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Mathematical Modeling of Human Emotions using Neural Network - A Qualitative Approach

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Abstract

In this article, a model is proposed to study the impact of concepts stored in the memory from earlier experiences on the output of emotions. The model with constant and time-varying inputs has been considered. To show that the systems are well behaved, sufficient conditions for global asymptotic stability are derived for the system with constant inputs, and sufficient conditions for asymptotic nearness and boundedness are derived for the system with time-varying inputs. Numerical examples with simulations are illustrated to support the theory.

Keywords: Cooperative and supportive networks, Time delays, Equilibria, Global stability, Emotions, Memory.

Mathematics Subject Classification (2010) :34D23, 34K20, 91E40, 92B20.

1 Introduction

Emotions are a class of feelings which act as a source of non-verbal information to better understand what is being communicated. A human being is exposed to different kinds of emotions with different degrees of intensity by various experiences in life. Emotions have a huge impact on the behavior of a person as they get influenced through motivation and aggression. Emotions steer the decision-making process by creating certain feelings. Different emotions affect decisions in different ways, like anger can lead to impatience, rash decision-making, and if the person is afraid, the decisions may be uncertain, and it might take them longer to choose. Emotions and social life are intimately connected; as to how a person judges or understands or forms a relationship with another person depends on his/her emotions. Thus emotions have a vital role in day to day life of a human being.
The role of emotions from their perspective is being studied by several researchers from a variety of disciplines, including the social sciences, biological sciences, mathematical sciences, engineering sciences, and many more. The psychological aspects of the causes and

effects of these emotions have been the attention of psychologists. They have described a number of emotional theories to anticipate or identify an individual's emotion. The impact of brain activity on emotions and their reactions has been investigated by biologists. Computer scientists have attempted to use artificial neural networks to identify or predict emotions. Various methods have been used by mathematicians to attempt to formulate a mathematical model for emotions.

In psychology, according to the classical view of emotion, emotions can be assessed objectively and accurately through facial expressions. But in recent literature it is said that the facial expressions or heart rate variability or body movements don't give fingerprints to a particular emotion, i.e., the particular pattern of body movement or facial expression may not represent a particular emotion i.e., a smile on a face or furrow on your brow may not always represent happiness or anger [2, 9, 10]. When a person is angry, the way he gives response varies from person to person, they may express it with a scowl or with glowering looks or by shouting or they may become quiet. Thus, there is no particular fingerprint for any emotion, it varies from person to person and time to time. But how does the brain guide us to take a particular action for different emotions based on the situation? This is a key point on which many researchers are working. It is said in [2] that these actions for emotions are generated by the brain by using the concepts stored in memory from earlier experiences. We will try to model this idea of how the past experiences will contribute to the present outcome of emotion.

As we will be dealing with mathematical models and neural networks, we will see some of the works of mathematicians and computer scientists on emotions. Mathematical models to study emotions are proposed in [3,7,11]. Various neural networks for studying emotions were introduced and studied in [4- 6, 8, 16-17, 21, 23]. Most of the research was to predict or recognize the emotions, which has a wide application in the fields of marketing, media and communication, economic theory, banking, hospitality industry, etc. We will focus on the study of emotions based on the previous experiences, which will help us to predict the outcome of it.

In this paper, we have tried to understand how the concepts stored in the memory of past experiences contribute to the output of an emotion using a Cooperative and Supportive Neural Network (CSNN) model. CSNN concentrates on the contribution of collective capabilities and distributive operations of neurons. CSNN is a more reliable model that can be used for classification and clustering problems, data mining, financial and economic systems, and industrial information systems. CSNN model was propounded in [19] and various modifications of the system were studied in [12-14]. This network was considered for estimation of key parameters in infectious disease models [17] and in a recent study, it was used to understand the interaction between the focal and non-focal parts of the human brain [15].

The paper is organised as follows. In Section 2, the generalised CSNN model with constant inputs has been described in terms of our preposition, and its behaviour has been studied by deriving sufficient conditions for global asymptotic stability. In Section 3, the generalised CSNN model with time-varying inputs has been described, and its behaviour has been studied by deriving conditions for asymptotic nearness and boundedness of solutions. Followed by a discussion in Section 4.

2 Description of the Model with Constant Inputs-Behavior of the Solutions

A CSNN model comprises of two neuronal fields F_x and F_y. The neurons in F_x are denoted by x_i's and the neurons in F_y are denoted by y_{i_k}'s, for $i = 1, 2, 3....n,\ k = 1, 2, 3.....r_i,\ 1 \leq r_i \leq n$. Each neuron x_i in F_x is connected to the other neurons x_j for $i \neq j$ in the same neuronal field and they are also connected to r_i number of neurons $(y_{i_1},\ y_{i_2}....y_{i_{r_i}})$ in the neuronal field F_y. The pictorial representation of this network can be seen in the Figure 1.

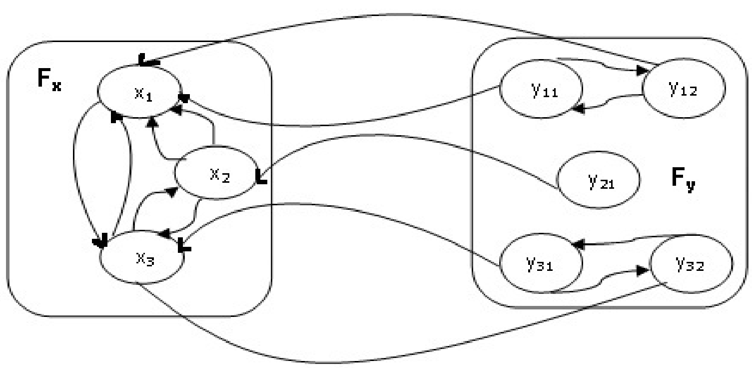

Figure 1

The dynamics of the model will be given by the following system of equations

$$x_i^{'} = -a_i x_i + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i, y_{i_k}(t)) + I_i,$$

$$y_{i_k}^{'} = -c_{i_k} y_{i_k} + \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}(t)) + J_{i_k},\ \ i = 1, 2, 3....n, k = 1, 2, 3.....r_i, 1 \leq r_i \leq .n \qquad (1)$$

Here x_i's be a state of particular emotion like happiness, anger, sadness, fear, etc., and y_{i_k}'s be the neurons corresponding to different memories stored from the previous experiences for a particular emotion x_i. a_i be the resting potential of an emotion x_i(i.e., the rate at which the neuron of a particular emotion is not active). c_{i_k} be the resting potential of y_{i_k} (i.e., the rate at which the neuron corresponding to a particular memory is not active). b_{ij} be the synaptic connection strength among emotions x_i and x_j. d_{i_l} be the synaptic connection strength among the memories of a particular emotion x_i. c_{ii_k} be the rate at which particular memory is influencing the emotion. The functions f_j be the functional relation among the emotions x_j's (for example, emotions like joy and happiness will inhibit the occurrence of anger or sadness, whereas emotion like enjoyment will enhance the occurrence of happiness and satisfaction [22]), g_{i_k} be the functional relation which shows how y_{i_k}'s are influencing x_i's, h_{i_l} be the response function of y_{i_l} towards y_{i_k} (i.e., how the memories of particular emotions are interrelated). I_i, J_{i_k} be the external inputs which may be information from the outside world or information from inside the body (or sensory inputs). They could as well be parts of the input that provoked and invoked particular emotion and its past memory.

It is noted that a person may not react to an emotion immediately, it may be because of the concurrency of many emotions at the same time, so a processing delay may arise among x_i's. Also, it's quite natural that the brain may take some time to recollect previously stored memory, which results in the processing delay among y_{i_k}'s. In certain situations, it may take some time for the concepts of previous memories to show their influence on emotions, as the brain may be engaged with another activity, which leads to transmission delays. To make the model more realistic, we will incorporate these delays in (1). Hence, we get

$$x_i^{'} = -a_i x_i + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j)) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i, y_{i_k}(t-\vartheta_{i_k})) + I_i,$$

$$y_{i_k}^{'} = -c_{i_k} y_{i_k} + \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}(t-\zeta_{i_l})) + J_{i_k}, i = 1,2,3....n, k = 1,2,3.....r_i, 1 \leq r_i \leq n. \quad (2)$$

Where in τ_i's and ζ_{i_k}'s are the processing delays among x_i's and y_{i_k}'s respectively. ϑ_{i_k} is the transmission delay from y_{i_k}'s to x_i's.

The occurrence of delay will depend on the requirement of the problem under study, so the presence of all the three delays may not require always. Hence, we can deduce different models by considering one or two delays to exist. Some of these deduced models are the open problems II, III, and IV, which are proposed in [19] and one of them is the model that has been studied in [12]. The results that are going to be derived for (2) will apply to all the deduced models and also to the basic CSNN model (1).

We assume the following Lipschitz conditions on the response functions, as they are needed for the solutions to exist.

$$\|g_{i_k}(x_i, y_{i_k}) - g_{i_k}(\overline{x}_i, \overline{y}_{i_k})\| \leq M_{1i_k}|y_{i_k} - \overline{y}_{i_k}| + M_{2i_k}|x_i - \overline{x}_i|,$$

$$|f_j(x_j) - f_j(\overline{x}_j)| \leq p_j|x_j - \overline{x}_j|,$$

$$|h_{i_k}(y_{i_k}) - h_{i_k}(\overline{y}_{i_k})| \leq q_{i_k}|y_{i_k} - \overline{y}_{i_k}|, \tag{3}$$

for some positive constants M_{1i_k}, M_{2i_k}, p_j and q_{i_k}.
By the theory of delay differential equations, we know that the local Lipschitz conditions on response functions guarantee the existence of solutions. Thus, the system (2) possesses unique solutions that are continuous in their maximal interval of existence with suitable initial conditions [12].
Now we know the solutions of the system (2) exist (i.e., concepts stored in the previous memory are provoking a person to respond with an emotion). But how do they behave? As we are considering system (2) to study the outcome of emotions based on the concepts stored in memory, solutions should be controllable, i.e., the solutions should be either bounded or converge to a particular solution (i.e., the outcome of emotion should not be too wild). As a system (2) is an autonomous system, it may have an equilibrium solution. We can consider that equilibrium is the optimal way of responding to emotion, showing one's emotional stability. So first, we will check for what conditions does the equilibrium of the system exist.

Since the existence of equilibria is not affected by the presence of time delays, as in [12, 13, 19], we may establish that

Theorem 2.1. *If the output functions satisfy conditions (3) and the parameters satisfy conditions*

$$\sum_{j=1}^{n} |b_{ji}|p_j + \sum_{k=1}^{r_i} |c_{ii_k}|M_{2i_k} < a_i, \qquad \frac{1}{a_i}\sum_{k=1}^{r_i} |c_{ii_k}|M_{1i_k} + \frac{1}{c_{i_k}}\sum_{k=1}^{r_i} |d_{i_k}|q_{i_k} < 1. \tag{4}$$

Then the system (2) has a unique equilibrium solution.

Thus, under conditions (4), system (2) possesses a unique equilibrium, and they may typically be represented by $(x_i^*, y_{i_k}^*)$. As we know that equilibria are stationary solutions of the system, we can write

$$\begin{aligned}
(x_i - x_i^*)' &= -a_i(x_i - x_i^*) + \sum_{j=1}^{n} b_{ij}(f_j(x_j(t-\tau_j)) - f_j(x_j^*)) \\
&\quad + \sum_{k=1}^{r_i} c_{ii_k}(g_{i_k}(x_i, y_{i_k}(t-\vartheta_{i_k})) - g_{i_k}(x_i^*, y_{i_k}^*)), \\
(y_{i_k} - y_{i_k}^*)' &= -c_{i_k}(y_{i_k} - y_{i_k}^*) + \sum_{l=1}^{r_i} d_{i_l}(h_{i_l}(y_{i_l}(t-\zeta_{i_l})) - h_{i_l}(y_{i_l}^*)),
\end{aligned} \tag{5}$$

where $i = 1, 2, 3....n$, $k = 1, 2, 3.....r_i$ and $1 \leq r_i \leq n$.
We will be using equation (5) whenever required in our results.

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Remark 2.2. From here on, we assume that the equilibrium always exists in our system. We notice that conditions (4) are sufficient and simpler conditions may exist that ensure existence. However, we may recall Theorem 2.1 whenever necessary. Now the question arises, under what conditions do the solutions of (2) converge to the equilibrium point. So next we try to derive sufficient conditions for the solutions to converge to equilibria, which are nothing but the conditions for global asymptotic stability of the system. Two types of conditions are derived here using the Lyapunov functional technique. First one is delay independent stability where the parametric conditions are derived without restricting the delays and the second one is delay-dependent stability where the conditions are derived by restricting the delay parameters to a particular region. Let us first derive the delay-independent conditions.

2.1 Delay Independent

Here, we have obtained parametric conditions for asymptotic stability without restricting delay parameters. These conditions are for those situations where emotional stability is not affected by time delays. In this regard, we state the following theorem

Theorem 2.3. *Assume that the output function f_i, g_{i_k} and h_{i_l} satisfy the conditions (3) then the equilibrium $(x_i^*, y_{i_k}^*)$ of (2) is globally asymptotically stable, provided the parameters of the system satisfy the inequalities*

$$\begin{aligned} a_i &> \sum_{j=1}^{n} |b_{ji}| p_i + \sum_{k=1}^{r_i} |c_{ii_k}| M_{2i_k}, \\ c_{i_k} &> \sum_{k=1}^{r_i} |d_{i_k}| q_{i_k}. \end{aligned} \tag{6}$$

Proof. Consider the functional

$$V_1 = |y_{i_k} - y_{i_k}^*| + \sum_{l=1}^{r_i} |d_{i_l}| q_{i_l} \int_{t-\zeta_{i_l}}^{t} |y_{i_l}(z) - y_{i_l}^*| dz$$

Then the upper dini derivatives along the solutions of (2) is given by

$$\begin{aligned}
D^+V_1 &\leqslant -c_{i_k}|y_{i_k}-y^*_{i_k}| + \sum_{l=1}^{r_i}|d_{i_l}||h_{i_l}(y_{i_l}(t-\zeta_{i_l})) - h_{i_l}(y^*_{i_l})| \\
&\quad + \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}\Big[|y_{i_l}-y^*_{i_l}| - |y_{i_l}(t-\zeta_{i_l}) - y^*_{i_l}|\Big] \\
&\leqslant -\left[c_{i_k} - \sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}\right]|y_{i_k}-y^*_{i_k}| \\
&\leqslant -AV_1
\end{aligned}$$

Where A =min $\left\{c_{i_k} - \sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}, i = 1, 2...n, 1 \leq k \leq r_i\right\}$
By hypothesis $A > 0$, therefore we get $D^+V_1 < -AV_1 < 0 \Rightarrow D^+V_1 + AV_1 < 0$.
By comparision theorem using $V_1 \geq 0$, we have $V_1 \longrightarrow 0$ as $t \longrightarrow \infty$.
Thus for sufficiently large t, say $t > t^*$, we have $y_{i_k} \longrightarrow y^*_{i_k}$
Now in order to prove $x_i \longrightarrow x^*_i$, we consider

$$V_2 = \sum_{i=1}^{n}\left[|x_i - x^*_i| + \sum_{j=1}^{n}|b_{ij}|p_j\int_{t-\tau_j}^{t}|x_j(z)-x^*_j|dz + \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\int_{t-\vartheta_{i_k}}^{t}|y_{i_k}(z)-y^*_{i_k}|dz\right]$$

for $t > T = t^* + Max\{\tau_j\}$ for all j and proceeding as above we get

$$\begin{aligned}
D^+V_2 &\leq -\sum_{i=1}^{n}\left[a_i - \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} - \sum_{j=1}^{n}|b_{ji}|p_i\right]|x_i - x^*_i| \\
&= -\sum_{i=1}^{n}B|x_i - x^*_i| \qquad (7)
\end{aligned}$$

where $B = min\left\{a_i - \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} - \sum_{j=1}^{n}|b_{ji}|p_i\right\}$
By hypothesis $B > 0$, therefore $D^+V_2 < 0$.
If we integrate (7) from 0 to t with respect to t we get

$$V_2(t) + \int_0^t\sum_{i=1}^{n}B|x_i - x^*_i|dt \leq V_2(0) < \infty$$

Therefore we can say that $V_2(t)$ & x_i and there derivatives are bounded on $[0, \infty)$. Thus x_i's are uniformly continuous. Applying Barbalat's Lemma we get $|x_i - x^*_i| \longrightarrow 0$ as $t \longrightarrow \infty$.
Thus $x_i(t) \longrightarrow x^*_i$ as $t \longrightarrow \infty$.
Hence the equilibrium $(x^*_i, y^*_{i_k})$ is globally asymptotically stable. □

Remark 2.4. Theorem 2.3 gives sufficient conditions for the asymptotic stability of the system (2). These conditions depend on the Lyapunov functions that have been used. So, as the Lyapunov function varies, these conditions also vary.
In the next section, we derive the delay-dependent stability conditions.

2.2 Delay Dependent

Emotions are interrelated to each other, for instance, if a person is having the guilt of hurting his friend, as long as he overcomes it by expressing or giving an apology to his friend, he will not be happy and joyful (processing delays among x_i's). Thus, the delay in overcoming the feeling of guilt will disturb him mentally. It is quite natural for a human being to forget things, like remembering birthdays. Many of us remember the month of the birthday or anniversary of our friends, but don't remember the exact date (processing delays among y_{i_k}'s). Many times, humans tend to neglect certain things when they are indulging in many activities, or maybe because of priorities. For instance, a person may know that he will feel happy by talking to his friends or family, but he may not do it because of his work schedule (transmission delays). Which may lead him to feel lonely and depressed. Thus, delay affects the emotions in many ways. In this section, we have obtained conditions on parameters for global asymptotic stability of the system with suitable restrictions on delay parameters. These types of conditions were not discussed in the previous study of this model.

We propose the following change of variables, for simplicity
We let $z_i(t) = x_i - x_i^*$, $w_{i_k} = y_{i_k} - y_{i_k}^*$, $H_{i_k}(w_{i_k}) = h_{i_k}(y_{i_k}) - h_{i_k}(y_{i_k}^*)$
$F_i(z_i) = f_i(x_i) - f_i(x_i^*)$, and $G_{i_k}(z_i, w_{i_k}) = g_{i_k}(x_i, y_{i_k}) - g_{i_k}(x_i^*, y_{i_k}^*)$
Then, using (4,) system (2) can be written as

$$\begin{aligned} z_i^{'} &= -a_i z_i^{'} + \sum_{j=1}^{n} b_{ij} F_j(z_j(t-\tau_j)) + \sum_{k=1}^{r_i} c_{ii_k} G_{i_k}(z_i, w_{i_k}(t-\vartheta_{i_k})) \\ w_{i_k}^{'} &= -c_{i_k} w_{i_k} + \sum_{l=1}^{r_i} d_{i_l} H_{i_l}(w_{i_l}(t-\zeta_{i_l})) \end{aligned} \tag{8}$$

Further conditions (3) reduce to

$$\begin{aligned} |F_j(z_j(t-\tau_j))| &\le p_j |z_j(t-\tau_j)| \\ |G_{i_k}(z_i, w_{i_k}(t-\vartheta_{i_k}))| &\le M_{1i_k} |w_{i_k}(t-\vartheta_{i_k})| + M_{2i_k}|z_i| \\ |H_{i_l}(w_{i_l}(t-\zeta_{i_l}))| &\le q_{i_l} |w_{i_l}(t-\zeta_{i_l})| \end{aligned} \tag{9}$$

Theorem 2.5. *Suppose the output functions of the system (2) satisfy conditions(3) and* *the parameters satisfy.*

$$A = Min\left[a_i - \sum_{j=1}^{n} |b_{ji}|p_i - \sum_{k=1}^{r_i} |c_{ii_k}|M_{2i_k}\right] > 0$$
$$B = Min\left[c_{i_k} - |c_{ii_k}|M_{1i_k} - \sum_{k=1}^{r_i} |d_{i_k}|q_{i_k}\right] > 0$$
$$C = Min\left[\sum_{j=1}^{n} |b_{ji}|p_i\Big(a_i + \sum_{j=1}^{n} |b_{ji}|p_i + \sum_{k=1}^{r_i} |c_{ii_k}|M_{2i_k}\Big)\right]$$
$$D = Min\left[\sum_{j=1}^{n} |b_{ji}|p_i|c_{ii_k}|M_{1i_k}\right]$$
$$E = Min\left[|c_{ii_k}|M_{1i_k}\Big(c_{i_k} + \sum_{k=1}^{r_i} |d_{i_k}|q_{i_k}\Big)\right]$$
$$F = Min\left[\sum_{k=1}^{r_i} |d_{i_k}|q_{i_k}\Big(c_{i_k} + \sum_{k=1}^{r_i} |d_{i_k}|q_{i_k}\Big)\right]$$

Let $\tau^* = Max\Big\{\tau_i, 1 \le i \le n\Big\}, \vartheta^* = Max\Big\{\vartheta_{i_k}, 1 \le i \le n, 1 \le k \le r_i\Big\}$ *and* $\zeta^* = Max\Big\{\zeta_{i_k}, 1 \le i \le n, 1 \le k \le r_i\Big\}$. *Then the equilibrium* $(x_i^*, y_{i_k}^*)$ *of (2) is globally asymptotically stable for* $0 < \tau^* < r, 0 < \vartheta^* < s$ *and* $0 < \zeta^* < p$ *where* $r = Min\{\frac{A}{C}, \frac{B}{D}\}, s = \frac{B}{E}$ *and* $p = \frac{B}{F}$.

Proof. Let $V_1 = \sum_{i=1}^{n} |x_i(t) - x_i^*| = \sum_{i=1}^{n} |z_i(t)|$

The upper dini derivative of V_1 along the solutions of (7) is given by

$$D^+V_1(t) \leq \sum_{i=1}^{n}\Big[-a_i|z_i(t)| + \sum_{j=1}^{n}|b_{ji}|p_i|z_i(t-\tau_i)| + \sum_{k=1}^{r_i}|c_{ii_k}|\Big[M_{1i_k}|w_{i_k}(t-\vartheta_{i_k})| + M_{2i_k}|z_i(t)|\Big]\Big] \tag{10}$$

$$\begin{aligned} z_i(t-\tau_i) &= z_i(t) - \int_{t-\tau_i}^{t} z_i'(s)ds \\ &= z_i(t) - \int_{t-\tau_i}^{t}\Big[-a_i z_i(s) + \sum_{j=1}^{n} b_{ij}F_j(z_j(s-\tau_j)) + \sum_{k=1}^{r_i} c_{ii_k}G_{i_k}(z_i(s), w_{i_k}(s-\vartheta_{i_k}))\Big]ds \end{aligned} \tag{11}$$

$$\begin{aligned} w_{i_k}(t-\vartheta_{i_k}) &= w_{i_k}(t) - \int_{t-\vartheta_{i_k}}^{t} w_{i_k}'(s)ds \\ &= w_{i_k}(t) - \int_{t-\vartheta_{i_k}}^{t}\big[-c_{i_k}w_{i_k}(s) + \sum_{l=1}^{r_i} d_{i_l}H_{i_l}(w_{i_l}(s-\zeta_{i_l}))\big]ds \end{aligned} \tag{12}$$

Substituting (10) and (11) in (9)

$$\begin{aligned} D^+V_1(t) \leq \sum_{i=1}^{n}\Big[&-a_i|z_i(t)| + \sum_{j=1}^{n}|b_{ji}|p_i\Big[|z_i(t)| + \int_{t-\tau_i}^{t} a_i|z_i(s)|ds + \sum_{j=1}^{n}|b_{ij}|\int_{t-\tau_i}^{t}|F_j(z_j(s-\tau_j))|ds \\ &+ \sum_{k=1}^{r_i}|c_{ii_k}|\int_{t-\tau_i}^{t}|G_{i_k}(z_i(s), w_{i_k}(s-\vartheta_{i_k})|ds\Big] \\ &+ \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\Big[|w_{i_k}(t)| + \int_{t-\vartheta_{i_k}}^{t} c_{i_k}|w_{i_k}(s)|ds + \sum_{l=1}^{r_i}|d_{i_l}|\int_{t-\vartheta_{i_k}}^{t}|H_{i_l}(w_{i_l}(s-\zeta_{i_l}))|ds\Big] \\ &+ \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k}|z_i(t)|\Big] \end{aligned}$$

Using conditions (8), we get

$$\begin{aligned} D^+V_1(t) \leq \sum_{i=1}^{n}\Big[&-a_i|z_i(t)| + \sum_{j=1}^{n}|b_{ji}|p_i\Big[|z_i(t)| + \int_{t-\tau_i}^{t} a_i|z_i(s)|ds + \sum_{j=1}^{n}|b_{ji}|\int_{t-\tau_i}^{t} p_i|(z_i(s-\tau_i))|ds \\ &+ \sum_{k=1}^{r_i}|c_{ii_k}|\int_{t-\tau_i}^{t}(M_{2i_k}|z_i(s)| + M_{i_k}|w_{i_k}(s-\vartheta_{i_k})|)ds\Big] \end{aligned}$$

$$+\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\Big[|w_{i_k}(t)|+\int_{t-\vartheta_{i_k}}^{t}c_{i_k}|w_{i_k}(s)|ds+\sum_{l=1}^{r_i}|d_{i_l}|\int_{t-\vartheta_{i_k}}^{t}q_{i_l}|(w_{i_l}(s-\zeta_{i_l}))|ds\Big]$$
$$+\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k}|z_i(t)|\Bigg] \tag{13}$$

$$Let\ V_2(t)=\sum_{i=1}^{n}\Bigg[\sum_{j=1}^{n}|b_{ji}|p_i\Big[a_i\int_{t-\tau_i}^{t}ds\int_{s}^{t}z_i(u)du+\sum_{j=1}^{n}|b_{ji}|p_i\int_{t-\tau_i}^{t}ds\int_{s}^{t}|(z_i(u-\tau_i))|du$$
$$+\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k}\int_{t-\tau_i}^{t}ds\int_{s}^{t}|(z_i(u))|du+\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\int_{t-\tau_i}^{t}ds\int_{s}^{t}|w_{i_k}(u-\vartheta_{i_k})|du$$
$$+\tau_i\sum_{j=1}^{n}|b_{ji}|p_i\int_{t-\tau_i}^{t}|(z_i(s))|ds+\tau_i\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\int_{t-\vartheta_{i_k}}^{t}|w_{i_k}(s)|ds\Big]$$
$$+\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\Big[c_{i_k}\int_{t-\vartheta_{i_k}}^{t}ds\int_{s}^{t}|w_{i_k}(u)|du+\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}\int_{t-\vartheta_{i_k}}^{t}ds\int_{s}^{t}|(w_{i_l}(u-\zeta_{i_l}))|du$$
$$+\vartheta_{i_k}\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}\int_{t-\zeta_{i_l}}^{t}|w_{i_l}(s)|ds\Big]\Bigg]$$

$$D^{+}V_2(t)\le\sum_{i=1}^{n}\Bigg[\sum_{j=1}^{n}|b_{ji}|p_i\Big[\tau_i\Big(a_i|z_i(t)|+\sum_{j=1}^{n}|b_{ji}|p_i|(z_i(t))|+\sum_{k=1}^{r_i}|c_{ii_k}|(M_{2i_k}|z_i(t)|+M_{1i_k}|w_{i_k}(t)|)\Big)$$
$$-a_i\int_{t-\tau_i}^{t}|z_i(s)|ds-\sum_{j=1}^{n}|b_{ji}|\int_{t-\tau_i}^{t}p_i|(z_i(s-\tau_i))|ds-\sum_{k=1}^{r_i}|c_{ii_k}|\int_{t-\tau_i}^{t}M_{2i_k}|z_i(s)|ds$$
$$-\sum_{k=1}^{r_i}|c_{ii_k}|\int_{t-\tau_i}^{t}M_{1i_k}|w_{i_k}(s-\vartheta_{i_k})|ds\Big]+\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\Big[\vartheta_{i_k}\Big(c_{i_k}w_{i_k}(t)+\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}w_{i_l}(t)\Big)$$
$$-c_{i_k}\int_{t-\vartheta_{i_k}}^{t}|w_{i_k}(s)|ds-\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}\int_{t-\vartheta_{i_k}}^{t}|(w_{i_l}(s-\zeta_{i_l}))|ds\Bigg] \tag{14}$$

Let $V_3(t)=\sum_{i=1}^{n}\sum_{k=1}^{r_i}|w_{i_k}(t)|$. Then the dini derivative along the solutions of (7) is given by

$$D^{+}V_3(t)\le\sum_{i=1}^{n}\sum_{k=1}^{r_i}\Big[-c_{i_k}|w_{i_k}(t)|+\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}|w_{i_l}(t-\zeta_{i_l})|\Big] \tag{15}$$

$$
\begin{aligned}
w_{i_l}(t-\zeta_{i_l}) &= w_{i_l}(t) - \int_{t-\zeta_{i_l}}^{t} w_{i_l}^{'}(s)ds \\
&= w_{i_l}(t) - \int_{t-\zeta_{i_l}}^{t} \big[-c_{i_l}w_{i_l}(s) + \sum_{l=1}^{r_i} d_{i_l}H_{i_l}(w_{i_l}(s-\zeta_{i_l}))\big]ds \qquad (16)
\end{aligned}
$$

Substituting (15) in (14)

$$
D^+V_3(t) \leq \sum_{i=1}^{n}\sum_{k=1}^{r_i}\Big[-c_{i_k}|w_{i_k}(t)| + \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}\Big(|w_{i_l}(t)| + \int_{t-\zeta_{i_l}}^{t}\Big(c_{i_l}|w_{i_l}(s)| + \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}|w_{i_l}(s-\zeta_{i_l})|\Big)ds\Big)|\Big] \qquad (17)
$$

$$
\begin{aligned}
V_4(t) = &\sum_{i=1}^{n}\sum_{k=1}^{r_i}\Big[\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}\Big(\int_{t-\zeta_{i_l}}^{t} ds\int_{s}^{t} c_{i_l}|w_{i_l}(u)|du + \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}\int_{t-\zeta_{i_l}}^{t} ds\int_{s}^{t}|(w_{i_l}(u-\zeta_{i_l}))|du \\
&+ \zeta_{i_l}\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}\int_{t-\zeta_{i_l}}^{t}|w_{i_l}(s)|ds\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
D^+V_4(t) \leq &\sum_{i=1}^{n}\sum_{k=1}^{r_i}\Big[\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}\Big(\zeta_{i_l}\Big(c_{i_l}|w_{i_l}(t)| + \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}|w_{i_l}(t)|\Big) - \int_{t-\zeta_{i_l}}^{t} c_{i_l}|w_{i_l}(s)|ds \\
&- \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}\int_{t-\zeta_{i_l}}^{t}|(w_{i_l}(s-\zeta_{i_l}))|ds\Big)\Big] \qquad (18)
\end{aligned}
$$

Let $V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t)$
Using (12),(13),(16) and (17) we get

$$
\begin{aligned}
D^+V(t) \leq &\sum_{i=1}^{n}\left[-a_i|z_i(t)| + \sum_{j=1}^{n}|b_{ji}|p_i|z_i(t)| + \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|w_{i_k}(t)| + \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k}|z_i(t)|\right] \\
&+ \sum_{i=1}^{n}\left[\sum_{j=1}^{n}|b_{ji}|p_i\tau_i\Big(a_i|z_i(t)| + \sum_{j=1}^{n}|b_{ij}|p_j|(z_j(t))| + \sum_{k=1}^{r_i}|c_{ii_k}|(M_{2i_k}|z_i(t)| + M_{1i_k}|w_{i_k}(t)|)\Big)\right. \\
&\left.+ \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\vartheta_{i_k}\Big(c_{i_k}w_{i_k}(t) + \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}w_{i_l}(t)\Big)\right] + \sum_{i=1}^{n}\sum_{k=1}^{r_i}\Big[-c_{i_k}|w_{i_k}(t)| \\
&+ \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}|w_{i_l}(t)|\Big] + \sum_{i=1}^{n}\sum_{k=1}^{r_i}\Big[\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}\zeta_{i_l}\Big(c_{i_l}w_{i_l}(t) + \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}|w_{i_l}(t)|\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
D^+V(t) \leq &-\sum_{i=1}^{n}\Bigg[\Bigg(\Big(a_i-\sum_{j=1}^{n}|b_{ji}|p_i-\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k}\Big)-\tau_i\sum_{j=1}^{n}|b_{ji}|p_i\Big(a_i+\sum_{j=1}^{n}|b_{ij}|p_j+\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k}\Big)\Bigg)|z_i(t)| \\
&+\sum_{k=1}^{r_i}\Bigg(\Big(c_{i_k}-|c_{ii_k}|M_{1i_k}-\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}\Big)-\tau_i\sum_{j=1}^{n}|b_{ji}|p_i|c_{ii_k}|M_{1i_k} \\
&-\vartheta_{i_k}|c_{ii_k}|M_{1i_k}\Big(c_{i_k}+\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}\Big)-\zeta_{i_k}(\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}\Big(c_{i_k}+\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}\Big)\Bigg)|w_{i_k}(t)|\Bigg] \\
&\leq -\sum_{i=1}^{n}\Bigg[\Big(A-C\tau^*\Big)|z_i(t)|+\sum_{k=1}^{r_i}\Big(B-D\tau^*-E\vartheta^*-F\zeta^*\Big)|w_{i_k}(t)|\Bigg] \qquad (19)
\end{aligned}
$$

Let $r = Min\{\frac{A}{C}, \frac{B}{D}\}, s = \frac{B}{E}$ and $p = \frac{B}{F}$. We take $\tau^* = Max\Big\{\tau_i, 1 \leq i \leq n\Big\}, \vartheta^* = Max\Big\{\vartheta_{i_k}, 1 \leq i \leq n, 1 \leq k \leq r_i\Big\}$ and $\zeta^* = Max\Big\{\zeta_{i_k}, 1 \leq i \leq n, 1 \leq k \leq r_i\Big\}$ Clearly from hypothesis and (18), we have $0 < \tau^* < r, 0 < \vartheta^* < s, 0 < \zeta^* < p$. Thus we have $D^+V(t) < 0$. Therefore, the equilibrium $(x_i^*, y_{i_k}^*)$ is globally asymptotically stable (See [19, 13]). □

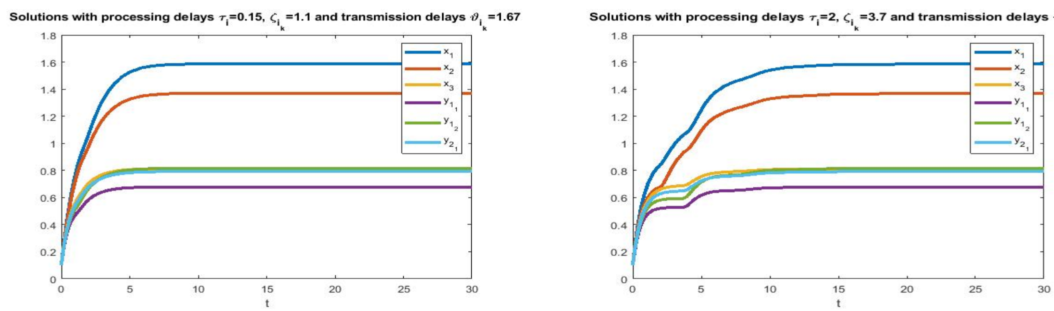
Remark 2.6. From the above results, we have obtained two types of conditions for global asymptotic stability of the system (2). One is by restricting the delays, and the other is by not restricting delays but putting some limitations on the parameters of the system. Thus, the solutions of the system (2) will converge to its equilibrium point under given conditions. So we can say our system (2) with constant inputs is well-behaved and controllable under certain conditions.

We illustrate the above results by using the numerical example,

Example 2.7. Consider the system of equations with the 2 main components and two sub-components attached to the main components

$$
\begin{aligned}
x_1' &= -1.9x_1 + 0.2f_1(x_1(t-\tau_1)) + 0.32f_2(x_2(t-\tau_2)) + 0.23g_{1_1}(x_1, y_{1_1}(t-\vartheta_{1_1})) \\
&\quad + 0.44g_{1_2}(x_1, y_{1_2}(t-\vartheta_{1_2})) + 1 \\
x_2' &= -2.2x_2 + 0.6f_1(x_1(t-\tau_1)) + 0.25f_2(x_2(t-\tau_2)) + 0.31g_{2_1}(x_2, y_{2_1}(t-\vartheta_{2_1})) \\
&\quad + 0.26g_{2_2}(x_2, y_{2_2}(t-\vartheta_{2_2})) + 1 \\
y_{1_1}' &= -1.5y_{1_1} + 0.22h_{1_1}(y_{1_1}(t-\zeta_{1_1})) + 0.12h_{1_2}(y_{1_2}(t-\zeta_{1_2})) + 1 \\
y_{1_2}' &= -2y_{1_2} + 0.31h_{1_1}(y_{1_1}(t-\zeta_{1_1})) + 0.25h_{1_2}(y_{1_2}(t-\zeta_{1_2})) + 1 \\
y_{2_1}' &= -1.8y_{2_1} + 0.5h_{2_1}(y_{2_1}(t-\zeta_{2_1})) + 0.2h_{2_2}(y_{2_2}(t-\zeta_{2_2})) + 1 \\
y_{2_2}' &= -1.6y_{2_1} + 0.3h_{2_1}(y_{2_1}(t-\zeta_{2_1})) + 0.1h_{2_2}(y_{2_2}(t-\zeta_{2_2})) + 1
\end{aligned}
$$

Choose the response function as $f_i(x_i) = tanh(x_i), h_{i_l}(y_{i_l}) = tanh(y_{i_l})$ and $g_{i_k}(x_i, y_{i_k}) = x_i + y_{i_k}$. For this choice of the functions $p_j = q_{i_l} = M_{1i_k} = M_{2i_k} = 1$ for $i = 1, 2, 3, k = 1, 2, 3$. The equilibrium point of this system is (1.5853, 1.3682, 0.8123, 0.6777, 0.8157, 0.7924). Substituting the parameters in the above Theorem 2.2.1, we get $A = 0.43, B = 0.79, C = 1.9038, D = 0.1482, E = 0.4232, F = 0.6256$ and $r = Min\{\frac{A}{C}, \frac{B}{D}\} = 0.229, s = \frac{B}{E} = 1.8667, p = \frac{B}{F} = 1.2628$ $0 < \tau^* < r, 0 < \vartheta^* < s, 0 < \zeta^* < p$. The following (Figure 2) is the simulation when the delays lie within and outside the region derived in Theorem 2.5



(a) (b)

Figure 2

Remark 2.8. This example satisfies the conditions of both Theorem 2.1 and Theorem 2.3. As it satisfies the conditions of Theorem 2.3, in spite of the presence of delays the system will be globally asymptotically stable. We can also observe that when the delays are outside the range of our results in Theorem 2.5, there is a disturbance in the solutions to reach the equilibria when compared to that of when they are within the range. Thus, our system (2) is in a controllable state under specific conditions. Hence, it is suitable for our proposition.

We further see that model (2) under consideration has constant exogenous inputs, but in some situations, the inputs may vary according to time. So, to suit such situations, we consider the model to have varying input in the next section.

3 Model with Time Varying Inputs-Behavior of the Solutions

The external inputs, like sensory inputs, information from the outside world, etc., may not always be constant. It may vary according to time. In order to study the impact of such time-varying inputs on the output of emotions, we modify the model (2) by changing the exogenous inputs from constant functions to a function of time t, i.e., we take the exogenous inputs I_i and J_{i_k} as a function of t then the system will take the form

$$x_i^{'} = -a_i x_i + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j)) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i, y_{i_k}(t-\vartheta_{i_k})) + I_i(t)$$
$$y_{i_k}^{'} = -c_{i_k} y_{i_k} + \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}(t-\zeta_{i_l})) + J_{i_k}(t), \tag{20}$$

where $i = 1, 2, 3....n$, $k = 1, 2, 3.....r_i$ and $1 \le r_i \le n$.
Under the conditions (3) on response functions and assuming the inputs $I_i(t)$ and $J_{i_k}(t)$ to be bounded and continuous on $[0, \infty)$, we can say that (19) poses unique solutions in there maximal intervals of existence [14, 20].
When we take $\tau_i = 0$ in (19) it will be deduce to the model which was studied in [14]. And if we take $\tau_i = 0, \vartheta_{i_k} = 0$ and $\zeta_{i_k} = 0$ for $i = 1, 2, ...n$ and $k = 1, 2, 3.....r_i$ in (19) it will deduce to the modified model that was mentioned as the open problem (V) in [19].

As (19) is a non-autonomous system, it may not possess equilibrium solutions. So we study the behaviour of solutions of the system based on the asymptotic nearness and boundedness of solutions. Asymptotic nearness or closeness shows that the solutions have similar or predictable behaviour with respect to one another. In other words, if one solution is controllable, so are the remaining. On the other hand, if one of the solutions is wild, the system as a whole may be regarded as wild. First, we start with the asymptotic nearness of solutions.

Theorem 3.1. *For any pair of solutions* (x_i, y_{i_k}) *and* $(\overline{x}_i, \overline{y}_{i_k})$ *of (19), we have* $\lim_{t\to\infty} |(x_i, y_{i_k}) - (\overline{x}_i, \overline{y}_{i_k})| = 0$ *provided the response functions satisfy (3) and the parameters satisfy* $\overline{A} = min\{A, B\} > 0$*, where*

$$A = min\Big\{a_i - \sum_{j=1}^{n} |b_{ji}| p_i - \sum_{k=1}^{r_i} \Big(|c_{ii_k}| M_{2i_k}\Big)\Big\}$$
$$B = min\Big\{c_{i_k} - \sum_{l=1}^{r_i} |d_{i_l}| q_{i_l} - |c_{ii_k}| M_{1i_k}\Big\}, \; for \; i = 1, 2, ..., n \; and \; k = 1, 2, ..., r_i. \tag{21}$$

Proof. By considering the functional

$$V(t) = \sum_{i=1}^{n} \Bigg[|x_i - \overline{x}_i| + \sum_{j=1}^{n} |b_{ij}| p_j \int_{t-\tau_j}^{t} |x_j(z) - \overline{x}_j(z)| dz + \sum_{k=1}^{r_i} |c_{ii_k}| M_{1i_k} \int_{t-\vartheta_{i_k}}^{t} |y_{i_k}(z) - \overline{y}_{i_k}(z)| dz$$
$$+ \sum_{k=1}^{r_i} \Big[|y_{i_k} - \overline{y}_{i_k}| + \sum_{l=1}^{r_i} |d_{i_l}| q_{i_l} \int_{t-\zeta_{i_l}}^{t} |y_{i_l}(z) - \overline{y}_{i_l}(z)| dz\Big]\Bigg].$$

and proceeding as in Theorem 2.1 of [14] we can prove the result. □

Now we will obtain conditions for the solutions of the system to be bounded under the conditions of Theorem 4.1, so that all the solutions stay near a bounded solution and hence, we may predict the behaviour of the system.

Theorem 3.2. *Assume that the parameters satisfy the condition (19) and let the response functions, besides (3), satisfy* $f_i(0) = 0$, $g_{i_k}(0,0) = 0$, *and* $h_{i_l}(0) = 0$ *for* $i = 1,2,3....,n$, $k = l = 1,2,3....,r_i$ *where* $1 \le r_i$. *Further if the inputs satisfy* $\int_0^\infty \sum_{i=1}^n |I_i(s) + \sum_{k=1}^{r_i} J_{i_k}(s)|ds < \infty$, *then all the solutions of (19) are bounded.*

Proof. We employ the functional

$$V(t) = \sum_{i=1}^n \Bigg[|x_i(t)| + \sum_{j=1}^n |b_{ij}| \int_{t-\tau_j}^t |f_j(x_j(z))|dz + \sum_{k=1}^{r_i} |c_{ii_k}| \int_{t-\vartheta_{i_k}}^t |g_{i_k}(y_{i_k}(z))|dz + \sum_{k=1}^{r_i} \Big[|y_{i_k}(t)| + \sum_{l=1}^{r_i} |d_{i_l}| \int_{t-\zeta_{i_l}}^t h_{i_l}|y_{i_l}(z)|dz\Big]\Bigg]$$

Doing the upper Dini derivative of V along the solutions of (19) and rearranging the terms after using conditions(3), we get

$$\begin{aligned} D^+V(t) &\le -\sum_{i=1}^n \Bigg[\Big[a_i - \sum_{j=1}^n |b_{ji}|p_i - \sum_{k=1}^{r_i}\Big(|c_{ii_k}|M_{2i_k}\Big)\Big]|x_i| \\ &\quad + \sum_{k=1}^{r_i}\Big[c_{i_k} - \sum_{l=1}^{r_i} |d_{i_l}|q_{i_l} - |c_{ii_k}|M_{1i_k}\Big]|y_{i_l}|\Bigg] \\ &\le -\overline{A} \sum_{i=1}^n \Big[|x_i(t)| + \sum_{k=1}^{r_i} |y_{i_k}(t)|\Big] + \sum_{i=1}^n \Big[|I_i(t) + \sum_{k=1}^{r_i} J_{i_k}(t)|\Big]. \end{aligned}$$

Integrating on both sides from 0 to t, we get

$$V(t) + \overline{A} \int_0^t \sum_{i=1}^n \Big[|x_i(s)| + \sum_{k=1}^{r_i} |y_{i_k}(s)|\Big] \le V(0) + \int_0^t \sum_{i=1}^n |I_i(s) + \sum_{k=1}^{r_i} J_{i_k}(s)|ds.$$

From our assumptions on the inputs and parameters, it is easy to see that $V(t)$, x_i's and y_{i_k}'s are bounded (See [20], for argument). Thus, the solutions of (19) are bounded. □

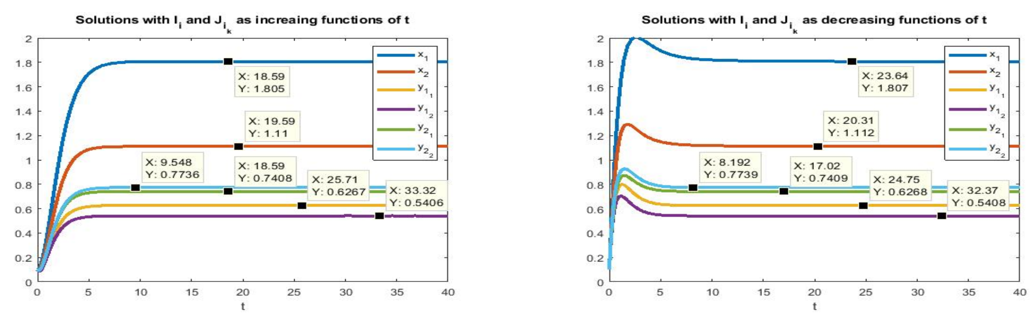
Remark 3.3. From Theorem 3.1 and Theorem 3.2, it is clear that the solutions of the system (19) are near to each other and bounded, so the system is well behaved and under control (i.e, as the input varies the response to emotion will not differ drastically). The effectiveness of the above result can be shown through the following example

Example 3.4.

$$
\begin{aligned}
x_1^{'} &= -1.65x_1 + 0.21f_1(x_1(t-\tau_1)) + 0.29f_2(x_2(t-\tau_2)) + 0.25g_{1_1}(x_1, y_{1_1}(t-\vartheta_{1_1})) \\
&\quad + 0.4g_{1_2}(x_1, y_{1_2}(t-\vartheta_{1_2})) + I_1(t) \\
x_2^{'} &= -2.38x_2 + 0.5f_1(x_1(t-\tau_1)) + 0.2f_2(x_2(t-\tau_2)) + 0.3g_{2_1}(x_2, y_{2_1}(t-\vartheta_{2_1})) \\
&\quad + 0.24g_{2_2}(x_2, y_{2_2}(t-\vartheta_{2_2})) + I_2(t) \\
y_{1_1}^{'} &= -2y_{1_1} + 0.19h_{1_1}(y_{1_1}(t-\zeta_{1_1})) + 0.3h_{1_2}(y_{1_2}(t-\zeta_{1_2})) + J_{1_1}(t) \\
y_{1_2}^{'} &= -2.4y_{1_2} + 0.27h_{1_1}(y_{1_1}(t-\zeta_{1_1})) + 0.3h_{1_2}(y_{1_2}(t-\zeta_{1_2})) + J_{1_2}(t) \\
y_{2_1}^{'} &= -2.8y_{2_1} + 1.5h_{2_1}(y_{2_1}(t-\zeta_{2_1}))) + 0.2h_{2_2}(y_{2_2}(t-\zeta_{2_2})) + J_{2_1}(t) \\
y_{2_2}^{'} &= -1.8y_{2_2} + 0.5h_{2_1}(y_{2_1}(t-\zeta_{2_1})) + 0.12h_{2_2}(y_{2_2}(t-\zeta_{2_2})) + J_{2_2}(t) \qquad (22)
\end{aligned}
$$

Choose the response function as $f_i(x_i) = tanh(x_i), h_{i_l}(y_{i_l}) = tanh(y_{i_l})$ and $g_{i_k}(x_i, y_{i_k}) = x_i + y_{i_k}$. For this choice of the functions, we have $p_j = q_{i_l} = M_{1i_k} = M_{2i_k} = 1$, & we choose the delays as $\tau_i = \vartheta_{i_k} = \zeta_{i_k} = 1$ for $i = 1, 2,\ k = 1, 2$.

Clearly all the parametric conditions of Theorem 3.1 and Theorem 3.2 are satisfied and we choose exogenous varying inputs $I_i(t)$ and J_{i_k} in such a way that they are bounded, then the simulations of the system (22) with different choices of input functions is as follows Figure 3 (a) shows the simulations when we take $I_i(t) = I_i - \dfrac{1}{1+t^2}$ and $J_{i_k}(x_i) = J_{i_k} - e^{-t}$ (increasing functions), where $I_i = 1$ and $J_{i_k} = 1$. And Figure 3 (b) shows the simulations when we take $I_i(t) = I_i + \dfrac{1}{1+t^2}$ and $J_{i_k}(x_i) = J_{i_k} + e^{-t}$ (decreasing functions), where $I_i = 1$ and $J_{i_k} = 1$



(a) (b)

Figure 3

Remark 3.5. We can observe that both the solutions of Figure 3 (a) and Figure 3 (b) are converging. But we have noticed an interesting point here, that there is a slight variation

in Figure 3 (b), i.e., when we are taking the inputs as decreasing function there is a disturbance in the solutions before it converge when compared to that of taking an increasing function.
If we take the increasing function as positive inputs and the decreasing function as negative inputs, then we can say from our observation that the emotional stability of a person may not deviate if he is getting positive inputs. But if he is getting negative inputs, then he needs to struggle to maintain it.

4 Discussion

In this article, the CSNN model is used to understand how the concepts stored in the memory contribute to the outcome of emotions. The dynamical system of the model with constant inputs and time-varying inputs has been considered. For a system with constant input, conditions for the existence and uniqueness of equilibria, delay-independent and delay-dependent conditions for global asymptotic stability have been derived. For a system with time-varying input, conditions for asymptotic nearness and boundedness have been derived. So, both in the case of constant input and time varying input, our systems are well behaved and under control, hence they can be used for proposition. Here, a point is noted that when the input function is an increasing function, the stability will not be disturbed compared to when the input is a decreasing function. Numerical examples are illustrated to support our results.

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