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Caputo Derivative Formulas of Hurwitz-Lerch Zeta Function and Applications

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Abstract: In this paper, we find the fractional derivative formulas of Hurwitz-Lerch Zeta function. Further, we compute the solution of fractional differential equations involving Hurwitz-Lerch Zeta function.

Keywords: Hurwitz-Lerch Zeta function, hypergeometric function and Fractional derivatives.

1. Introduction

Fractional Calculus serves as an excellent tool for studying fractional order integrals and derivatives. There are a lot of disciplines in science and engineering that benefit from fractional calculus. In a variety of fields, fractional differential equations and their applications have played a significant role. These include applied science, physics, biology, chemistry and engineering science. As a system of differential, kinetic equations provide a description of the rate at which changes in the chemical composition of a star occur. Fractional differential equations have been widely and successfully used to describe and solve many problems in physics and astrophysics over the past several decades. In mathematics and mathematical physics, the special functions are useful for the solution of fractional integral and differential equation problems. In order to incorporate fractional derivatives into differential equations, fractional differential equations were developed. When f(w) is a function of order alpha. Its fractional derivatives are represented mathematically as $D^{\alpha}f(w)$, where α is a non integer.

Many fractional derivatives exist, including Riemann-Lionville, Caputo and Grunwald-Letnikov derivatives. They each have their own advantage and uses.

The fractional derivative of f(w) in the Caputo sense is defined as:

$$D^{\alpha}f(w) = I^{l-\alpha}D^{l}f(w) \tag{1}$$

$$D^{\alpha}f(\omega) = \frac{1}{\Gamma(l-\alpha)}\int_0^{w}(\omega-u)^{l-\alpha-1}f^{l}\,(u)du.\,. \tag{2}$$

For $l-1 < \alpha \leq l, l \in N, \ \omega > 0$.

The Caputo derivative, we have $D^{\alpha}C = 0$, C is constant.

$$\mathrm{D}^{\alpha}\mathrm{r}^{\mathrm{r}}=\begin{cases}0 \ , \ \mathrm{r}\leq\alpha-1\\ \frac{\Gamma(\mathrm{r}+1)}{\Gamma(\mathrm{r}-\alpha+1)}\ \mathrm{t}^{\mathrm{r}-\alpha} \ ,\mathrm{r}>(\alpha-1)\end{cases} . \tag{3}$$

Mathematical special functions or SFs date back to the nineteenth century, when they were developed as a unified and complete theory. Researchers and engineers working with differential equations are well aware of the value of SFs as a tool for mathematical analysis. Several of these named functions are formulated as mathematical models by solving differential equations and systems of integer order. As a result of the growing by interest and widespread application of differential equations and fractional order system, many physical, engineering, automation, biological, chemical, earth science, economic phenomena have been better represented in the function, Euler beta function, and many more function have all recently seen extensions created by numerous writers. We are familiar with the Hurwitz-Lerch zeta function $\varphi(\mathrm{w},\mathrm{k},\mathrm{r})$ defined as:

$$\varphi(w,k,t)=\sum_{m=0}^{\infty}\frac{w^{m}}{(m+r)^{k}}, \tag{4}$$

$$(r\in Z^{+}\colon k\in C \ when \ |w|<1\colon R(k)>1 \ when \ |w|=1).$$

Various generalizations of the Hurwitz-Lerch zeta functions have been given by the researchers. For example, Goyal et.al. has introduced an extension of Hurwitz-Lerch zeta function defined as:

$$\varphi_{\theta_1}^{*}(w,k,t)=\sum_{m=0}^{\infty}\frac{(\theta_1)_m}{m!}\frac{w^m}{(m+r)^k}, \tag{5}$$

$$(\theta_1\in C; r\in Z^{+}; k\in C \ \ if \ |w|<1 \, ; R(k-\delta)>1 \ if \ \ |w|=1).$$

Lin et. al. [13] also defined the Hurwitz-Lerch zeta function as:

$$\varphi_{\theta_{1,}\theta_{2,}}^{\beta,\delta}(w,k,p)=\sum_{m=0}^{\infty}\frac{(\theta_1)_{\beta m}}{(\theta_2)_{\delta m}}\frac{w^m}{(m+p)^k}, \tag{6}$$

$$(\theta_1\in C; p,\theta_2\in Z^{+}; \beta,w\in R^{\,+};\ \beta<w \ ifs,w\in C; \beta=\delta, s\,\epsilon\, C \ if \ \ |w|<1 \, ; R(s-\theta_1+\theta_2) > 1 \, , |w|=1)$$

Garg et. al. [3] also introduced Hurwitz-Lerch zeta function as:

$$\varphi_{\theta_{1,}\theta_{2,}\theta_3}^{\beta,\gamma,\delta}(w,s,p)=\sum_{m=0}^{\infty}\frac{(\theta_1)_{r\beta}}{(\theta_2)_{r\delta}}\frac{(\theta_2)_{r\gamma}(w^{\alpha})^{r}}{(r+p)^{s}}, \tag{7}$$

$$(\theta_1,\theta_2\in C;\theta_3,p\in Z^{+}; s\in C;\ if \ \ |w|<1 \, ; R(s+\theta_3-\theta_1-\theta_2)>1 \, , |w|=1)$$

In order to address the short comings associated with fractional derivatives in mathematics and physics, the author developed formulas for the Caputo derivative of the Hurwitz-Lerch Zeta function. As a result of these derivatives formulas, we are able to compute solutions to fractional differential equations.

2. Analysis of Method

In mathematics and mathematics physics, the special functions are useful for the solution of fractional integral and differential equation problems. In order to incorporate fractional derivatives into differential equations, fractional differential equations were developed [2, 12, 15]. The Hurwitz-Lerch zeta function defined by power series (7) has efficiency as solution of fractional order differential and integral equations and thus have important role of the fractional calculus theory and applications. In this section, we consider few examples that demonstrate the performance and efficiency of Hurwitz-Lerch zeta function for solving linear fractional differential equations with fractional derivatives. The Hurwitz-Lerch zeta function suggests that the linear term $y(x)$ is decomposed by an power series:

$$f(w)=\varphi^{\beta,\gamma,\delta}_{\theta_1,\theta_2,\theta_3}(w,s,p)=\sum_{r=0}^{\infty}\frac{(\theta_1)_{r\beta}}{(\theta_3)_{r\delta}}\frac{(\theta_2)_{r\gamma}}{(r+p)^s}(Aw^{\alpha})^r \tag{8}$$

$$f(w)=\frac{1}{p^s}+\frac{(\theta_1)_{\beta}}{(\theta_3)_{\delta}}\frac{(\theta_2)_{\gamma}}{(1+p)^s}(Aw^{\alpha})^1+\ldots. \tag{9}$$

Theorem 1. The following derivative formula holds:

$$D^{\alpha}f(w)=\sum_{r=1}^{\infty}\frac{(\theta_1)_{r\beta}}{(\theta_3)_{r\delta}}\frac{(\theta_2)_{r\gamma}}{(r+p)^s}\frac{A^r\Gamma(r\alpha+1)}{\Gamma(\alpha(r-1)+1)}w^{\alpha(r-1)} \tag{10}$$

Proof. From (1) and (4), we have

$$D^{\alpha}f(w)=\frac{1}{\Gamma(l-\alpha)}\int_0^w(w-u)^{l-\alpha-1}\sum_{r=1}^{\infty}\frac{(\theta_1)_{r\beta}}{(\theta_3)_{r\delta}}\frac{(\theta_2)_{r\gamma}}{(r+p)^s}A^{\alpha}D^t u^{r\alpha}d$$

$$=\frac{1}{\Gamma(l-\alpha)}\int_0^w(w-u)^{l-\alpha-1}\sum_{r=1}^{\infty}\frac{(\theta_1)_{r\beta}(\theta_2)_{r\gamma}}{(\theta_3)_{r\delta}}\frac{A^{\alpha}}{(r+p)^s}\frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha-l+1)}u^{r\alpha-l}du$$

$$=\frac{1}{\Gamma(l-\alpha)}\sum_{r=1}^{\infty}\frac{(\theta_1)_{r\beta}(\theta_2)_{r\gamma}}{(\theta_3)_{r\delta}}\frac{A^{\alpha}}{(r+p)^s}\frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha-l+1)}\int_0^w(w-u)^{l-\alpha-1}u^{r\alpha-l}du$$

$$=\frac{1}{\Gamma(l-\alpha)}\sum_{r=1}^{\infty}\frac{(\theta_1)_{r\beta}(\theta_2)_{r\gamma}}{(\theta_3)_{r\delta}}\frac{A^{\alpha}}{(r+p)^s}\frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha-l+1)}\int_0^w w^{l-\alpha-1}(1-\frac{u}{w})^{l-\alpha-1}u^{r\alpha-l}du$$

Now let $\frac{u}{w}=v$

$du=wdv.$

Then

$$=\frac{1}{\Gamma(l-\alpha)}\sum_{r=1}^{\infty}\frac{(\theta_1)_{r\beta}(\theta_2)_{r\gamma}}{(\theta_3)_{r\delta}}\frac{A^{\alpha}}{(r+p)^s}\frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha-l+1)}w^{\alpha(r-1)}\frac{\Gamma(r\alpha-l+1)\Gamma(l-\alpha)}{\Gamma(\alpha(r-1)+1)}$$

On solving we get the desired result (10).

Theorem 2. The following derivative formula holds:

$$D^{2\alpha}f(w)=\sum_{r=2}^{\infty}\frac{(\theta_1)_{r\beta}}{(\theta_3)_{r\delta}}\frac{(\theta_2)_{r\gamma}}{(r+p)^s}\frac{A^r\Gamma(r\alpha+1)}{\Gamma(\alpha(r-2)+1)}w^{\alpha(r-2)}\ . \qquad (11)$$

Similarly, we can proof the above result as a proof of Theorem 1.

3. Numerical applications

In this section, we consider few examples that demonstrate the extended Hurwitz-Lerch Zeta function for solving linear differential equation with fractional derivative.

Example 1. The solution of following fractional differential equation

$$D^{\alpha}f(w)-Cf(w)=0.$$

By equation (4) and theorem (1)

$$\sum_{r=1}^{\infty}\frac{(\theta_1)_{r\beta}}{(\theta_3)_{r\delta}}\frac{(\theta_2)_{r\gamma}}{(r+p)^s}\frac{A^r\Gamma(r\alpha+1)}{\Gamma(\alpha(r-1)+1)}w^{\alpha(r-1)}-C\sum_{r=0}^{\infty}\frac{(\theta_1)_{r\beta}}{(\theta_3)_{r\delta}}\frac{(\theta_2)_{r\gamma}}{(r+p)^s}(Aw^{\alpha})^r=0$$

Replace r by r+1 in first summation

$$\sum_{r=0}^{\infty}\frac{(\theta_1)_{(r+1)\beta}}{(\theta_3)_{(r+1)\delta}}\frac{(\theta_2)_{(r+1)\gamma}}{(r+1+p)^s}\frac{A^{r+1}\Gamma((r+1)\alpha+1)}{\Gamma(\alpha r+1)}w^{\alpha r}-C\sum_{r=0}^{\infty}\frac{(\theta_1)_{r\beta}}{(\theta_3)_{r\delta}}\frac{(\theta_2)_{r\gamma}}{(r+p)^s}(Aw^{\alpha})^r=0$$

$$\sum_{r=0}^{\infty}[\frac{(\theta_1)_{(r+1)\beta}}{(\theta_3)_{(r+1)\delta}}\frac{(\theta_2)_{(r+1)\gamma}}{(r+1+p)^s}\frac{A^{1}\Gamma((r+1)\alpha+1)}{\Gamma(\alpha r+1)}w^{\alpha r}-C\sum_{r=0}^{\infty}\frac{(\theta_1)_{r\beta}}{(\theta_3)_{r\delta}}\frac{(\theta_2)_{r\gamma}}{(r+p)^s}]A^r w^{r\alpha}=0$$

Now equating to zero the coefficient of $w^{r\alpha}$, we get

$$\frac{(\theta_1)_{(r+1)\beta}(\theta_2)_{(r+1)\gamma}}{(\theta_3)_{(r+1)\delta}}\frac{A}{(r+1+p)^s}\frac{\Gamma((r+1)\alpha+1)}{\Gamma(\alpha r+1)}=C\frac{(\theta_1)_{r\beta}}{(\theta_3)_{r\delta}}\frac{(\theta_2)_{r\gamma}}{(p+r)^s}$$

$$at\ r=0,\qquad \frac{(\theta_1)_{\beta}(\theta_2)_{\gamma}}{(\theta_3)_{\delta}(p+1)^s}\frac{\Gamma(\alpha+1)}{\Gamma(1)}A=C\frac{1}{p^s}$$

$$\frac{(\theta_1)_{\beta}(\theta_2)_{\gamma}}{(\theta_3)_{\delta}(p+1)^s}A=C\frac{1}{p^s\Gamma(\alpha+1)}$$

$$at\ r=1,\qquad \frac{(\theta_1)_{2\beta}(\theta_2)_{2\gamma}}{(\theta_3)_{2\delta}(p+2)^s}\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}A=C\frac{(\theta_1)_{\beta}(\theta_2)_{\gamma}}{(\theta_3)_{\delta}(p+1)^s}$$

$$\frac{(\theta_1)_{2\beta}(\theta_2)_{2\gamma}}{(\theta_3)_{2\delta}(p+2)^s}\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}AA=C\frac{(\theta_1)_{\beta}(\theta_2)_{\gamma}}{(\theta_3)_{\delta}(p+1)^s}A$$

$$\frac{(\theta_1)_{2\beta}(\theta_2)_{2\gamma}}{(\theta_3)_{2\delta}(p+2)^s}\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}AA=CC\frac{1}{p^s\Gamma(\alpha+1)}=C^2\frac{1}{p^s\Gamma(\alpha+1)}$$

$$thus\qquad \frac{(\theta_1)_{2\beta}(\theta_2)_{2\gamma}}{(\theta_3)_{2\delta}(p+2)^s}A^2=C^2\frac{1}{p^s\Gamma(2\alpha+1)}$$

$$at\ r=2,\qquad \frac{(\theta_1)_{3\beta}(\theta_2)_{3\gamma}}{(\theta_3)_{3\delta}(p+3)^s}\frac{\Gamma(3\alpha+1)}{2\Gamma(\alpha+1)}A=C\frac{(\theta_1)_{2\beta}(\theta_2)_{2\gamma}}{(\theta_3)_{2\delta}(p+2)^s}$$

$$on\ solving \qquad \frac{(\theta_1)_{3\beta}(\theta_2)_{3\gamma}(\theta_3)_{3\delta}(p+3)^s}{A} = C^3\frac{1}{p^s\Gamma(3\alpha+1)}$$

Substituting these values in equation (5), we get

$$f(w) = \frac{1}{p^s} + C\frac{1}{p^s\Gamma(\alpha+1)}w^{\alpha} + C^2\frac{1}{p^s\Gamma(2\alpha+1)}w^{2\alpha} + \cdots \tag{10}$$

Example 2. Again we take a fractional differential equation

$$D^{2\alpha}f(w) - Bf(w) = 0 \tag{11}$$

From eq. (4) and theorem (2), we get

$$\sum_{r=2}^{\infty}\frac{(\theta_1)_{r\beta}}{(\theta_3)_{r\delta}}\frac{(\theta_2)_{r\gamma}}{(r+p)^s}\frac{A^r\Gamma(r\alpha+1)}{\Gamma(\alpha(r-2)+1)}w^{\alpha(r-2)} - B\sum_{r=0}^{\infty}\frac{(\theta_1)_{r\beta}}{(\theta_3)_{r\delta}}\frac{(\theta_2)_{r\gamma}}{(r+p)^s}A^r(w^{\alpha})^r = 0$$

Replace r by r+2 in the above equation (only in first summation) then

$$\sum_{r=0}^{\infty}\frac{(\theta_1)_{(r+2)\beta}}{(\theta_3)_{(r+2)\delta}}\frac{(\theta_2)_{(r+2)\gamma}}{(r+2+p)^s}\frac{A^{r+2}\Gamma((r+2)\alpha+1)}{\Gamma(\alpha r+1)}w^{\alpha r} - B\sum_{r=0}^{\infty}\frac{(\theta_1)_{r\beta}}{(\theta_3)_{r\delta}}\frac{(\theta_2)_{r\gamma}}{(r+p)^s}A^r(w^{\alpha})^r = 0$$

$$\sum_{r=0}^{\infty}[\frac{(\theta_1)_{(r+2)\beta}}{(\theta_3)_{(r+2)\delta}}\frac{(\theta_2)_{(r+2)\gamma}}{(r+2+p)^s}\frac{A^2\Gamma((r+2)\alpha+1)}{\Gamma(\alpha r+1)}A^2 - B\frac{(\theta_1)_{r\beta}}{(\theta_3)_{r\delta}}\frac{(\theta_2)_{r\gamma}}{(r+p)^s}]A^r(w^{\alpha})^r = 0$$

Equating to zero the coefficient of $w^{r\alpha}$ in the above equation

$$\frac{(\theta_1)_{(r+2)\beta}}{(\theta_3)_{(r+2)\delta}}\frac{(\theta_2)_{(r+2)\gamma}}{(r+2+p)^s}\frac{\Gamma((r+2)\alpha+1)}{\Gamma(\alpha r+1)}A^2 - B\frac{(\theta_1)_{r\beta}}{(\theta_3)_{r\delta}}\frac{(\theta_2)_{r\gamma}}{(r+p)^s} = 0$$

$$\frac{(\theta_1)_{(r+2)\beta}}{(\theta_3)_{(r+2)\delta}}\frac{(\theta_2)_{(r+2)\gamma}}{(r+2+p)^s}\frac{\Gamma((r+2)\alpha+1)}{\Gamma(\alpha r+1)}A^2 = B\frac{(\theta_1)_{r\beta}}{(\theta_3)_{r\delta}}\frac{(\theta_2)_{r\gamma}}{(r+p)^s}$$

$$at\ r=0, \qquad \frac{(\theta_1)_{2\beta}}{(\theta_3)_{2\delta}}\frac{(\theta_2)_{2\gamma}}{(2+p)^s}\frac{\Gamma(2\alpha+1)}{\Gamma(1)}A^2 = \frac{B}{p^s}$$

$$\frac{(\theta_1)_{2\beta}}{(\theta_3)_{2\delta}}\frac{(\theta_2)_{2\gamma}}{(2+p)^s}A^2 = \frac{B}{p^s\Gamma(2\alpha+1)}$$

$$at\ r=1, \qquad \frac{(\theta_1)_{3\beta}}{(\theta_3)_{3\delta}}\frac{(\theta_2)_{3\gamma}}{(3+p)^s}A^3 = AB\frac{(\theta_1)_{\beta}(\theta_2)_{\gamma}}{(\theta_3)_{\delta}(p+1)^s}$$

$$at\ r=2, \qquad \frac{(\theta_1)_{4\beta}}{(\theta_3)_{4\delta}}\frac{(\theta_2)_{4\gamma}}{(4+p)^s}\frac{\Gamma(4\alpha+1)}{\Gamma(2\alpha)}A^2 = B\frac{(\theta_1)_{2\beta}(\theta_2)_{2\gamma}}{(\theta_3)_{2\delta}(p+2)^s}$$

Multiplying above equation by A^2

$$\frac{(\theta_1)_{4\beta}}{(\theta_3)_{4\delta}}\frac{(\theta_2)_{4\gamma}}{(4+p)^s}\frac{\Gamma(4\alpha+1)}{\Gamma(2\alpha)}A^4 = B\frac{(\theta_1)_{2\beta}(\theta_2)_{2\gamma}}{(\theta_3)_{2\delta}(p+2)^s}A^2$$

$$\frac{(\theta_1)_{4\beta}}{(\theta_3)_{4\delta}}\frac{(\theta_2)_{4\gamma}}{(4+p)^s}\frac{\Gamma(4\alpha+1)}{\Gamma(2\alpha)}A^4=B^2\frac{1}{p^s\Gamma(4\alpha+1)}$$

Now put these values in equation (5) then,

$$f(w)=\frac{1}{p^s}+\frac{(\theta_1)_\beta(\theta_2)_\gamma}{(\theta_3)_\delta(p+1)^s}Aw^\alpha+B\frac{1}{p^s\Gamma(2\alpha+1)}w^{2\alpha}+B\frac{(\theta_1)_\beta(\theta_2)_\gamma}{(\theta_3)_\delta(p+1)^s}Aw^{3\alpha}+B^2\frac{1}{p^s\Gamma(4\alpha+1)}w^{4\alpha}+\cdots \quad (12)$$

Example 3. Consider fractional differential equation

$$\mathrm{D}^{2\alpha}\mathrm{f(w)}+\mathrm{D}^{\alpha}\mathrm{f(w)}-3\mathrm{f(w)}=0\,. \qquad (13)$$

Then by equation (4), theorem (1) and theorem (2),

$$\sum_{k=2}^{\infty}\frac{(\theta_1)_{k\beta}}{(\theta_3)_{k\delta}}\frac{(\theta_2)_{k\gamma}}{(k+p)^s}\frac{A^k\Gamma(k\alpha+1)}{\Gamma(\alpha(k-2)+1)}w^{\alpha(k-2)}+\sum_{k=1}^{\infty}\frac{(\theta_1)_{k\beta}}{(\theta_3)_{k\delta}}\frac{(\theta_2)_{k\gamma}}{(k+p)^s}\frac{A^k\Gamma(k\alpha+1)}{\Gamma(\alpha(k-1)+1)}w^{\alpha(k-1)}-3\sum_{k=0}^{\infty}\frac{(\theta_1)_{k\beta}}{(\theta_3)_{k\delta}}\frac{(\theta_2)_{k\gamma}}{(k+p)^s}A^k(w^\alpha)^k=0$$

Replacing k by k+2 in the first summation and k by k+1 in second summation respectively,

$$\sum_{k=0}^{\infty}\frac{(\theta_1)_{(k+2)\beta}}{(\theta_3)_{(k+2)\delta}}\frac{(\theta_2)_{(k+2)\gamma}}{((k+2)+p)^s}\frac{A^{k+2}\Gamma((k+2)\alpha+1)}{\Gamma(\alpha(k)+1)}w^{\alpha k}+\sum_{k=0}^{\infty}\frac{(\theta_1)_{(k+1)\beta}}{(\theta_3)_{(k+1)\delta}}\frac{(\theta_2)_{(k+1)\gamma}}{(k+1+p)^s}\frac{A^{k+1}\Gamma((k+1)\alpha+1)}{\Gamma(\alpha k+1)}w^{\alpha k}-3\sum_{k=0}^{\infty}\frac{(\theta_1)_{k\beta}}{(\theta_3)_{k\delta}}\frac{(\theta_2)_{k\gamma}}{(k+p)^s}A^k(w^\alpha)^k=0$$

$$\sum_{k=0}^{\infty}[\frac{(\theta_1)_{(k+2)\beta}}{(\theta_3)_{(k+2)\delta}}\frac{(\theta_2)_{(k+2)\gamma}}{((k+2)+p)^s}\frac{A^2\Gamma((k+2)\alpha+1)}{\Gamma(\alpha(k)+1)}+\frac{(\theta_1)_{(k+1)\beta}}{(\theta_3)_{(k+1)\delta}}\frac{(\theta_2)_{(k+1)\gamma}}{(k+1+p)^s}\frac{A^1\Gamma((k+1)\alpha+1)}{\Gamma(\alpha k+1)}-3\frac{(\theta_1)_{k\beta}}{(\theta_3)_{k\delta}}\frac{(\theta_2)_{k\gamma}}{(k+p)^s}]A^k(w^\alpha)^k=0$$

Now equating to zero the coefficient of $w^{k\alpha}$

$$\frac{(\theta_1)_{(k+2)\beta}}{(\theta_3)_{(k+2)\delta}}\frac{(\theta_2)_{(k+2)\gamma}}{((k+2)+p)^s}\frac{A^2\Gamma((k+2)\alpha+1)}{\Gamma(\alpha(k)+1)}+\frac{(\theta_1)_{(k+1)\beta}}{(\theta_3)_{(k+1)\delta}}\frac{(\theta_2)_{(k+1)\gamma}}{(k+1+p)^s}\frac{A^1\Gamma((k+1)\alpha+1)}{\Gamma(\alpha k+1)}-3\frac{(\theta_1)_{k\beta}}{(\theta_3)_{k\delta}}\frac{(\theta_2)_{k\gamma}}{(k+p)^s}=0$$

$$at\ k=0,\qquad \frac{(\theta_1)_{2\beta}}{(\theta_3)_{2\delta}}\frac{(\theta_2)_{2\gamma}}{(2+p)^s}\frac{A^2\Gamma(2\alpha+1)}{\Gamma(1)}+\frac{(\theta_1)_{\beta}}{(\theta_3)_{\delta}}\frac{(\theta_2)_{\gamma}}{(1+p)^s}\frac{A^1\Gamma(\alpha+1)}{\Gamma(1)}-3\frac{1}{p^s}=0$$

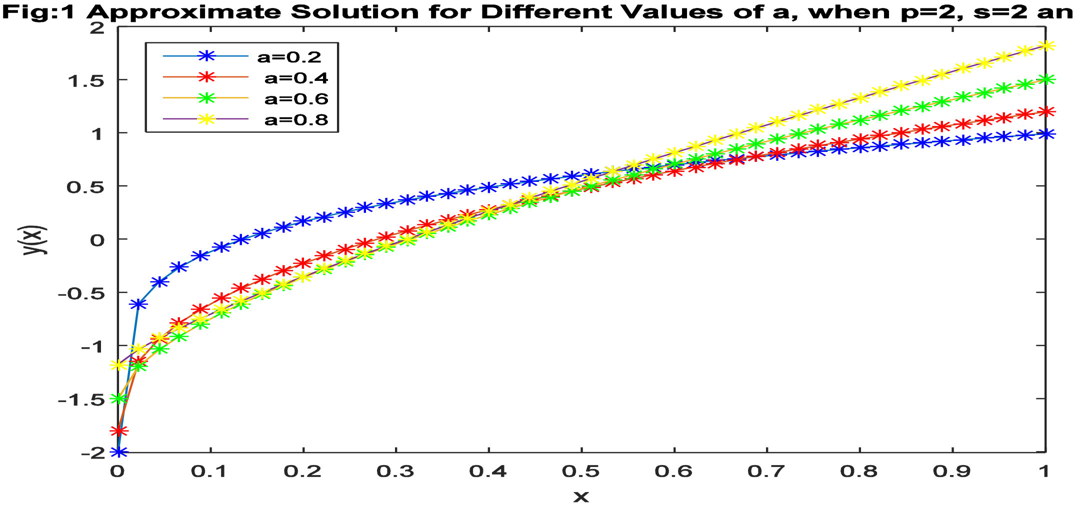
$$at\ k=1,\quad \frac{(\theta_1)_{3\beta}}{(\theta_3)_{3\delta}}\frac{(\theta_2)_{3\gamma}}{(3+p)^s}\frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)}A^3+\frac{(\theta_1)_{2\beta}}{(\theta_3)_{2\delta}}\frac{(\theta_2)_{2\gamma}}{(2+p)^s}\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}A^2-3\frac{(\theta_1)_{\beta}}{(\theta_3)_{\delta}}\frac{(\theta_2)_{\gamma}}{(1+p)^s}A=0$$

$$at\ k=2,\quad \frac{(\theta_1)_{4\beta}}{(\theta_3)_{4\delta}}\frac{(\theta_2)_{4\gamma}}{(4+p)^s}\frac{\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)}A^2+\frac{(\theta_1)_{2\beta}}{(\theta_3)_{2\delta}}\frac{(\theta_2)_{2\gamma}}{(3+p)^s}\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}A^1-3\frac{(\theta_1)_{\beta}}{(\theta_3)_{\delta}}\frac{(\theta_2)_{\gamma}}{(2+p)^s}=0$$

And so on. By equation (5) , we get following solution

$$f(w)=\frac{1}{p^s}+Bw^{\alpha}+\frac{1}{\Gamma(2\alpha+1)}\left[\frac{3}{p^s}-B\Gamma(\alpha+1)\right]+\frac{1}{\Gamma(3\alpha+1)}\left[-\frac{3}{p^s}+4B\Gamma(\alpha+1)\right]+\frac{1}{\Gamma(4\alpha+1)}\left[\frac{12}{p^s}-5B\Gamma(\alpha+1)\right]+\cdots$$

Fig:1 Approximate Solution for Different Values of a, when p=2, s=2 and B=3



4. Conclusion

We compute Caputo derivative formula of the Extended Hurwitz-Lerch Zeta function. Moreover, we obtained the solution of fractional differential equation involving Hurwitz-Lerch Zeta function. On the basis of the above result. We should be able to solve fractional differential equations involving other special functions, such as Mittag-Leffler functions, hypergeometric polynomials, and Jacobi polynomials.

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