{J}_x u\right)_{\mathbb{L}^2(\Omega)} + \gamma(w, \mathfrak{J}_x u)_{\mathbb{L}^2(\Omega)} - \left(\int_0^t a(t-s)w(x,s)ds\,, \mathfrak{J}_x u\right)_{\mathbb{L}^2(\Omega)}. \tag{84}$$

Where:

$$\left({}_0^c\partial_t^\delta w, \mathfrak{J}_x u\right)_{\mathbb{L}^2(\Omega)} = -\left({}_0^c\partial_t^\delta \mathfrak{J}_x w, u\right)_{\mathbb{L}^2(\Omega)} \tag{85}$$

$$-\alpha\left(\frac{\partial^2 w}{\partial x^2}, \mathfrak{J}_x u\right)_{\mathbb{L}^2(\Omega)} = -\alpha\left(\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right), \mathfrak{J}_x u\right)_{\mathbb{L}^2(\Omega)}$$

$$= \alpha\left(\frac{\partial w}{\partial x}, u\right)_{\mathbb{L}^2(\Omega)} \tag{86}$$

$$-\beta\left(\frac{\partial^3 w}{\partial t\,\partial x^2}, \mathfrak{J}_x u\right)_{\mathbb{L}^2(\Omega)} = -\beta\left(\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x}\right), u\right)_{\mathbb{L}^2(\Omega)} \tag{87}$$

$$-\left(\int_0^t a(t-s)w(x,s)ds\,, \mathfrak{J}_x u\right)_{\mathbb{L}^2(\Omega)} = \left(\int_0^t a(t-s)\mathfrak{J}_x w(x,s)ds\,, u\right)_{\mathbb{L}^2(\Omega)}. \tag{88}$$

We obtain:

$$-\left({}_0^c\partial_t^\delta \mathfrak{J}_x w, u\right)_{\mathbb{L}^2} + \alpha\left(\frac{\partial w}{\partial x}, u\right)_{\mathbb{L}^2} + \beta\left(\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x}\right), u\right)_{\mathbb{L}^2} + \left(\int_0^t a(t-s)\mathfrak{J}_x w(x,s)ds\,, u\right)_{\mathbb{L}^2(\Omega)} = (u, \mathfrak{J}_x \chi)_{\mathbb{L}^2(\Omega)} \tag{89}$$

Such that : $$\kappa(w, u) = (u, \mathfrak{J}_x \chi)_{\mathbb{L}^2(\Omega)} \tag{90}$$

Definition: A function $w \in \mathbb{L}^2(0, T, H^1(\Omega))$ is considered as the weak solution of the problem (79)-(81) if it satisfies (89) and (90) holds.

We will construct an iteration sequence as follows, let: $w(0) = 0$, and $\left(w^{(n)}\right)_n \in \mathbb{N}$, if $w^{(n-1)}$ is given, then for $n \in \mathbb{N}$ solve the following problem:

$$\begin{cases} \mathcal{L}w^{(n)} = {}_0^c\partial_t^\delta w^{(n)} - \alpha\frac{\partial^2 w^{(n)}}{\partial x^2} - \beta\frac{\partial^3 w^{(n)}}{\partial t\,\partial x^2} + \gamma w^{(n)} - \int_0^t a(t-s)w^{(n)}(x,s)ds = \chi(x,t,w^{(n-1)},\frac{\partial w^{(n-1)}}{\partial x}), & (91) \\ \ell w^{(n)} = w^{(n)}(x,0) = 0, \quad qw^{(n)} = \frac{\partial w^{(n)}(x,0)}{\partial t} = 0, \qquad 0 < x < 1. & (92) \\ \int_0^t w^{(n)}(x,t)dx = 0, \qquad \int_0^t xw^{(n)}(x,t)dx = 0, \qquad 0 < t \le T. & (93) \end{cases}$$

Theorem: For each fixed n assure that the solution of problem (91)-(93), $w^{(n)}(x,t)$ is unique.

We put: $W^{(n)}(x,t) = w^{(n+1)}(x,t) - w^{(n)}(x,t)$, we obtain:

$$\begin{cases} \mathcal{L}W^{(n)} = {}_0^c\partial_t^\delta W^{(n)} - \alpha\frac{\partial^2 W^{(n)}}{\partial x^2} - \beta\frac{\partial^3 W^{(n)}}{\partial t\,\partial x^2} + \gamma W^{(n)} - \int_0^t a(t-s)W^{(n)}(x,s)ds = N^{(n-1)}(x,t) & (94) \\ \ell W^{(n)} = W^{(n)}(x,0) = 0, \quad qW^{(n)} = \frac{\partial W^{(n)}(x,0)}{\partial t} = 0, \qquad 0 < x < 1. & (95) \\ \int_0^t W^{(n)}(x,t)dx = 0, \qquad \int_0^t xW^{(n)}(x,t)dx = 0, \qquad 0 < t \le T. & (96) \end{cases}$$

Such that: $N^{(n-1)}(x,t) = \chi\left(x,t,w^{(n)},\frac{\partial w^{(n)}}{\partial x}\right) - \chi\left(x,t,w^{(n-1)},\frac{\partial w^{(n-1)}}{\partial x}\right).$ (97)

Lemma 7.1

Supposing that the condition (82) holds, then for the linearized problem (94)-(96), we have:

$$\|W^{(n)}\|_{\mathbb{L}^2(0,T,H^1(\Omega))} \le C\|W^{(n-1)}\|_{\mathbb{L}^2(\Omega)}. \tag{98}$$

Where: $C > 0$.

Proof:

We put: $MW^{(n)} = -\Im_x{}^2\frac{\partial W^{(n)}}{\partial t}$, we get:

$$\left({}_0^c\partial_t^\delta W^{(n)}, -\mathfrak{I}_x{}^2 \frac{\partial W^{(n)}}{\partial t}\right)_{\mathbb{L}^2(\Omega)} + \alpha\left(\frac{\partial^2 W^{(n)}}{\partial x^2}, \mathfrak{I}_x{}^2 \frac{\partial W^{(n)}}{\partial t}\right)_{\mathbb{L}^2(\Omega)} + \beta\left(\frac{\partial^3 W^{(n)}}{\partial t\,\partial x^2}, \mathfrak{I}_x{}^2 \frac{\partial W^{(n)}}{\partial t}\right)_{\mathbb{L}^2(\Omega)} - \gamma\left(W^{(n)}, \mathfrak{I}_x{}^2 \frac{\partial W^{(n)}}{\partial t}\right)_{\mathbb{L}^2(\Omega)} + \left(\int_0^t a(t-s)W^{(n)}(x,s)ds, \mathfrak{I}_x{}^2 \frac{\partial W^{(n)}}{\partial t}\right)_{\mathbb{L}^2(\Omega)} = \left(N^{(n-1)}(x,t), -\mathfrak{I}_x{}^2 \frac{\partial W^{(n)}}{\partial t}\right)_{\mathbb{L}^2(\Omega)} \tag{99}$$

After integrating by parts all terms of (99) and using conditions (95) and (96), proceding as in the establishment of theorem 5.1

$$D_t{}^{\delta-1}\left\|\mathfrak{I}_x \frac{\partial W^{(n)}}{\partial t}\right\|^2_{\mathbb{L}^2(\Omega)} + \left(a_0 + \frac{\alpha}{2}\right)\left\|W^{(n)}(.,\tau)\right\|^2_{\mathbb{L}^2(\Omega)} \le \int_0^\tau \left\|\mathfrak{I}_x N^{(n-1)}(x,t)\right\|^2_{\mathbb{L}^2(\Omega)} dt + \left(\beta + \frac{\varepsilon}{2}\right)\int_0^\tau \left\|\mathfrak{I}_x \frac{\partial W^{(n)}(.,t)}{\partial t}\right\|^2_{\mathbb{L}^2(\Omega)} dt + \left(-\frac{\alpha}{2} + a_1 T^2\right)\int_0^\tau \left\|W^{(n)}(.,\tau)\right\|^2_{\mathbb{L}^2(\Omega)} dt \tag{100}$$

We apply $\mathfrak{I}_x$ to the equation (94), and we multiplying the resulting equation by $\frac{\partial W^{(n)}}{\partial x}$, and we integrate by parts over Ω :

$$\int_\Omega {}_0^c\partial_t^\delta \mathfrak{I}_x W^{(n)} . \frac{\partial W^{(n)}}{\partial x} dxdt - \alpha\int_\Omega \left(\frac{\partial W^{(n)}}{\partial x}\right)^2 dxdt - \beta\int_\Omega \frac{\partial}{\partial x}\left(\frac{\partial^2 W^{(n)}}{\partial t\,\partial x}\right) dxdt + \gamma\int_\Omega \mathfrak{I}_x W^{(n)} . \frac{\partial W^{(n)}}{\partial x} dxdt - \int_\Omega \int_0^t a(t-s)\mathfrak{I}_x W^{(n)}(x,s). \frac{\partial W^{(n)}}{\partial x} dsdxdt = \int_\Omega \mathfrak{I}_x N^{(n-1)}(x,t). \frac{\partial W^{(n)}}{\partial x} dxdt. \tag{101}$$

Where:

$$\int_\Omega {}_0^c\partial_t^\delta \mathfrak{I}_x W^{(n)} . \frac{\partial W^{(n)}}{\partial x} dxdt = -\int_\Omega {}_0^c\partial_t^\delta W^{(n)} . W^{(n)} dxdt \tag{102}$$

$$\alpha\int_\Omega \left(\frac{\partial W^{(n)}}{\partial x}\right)^2 dxdt = \alpha\int_0^\tau \left\|\frac{\partial W^{(n)}(.,t)}{\partial x}\right\|^2_{\mathbb{L}^2(\Omega)} dt \tag{103}$$

$$\beta\int_\Omega \frac{\partial}{\partial x}\left(\frac{\partial^2 W^{(n)}}{\partial t\,\partial x}\right) dxdt = \beta\left\|\frac{\partial W^{(n)}}{\partial x}\right\|^2_{\mathbb{L}^2(\Omega)} \tag{104}$$

$$\gamma\int_\Omega \mathfrak{I}_x W^{(n)} . \frac{\partial W^{(n)}}{\partial x} dxdt = -\gamma\int_0^\tau \left\|W^{(n)}\right\|^2_{\mathbb{L}^2(\Omega)} dt \tag{105}$$

$$\int_\Omega \int_0^t a(t-s)\mathfrak{I}_x W^{(n)}(x,s). \frac{\partial W^{(n)}}{\partial x} dsdxdt \le a_1 T^2 \int_0^\tau \left\|W^{(n)}(.,t)\right\|^2_{\mathbb{L}^2(\Omega)} dt \tag{106}$$

$$\int_{\Omega} \Im_x N^{(n-1)}(x,t).\frac{\partial W^{(n)}}{\partial x}\,dxdt \;\leq\; \frac{1}{2}\int_0^{\tau}\left\|N^{(n-1)}(x,t)\right\|^2_{\mathbb{L}^2(\Omega)}dt+\frac{1}{2}\int_0^{\tau}\left\|W^{(n)}(.,t)\right\|^2_{\mathbb{L}^2(\Omega)}dt \tag{107}$$

After integration by parts of all the terms of (101) and taking into consideration conditions (95), (96) and using inequality (8), we have :

$$\int_{\Omega} {}_0^c\partial_t^{\delta} W^{(n)}.W^{(n)}dxdt + \alpha\int_0^{\tau}\left\|\frac{\partial W^{(n)}(.,t)}{\partial x}\right\|^2_{\mathbb{L}^2(\Omega)}dt + \beta\left\|\frac{\partial W^{(n)}}{\partial x}\right\|^2_{\mathbb{L}^2(\Omega)}$$
$$\leq \frac{1}{2}\int_0^{\tau}\left\|N^{(n-1)}(x,t)\right\|^2_{\mathbb{L}^2(\Omega)}dt+\left(\gamma+\frac{1}{2}+a_1T^2\right)\int_0^{\tau}\left\|W^{(n)}(.,t)\right\|^2_{\mathbb{L}^2(\Omega)}dt \tag{108}$$

After combination of inequalities (100) and (108), we obtain:

$$D_t^{\delta-1}\left\|\Im_x\frac{\partial W^{(n)}}{\partial t}\right\|^2_{\mathbb{L}^2(\Omega)}+\int_{\Omega^{\tau}} {}_0^c\partial_t^{\delta}W^{(n)}.W^{(n)}dxdt+\alpha\int_0^{\tau}\left\|\frac{\partial W^{(n)}(.,t)}{\partial x}\right\|^2_{\mathbb{L}^2(\Omega)}$$
$$+\;\beta\left\|\frac{\partial W^{(n)}}{\partial x}\right\|^2_{\mathbb{L}^2(\Omega)}+\left(a_0+\frac{\alpha}{2}\right)\left\|W^{(n)}(.,\tau)\right\|^2_{\mathbb{L}^2(\Omega)}$$
$$\leq\int_0^{\tau}\left\|N^{(n-1)}(x,t)\right\|^2_{\mathbb{L}^2(\Omega)}dt$$
$$+\left(\beta+\frac{\varepsilon}{2}\right)\int_0^{\tau}\left\|\Im_x\frac{\partial W^{(n)}(.,t)}{\partial t}\right\|^2_{\mathbb{L}^2(\Omega)}dt+\left(a_1T^2-\frac{\alpha}{2}+\gamma+\frac{1}{2}\right)\int_0^{\tau}\left\|W^{(n)}(.,\tau)\right\|^2_{\mathbb{L}^2(\Omega)}dt \tag{109}$$

Now we will eliminate the last term in (109) by applying the Gronwall lemma:

$$D_t^{\delta-1}\left\|\Im_x\frac{\partial W^{(n)}}{\partial t}\right\|^2_{\mathbb{L}^2(\Omega)}+\int_{\Omega^{\tau}} {}_0^c\partial_t^{\delta}W^{(n)}.W^{(n)}dxdt+\alpha\int_0^{\tau}\left\|\frac{\partial W^{(n)}(.,t)}{\partial x}\right\|^2_{\mathbb{L}^2(\Omega)}+$$
$$\beta\left\|\frac{\partial W^{(n)}}{\partial x}\right\|^2_{\mathbb{L}^2(\Omega)}+\left(a_0+\frac{\alpha}{2}\right)\left\|W^{(n)}(.,\tau)\right\|^2_{\mathbb{L}^2(\Omega)}\leq$$
$$\exp c_0\left\{\int_0^{\tau}\left\|N^{(n-1)}(x,t)\right\|^2_{\mathbb{L}^2(\Omega)}dt+c_1\int_0^{\tau}\left\|\Im_x\frac{\partial W^{(n)}(.,t)}{\partial t}\right\|^2_{\mathbb{L}^2(\Omega)}dt\right\}. \tag{110}$$

Where:

$$\begin{cases} c_0=\beta+\dfrac{\varepsilon}{2}, \\ c_1=a_1T^2-\dfrac{\alpha}{2}+\gamma+\dfrac{1}{2}. \end{cases}$$

We apply **Gronwall** lemma to the last term of (110)

$$\int_0^\tau \left\| \Im_x \frac{\partial W^{(n)}(.,t)}{\partial t} \right\|^2_{\mathbb{L}^2(\Omega)} dt \leq \Gamma(\delta) E_{\delta\delta}\left(c_1 \exp(c_0 T)\, t^\delta\right) \exp(c_0 t)\,.\, D_t{}^{-\delta} \left\| N^{(n-1)}(x,t) \right\|^2_{\mathbb{L}^2(\Omega)} \tag{111}$$

On the other side, we applying the condition (82), we get :

$$\int_0^\tau \left\| N^{(n-1)}(x,t) \right\|^2_{\mathbb{L}^2(\Omega)} dt \leq 2M^2 \int_0^\tau \left(\left\| W^{(n-1)}(.,t) \right\|^2_{\mathbb{L}^2(\Omega)} + \left\| \frac{\partial W^{(n-1)}}{\partial x} \right\|^2_{\mathbb{L}^2(\Omega)} \right) dt \tag{112}$$

Combining (111)-(112) and using (49), we get:

$$\begin{aligned} D_t{}^{\delta-1} \left\| \Im_x \frac{\partial W^{(n)}}{\partial t} \right\|^2_{\mathbb{L}^2(\Omega)} + \int_{\Omega^\tau} {}_0^c\partial_t^\delta W^{(n)} . W^{(n)} dx dt + \int_0^\tau \left\| \frac{\partial W^{(n)}(.,t)}{\partial x} \right\|^2_{\mathbb{L}^2(\Omega)} \\ + \left\| \frac{\partial W^{(n)}}{\partial x} \right\|^2_{\mathbb{L}^2(\Omega)} \end{aligned}$$

$$+ \left\| W^{(n)}(.,\tau) \right\|^2_{\mathbb{L}^2(\Omega)} \leq C^* M^2 \int_0^\tau \left(\left\| W^{(n-1)}(.,t) \right\|^2_{\mathbb{L}^2(\Omega)} + \left\| \frac{\partial W^{(n-1)}}{\partial x} \right\|^2_{\mathbb{L}^2(\Omega)} \right) dt \tag{113}$$

Such that:

$$C^* = \exp(c_0 t) \left(1 + \Gamma(\delta) E_{\delta\delta}\left(c_1 \exp(c_0 T)\, t^\delta\right) \right) . \frac{T^\delta}{\Gamma(1+\delta)} \tag{114}$$

After discarding the first two terms on (113), we obtain:

$$\begin{aligned} \int_0^\tau \left\| \frac{\partial W^{(n)}(.,t)}{\partial x} \right\|^2_{\mathbb{L}^2(\Omega)} + \left\| \frac{\partial W^{(n)}}{\partial x} \right\|^2_{\mathbb{L}^2(\Omega)} + \left\| W^{(n)}(.,\tau) \right\|^2_{\mathbb{L}^2(\Omega)} \\ \leq C^* M^2 \int_0^\tau \left(\left\| W^{(n-1)}(.,t) \right\|^2_{\mathbb{L}^2(\Omega)} + \left\| \frac{\partial W^{(n-1)}}{\partial x} \right\|^2_{\mathbb{L}^2(\Omega)} \right) dt \end{aligned} \tag{115}$$

Here, the RHS doesn't depend on τ so, we can replace the LHS by upper bounds with respect to τ, we obtain:

$$\int_0^T \left\|\frac{\partial W^{(n)}(.,t)}{\partial x}\right\|^2_{\mathbb{L}^2(\Omega)} + \left\|\frac{\partial W^{(n)}}{\partial x}\right\|^2_{\mathbb{L}^2(\Omega)} + \left\|W^{(n)}(.,\tau)\right\|^2_{\mathbb{L}^2(\Omega)} \leq C^* M^2 \int_0^T \left(\left\|W^{(n-1)}(.,t)\right\|^2_{\mathbb{L}^2(\Omega)} + \left\|\frac{\partial W^{(n-1)}(.,t)}{\partial x}\right\|^2_{\mathbb{L}^2(\Omega)} \right) dt \tag{116}$$

We integrate over (0,T), we get:

$$\int_0^T \left\|\frac{\partial W^{(n)}(.,t)}{\partial x}\right\|^2_{\mathbb{L}^2(\Omega)} + \int_0^T \left\|W^{(n)}(.,\tau)\right\|^2_{\mathbb{L}^2(\Omega)} dt \leq \lambda M^2 \int_0^T \left(\left\|W^{(n-1)}(.,t)\right\|^2_{\mathbb{L}^2(\Omega)} + \left\|\frac{\partial W^{(n-1)}(.,t)}{\partial x}\right\|^2_{\mathbb{L}^2(\Omega)} \right) dt \tag{117}$$

Such that: $\lambda = \frac{C^* M^2 T}{\min(1,T)}$.

We obtain the inequality:

$$\left\|W^{(n)}\right\|^2_{\mathbb{L}^2\left(0,T,H^1(\Omega)\right)} \leq \lambda \left\|W^{(n-1)}\right\|^2_{\mathbb{L}^2\left(0,T,H^1(\Omega)\right)}. \tag{118}$$

Using the convergence of series criteria we conclude that $\sum_{n-1}^{\infty} W^{(n)}$ converges if $\lambda < 1$, in other words if $M < \sqrt{\frac{\min(1,T)}{C^* T}}$. Since: $W^{(n)} = w^{(n+1)}(x,t) - w^{(n)}(x,t)$, then $\left(w^{(n)}\right)_{n\in\mathbb{N}}$ converge to a function $w \in \mathbb{L}^2\left(0,T,H^1(\Omega)\right)$.

So to prove that w is the solution of problem (94)-(96), we have only to prove that w verifies (81) and (90).

We have from problem (91)-(93), that:

$$\kappa\left(w^{(n)},u\right) = \left(u, \Im_x \chi\left(x,t,w^{(n-1)},\frac{\partial w^{(n-1)}}{\partial x}\right)\right)_{\mathbb{L}^2(\Omega)} \tag{119}$$

More precisely

$$\kappa\left(w^{(n)}-w,u\right)+\kappa(w,u)=\left(u,\mathfrak{I}_x\chi\left(x,t,w^{(n-1)},\frac{\partial w^{(n-1)}}{\partial x}\right)-\mathfrak{I}_x\chi\left(x,t,w,\frac{\partial w}{\partial x}\right)\right)_{\mathbb{L}^2(\Omega)}$$

$$+\left(u,\mathfrak{I}_x\chi\left(x,t,w,\frac{\partial w}{\partial x}\right)\right)_{\mathbb{L}^2(\Omega)}, \tag{120}$$

After using (91), then (120) becomes:

$$\begin{aligned}\kappa&\left(w^{(n)}-w,u\right)\\&=-\left({}_0^c\partial_t^\delta\mathfrak{I}_x\left(w^{(n)}-w\right),u\right)_{\mathbb{L}^2(\Omega)}+\alpha\left(\frac{\partial\left(w^{(n)}-w\right)}{\partial x},u\right)_{\mathbb{L}^2(\Omega)}\\&+\beta\left(\frac{\partial}{\partial t}\left(\frac{\partial w^{(n)}}{\partial x}\right),u\right)_{\mathbb{L}^2(\Omega)}+\gamma\left(\mathfrak{I}_x\left(w^{(n)}-w\right),u\right)_{\mathbb{L}^2(\Omega)}\\&+\left(\int_0^t a(t-s)\mathfrak{I}_x\left(w^{(n)}-w\right)(x,s)ds\,,u\right)_{\mathbb{L}^2(\Omega)}\end{aligned} \tag{121}$$

After applying the integration by parts, and taking in consideration conditions on:u and w, (121) will be transformed as:

$$\begin{aligned}\kappa&\left(w^{(n)}-w,u\right)\\&=-\left({}_0^c\partial_t^\delta\left(w^{(n)}-w\right),\mathfrak{I}_x u\right)_{\mathbb{L}^2(\Omega)}+\alpha\left(\frac{\partial\left(w^{(n)}-w\right)}{\partial x},u\right)_{\mathbb{L}^2(\Omega)}\\&+\beta\left(\left(\frac{\partial w^{(n)}}{\partial x}\right),\frac{\partial u}{\partial t}\right)_{\mathbb{L}^2(\Omega)}+\gamma\left(\mathfrak{I}_x\left(w^{(n)}-w\right),u\right)_{\mathbb{L}^2(\Omega)}\\&+\left(\int_0^t a(t-s)\mathfrak{I}_x\left(w^{(n)}-w\right)(x,s)ds\,,u\right)_{\mathbb{L}^2(\Omega)}\end{aligned} \tag{122}$$

We will apply the inequality of Cauchy-Schwartz and lemma (2.4), we have:

$$\kappa\left(w^{(n)}-w,u\right)\le\zeta\left\|w^{(n)}-w\right\|_{\mathbb{L}^2\left(0,T,H^1(\Omega)\right)}\cdot\left[\|u\|_{\mathbb{L}^2(\Omega)}+\left\|\frac{\partial u}{\partial t}\right\|_{\mathbb{L}^2(\Omega)}\right] \tag{123}$$

Such that: $\zeta=\max\left(\alpha+\frac{T}{2}+\frac{\gamma}{2},\beta\right),$

And from (120), we have the following estimation:

$$\left(u, \mathfrak{I}_x\chi\left(x,t,w^{(n-1)},\frac{\partial w^{(n-1)}}{\partial x}\right) - \mathfrak{I}_x\chi\left(x,t,w,\frac{\partial w}{\partial x}\right)\right)_{\mathbb{L}^2(\Omega)} \leq \frac{M}{\sqrt{2}}\left\|w^{(n)} - w\right\|_{\mathbb{L}^2(0,T,H^1(\Omega))} \cdot \|u\|_{\mathbb{L}^2(\Omega)} \qquad (124)$$

When the limit $n \to \infty$ in (122) , and we take in consideration (123) and (124), we obtain

$$\kappa(w,u) = \left(u, \mathfrak{I}_x\chi\left(x,t,w,\frac{\partial w}{\partial x}\right)\right)_{\mathbb{L}^2(\Omega)}. \qquad (125)$$

So the problem (94)-(96) admit a weak solution.

Now, we will prove the uniqueness of problem (79)-(81).

Theorem:

Under condition of lemma (76), the problem (79)-(81) admits a unique solution.

Proof: We suppose that the problem (79)-(81) admit u_1 , u_2 solutions in$\mathbb{L}^2\left(0,T,H^1(\Omega)\right)$, and $W = u_1 - u_2$, and verifies:

$$\begin{cases} \mathcal{L}W = {}_0^c\partial_t^\delta W - \alpha\frac{\partial^2 W}{\partial x^2} - \beta\frac{\partial^3 W}{\partial t\,\partial x^2} + \gamma W - \int_0^t a(t-s)W(x,s)ds = N(x,t) & (126) \\ \ell W = W(x,0) = 0, \quad qW = \frac{\partial W(x,0)}{\partial t} = 0, \qquad 0 < x < 1. & (127) \\ \int_0^t W(x,t)dx = 0, \qquad \int_0^t xW(x,t)dx = 0, \qquad 0 < t \leq T\,. & (128) \end{cases}$$

Where: $N(x,t) = \chi(x,t,u_1,r_1) - \chi(x,t,u_2,r_2)$

This will be done by establishing the same proof of lemma (76), we obtain:

$$\|W\|_{\mathbb{L}^2(0,T,H^1(\Omega))} \leq C\|W\|_{\mathbb{L}^2(0,T,H^1(\Omega))}. \qquad (129)$$

Since $C < 1$, then: $(1-C)\|W\|_{\mathbb{L}^2(0,T,H^1(\Omega))} \leq 0$, we deduce finally that: $u_1 - u_2 = 0$, so $u_1 = u_2$ in $\mathbb{L}^2\left(0,T,H^1(\Omega)\right)$. ■

8. Conclusion

Based on the a priori estimate method and the density of the operator generated by the problem, and the iterative process, we have successfully proven the problem's existence and uniqueness of its solution. After carrying out our experimental research, we were able to confirm our initial hypotheses and respond to our problem. This work could be considered as a contribution to the development of the functional analysis method. And to conclude we argue that the research carried out in this article could serve as a contribution for possible studies in the field of applied mathematics.

Our study being based on the a priori estimation method is not exhaustive and opens several research perspectives such as: the application of the latter to ordinary physical problems and other purely fractional ones.

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