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Variational Analysis of a Dynamic Frictional Contact Problem with Adhesion and Long Memory in Viscoelastic Materials

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Abstract: We consider a mathematical model that describes a dynamic frictional contact between a foundation and a viscoelastic body with long memory. The contact is modelled with a normal compliance condition in that the penetration is limited and restricted to a unilateral constraint and associated to the nonlocal friction law with adhesion, where the coefficient of friction is an independent solution. The adhesion of the contact surfaces is considered and modelled with a surface variable. We derive a variational formulation written as the coupling between a variational inequality and a differential equation. The existence and uniqueness result of the weak solution under a smallness assumption on the coefficient of friction is established. The proof is based on arguments of nonlinear evolution equation with monotone operators, a classical existence, differential equations, and the Banach fixed point theorem.

Keywords: viscoelastic; normal compliance; dynamic process; adhesion; differential equations; friction; weak solution..

Mathematics subject classification 2010: 74M15, 74M10, 74F15, 49J40.

1. Introduction

Contact issues with deformable bodies are prevalent in industrial applications and daily life, significantly impacting structural and mechanical systems. The last fifty years, variational inequalities have become a formidable tool in the mathematical study of many non-linear problems in physics and mechanics and the analysis of mathematical models in contact mechanics has expanded quickly over the past few decades, the complexity of the boundary conditions and the diversity of the constituent equations leading to variational formulations of the inequation type. The state of the art in mathematics, mechanics and numerical analysis is contained in [18]. This reference finds numerical investigations and a thorough analysis of the adhesive contact problem. General models for unilateral and frictional contact problems with adhesion can be found in [5, 7, 9, 13, 19, 20]. In [21], a quasistatic viscoelastic unilateral and frictional contact problem with adhesion and long memory was studied. This paper aims to model and establish the variational analysis for a dynamic process of unilateral and frictional contact with adhesion and long memory. Remember that [2, 3, 4, 6, 12, 14, 15, 16] has studied models for dynamic or quasistatic processes of frictionless adhesive contact between a deformable body and foundation. Following [10,11], the bondind field is used as an extra variable β wich satisfies the restriction $0 \leq \beta \leq 1$, at a point of the contact surface of the boundary, when $\beta = 1$ all of the bonds are active and the adhesion is complete and for $\beta = 0$ the bonds are inactive, severed and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active.

We direct the reader's attention to the comprehensive bibgraphy on the topic in [1, 10, 17, 18, 19]. In this paper, we deal with the study of a dynamic problem, we derive a variational formulation of the problem which is set a system coupling a variational second order evolution inequality. We establish the existence and the uniqueness of a weak solution of the model. The idea is to reduce the second order evolution inequality of the system to first order evolution inequality. After this, we use classical results on first order evolution inequalities and differential equations and the fixed point arguments. The rest of the paper is structured as follows. In section 2 and 3 we present some notations and preliminaires, and the viscoelastic unilateral and frictional contact model with adhesion and long memory, and provide comments on the contact boundary conditions. In section 4, we list the assumptions on the data and derive the variational formulation. In section 5, we present our main results on existence and uniqueness which state the unique weak solvability.

2. Notations and preliminaries

Throughout this paper, $\mathbb{S}^d$ represents the space of second order symmetric tensors on $\mathbb{R}^d (d = 2,3)$ while $|.|$ represents the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$.

Thus, for every $u, v \in \mathbb{R}^d, u.v = u_i v_i$ and $|v| = (v,v)^{\frac{1}{2}}$, and for every $\sigma, \tau \in \mathbb{S}^d, \sigma.\tau = \sigma_{ij}\tau_{ij}$,

$|\tau| = (\tau.\tau)^{\frac{1}{2}}$.

The summing convention over represented indices is used here and below, where the indices i and j range from 1 to d. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary $\Gamma$and let ν denote the unit outer normal on Γ. For Lebesgue and Sololev spaces associated to Ω and $\Gamma$and introduce the spaces,

$$H = \mathbb{L}^2(\Omega)^d = \{u = (u_i)/u_i \in \mathbb{L}^2(\Omega)\},$$

$$\mathcal{H} = \{\sigma = (\sigma_{ij})\ /\sigma_{ij} = \sigma_{ji} \in \mathbb{L}^2(\Omega)\},$$

$$H_1 = \{u = (u_i)/\varepsilon(u) \in \mathcal{H}\},$$

$$\mathcal{H}_1 = \{\sigma \in \mathcal{H}/Div\sigma \in H\ \}.$$

Here The deformation ε and divergence Div are operators defined by

$$\varepsilon(u) = \left(\varepsilon_{ij}(u)\right), \quad \varepsilon_{ij}(u) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad Div\sigma = \left(\sigma_{ij,j}\right).$$

The spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canon- ical inner products given by

$$(u,v)_H = \int_\Omega u_i v_i \, dx \qquad \forall\, u,v \in H,$$

$$(\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\, dx \qquad \forall \sigma,\tau \in \mathcal{H},$$

$$(u,v)_{H1} = (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}\ \forall u,v \in H_1,$$

$$(\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (Div\sigma, Div\tau)_H \forall \sigma,\tau \in \mathcal{H}_1.$$

The associated norms on the spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are denoted by $|.|_H, |.|_{\mathcal{H}}, |.|_{H1}$ and $|.|_{\mathcal{H}_1}$, respectively. For every element $v \in H_1$ we also use the notation ν for the trace of ν on Γ and we denote by v_ν and v_τ the normal and the tangential components of v on Γ given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu. \qquad (2.1)$$

We also denote by v_ν and v_τ the normal and tangential traces of a function $\sigma \in \mathcal{H}_1$ and we recall that when σ is a regular function then

$$\sigma_\nu = (\sigma\nu).\nu, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu \nu. \qquad (2.2)$$

And the following Green's formula holds:

$$\big(\sigma, \varepsilon(v)\big)_{\mathcal{H}} + (Div\sigma, v)_H = \int_\Gamma \sigma\nu.v\, da \quad \forall v \in H_1, \qquad (2.3)$$

where the surface measure element is da. Let $T > 0$, for every real Hilbert space X we employ the usual notation for the spaces $\mathbb{L}^p(0,T;X)$, $1 \le p \le \infty$ and $W^{1,\infty}(0,T;X)$. Recall that the norm on the space $W^{1,\infty}(0,T;X)$ is given by

$$\|u\|_{W^{1,\infty}(0.T;X)} = \|u\|_{\mathbb{L}^\infty(0.T;X)} + \|\dot{u}\|_{\mathbb{L}^\infty(0.T,X)},$$

where $\dot{u}$ denote the first derivative of u with respect to time. Fanilly, the space of continuous functions from $[0,T]$ to X is denoted by $C([0,T];X)$ with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X.$$

Moreover, for a real number r, r_+ is used to represent its positive part, that is $r_+ = \max\{r,0\}$.

3. Problem statement

The physical setting is the following. A viscoelastic body with long memory occupies a bounded domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$ with a regular boundary Γ that is partitioned into three disjoint measurable parts Γ_1, Γ_2 and $\Gamma_3$such that *meas* $\Gamma_1 > 0$. The body is acted upon by a volume force of density $\varphi_1$on Ω and a surface traction of density $\varphi_2$on Γ_2 and it is in unilate contact with adhesion following the nonlocal friction law with a foundation, over the potential contact surfaceΓ_3. Therefore, the mechanical problem's classical formulation is written as follows.

Problem P_1.

Find a displacement field u: $\Omega \times [0,T] \to \mathbb{R}^d$, a stress field σ: $\Omega \times [0,T] \to \mathbb{S}^d$and a bonding field β: $\Gamma_3 \times [0,T] \to [0,1]$ such that for all $t \in [0,T]$:

$$\sigma(t) = \mathcal{A}\varepsilon\big(u(t)\big) + \mathcal{G}\varepsilon\big(\dot{u}(t)\big) + \int_0^t \mathcal{F}(t-s)\varepsilon\big(u(s)\big)ds \quad \text{in } \Omega, \qquad (3.1)$$

$$Div\sigma(t) + \varphi_1(t) = \rho\ddot{u} \quad \text{in } \Omega, \qquad (3.2)$$

$$u(t) = 0 \quad \text{on } \Gamma_1, \qquad (3.3)$$

$$\sigma(t)\nu = \varphi_2 \quad \text{on } \Gamma_2, \qquad (3.4)$$

$$\left.\begin{array}{c} u_\nu \leq g;\ \sigma_\nu(t) + p\big(u_\nu(t)\big) - c_\nu\beta^2(t)R_\nu\big(u_\nu(t)\big) \leq 0 \\ \Big(\sigma_\nu(t) + p\big(u_\nu(t)\big) - c_\nu\beta^2(t)R_\nu\big(u_\nu(t)\big)\Big)\big(u_\nu(t) - g\big) = 0 \end{array}\right\} \text{on } \Gamma_3, \qquad (3.5)$$

$$\left.\begin{array}{c} \big|\sigma_\tau(t) + c_\tau\beta^2(t)R_\tau\big(u_\tau(t)\big)\big| \leq \mu\big|R\sigma_\tau\big(u(t)\big)\big| \\ \big|\sigma_\tau(t) + c_\tau\beta^2(t)R_\tau\big(u_\tau(t)\big)\big| < \mu\big|R\sigma_\tau\big(u(t)\big)\big| \Rightarrow u_\tau(t) = 0 \\ \big|\sigma_\tau(t) + c_\tau\beta^2(t)R_\tau\big(u_\tau(t)\big)\big| = \mu\big|R\sigma_\tau\big(u(t)\big)\big| \Longrightarrow \\ \exists\lambda \geq 0 \text{ such that } u_\tau(t) = -\lambda\Big(\sigma_\tau(t) + c_\tau\beta^2(t)R_\tau\big(u_\tau(t)\big)\Big) \end{array}\right\} \text{on } \Gamma_3, \qquad (3.6)$$

$$\dot{\beta}(t) = -\left[\beta(t)\left(c_\nu\Big(R_\nu\big(u_\nu(t)\big)\Big)^2 + c_\tau|R_\nu u_\tau(t)|^2 - \varepsilon_a\right)\right]_+ \quad \text{on } \Gamma_3, \qquad (3.7)$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3, \qquad (3.8)$$

$$u(0) = u_0, \ \dot{u}(t) = u_1 \quad \text{in } \Omega. \qquad (3.9)$$

The viscoelastic constitutive law with long memory of the material is represented by the equation (3.1). Here $\mathcal{G}$ and $\mathcal{A}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively and $\int_0^t \mathcal{F}(t-s)\varepsilon\big(u(s)\big)ds$ is the memory term in which $\mathcal{F}$ denotes the tensor of relaxation, the stress $\sigma(t)$ at current instant t depends on the whole history of strains up to this moment of time. Equation (3.2) represents the equation of motion where ρ denotes the material mass density, while (3.3) and (3.4) are the displacement and traction boundary conditions, respectively, in which $\sigma\nu$ represents the Cauchy stress vector. The conditions (3.5) represents the unilateral contact with adhesion in which c_ν is a given a adhesion coefficient which may dependent on $x \in \Gamma_3$ and R_ν and R_τ are truncation operators defined by

$$R_\nu(s) = \begin{cases} L \ \ if\ s < -L \\ -s\ \ if\ -L \leq s \leq 0, \\ 0\ \ if\ s > 0 \end{cases}$$

$$R_\tau(s) = \begin{cases} v \quad if \quad |v| \leq L, \\ L\dfrac{v}{|v|} \quad if\ |v| > L. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which the latter has no additional traction (see [18]) and p is a normal compliance function which satisfies the assumption (4.14), g denotes the maximum value of the penetration which satisfies $g \geq 0$. When $u_\nu < 0$ i.e. when there is separation between the body and the foundation then the condition (3.5) combined with hypothese (3.23) and definition of R_ν shows that $\sigma_\nu = c_\nu\beta^2R_\nu(u_\nu)$ and does not exeed the value $L\,\|c_\nu\|_{L^\infty(\Gamma_3)}$. When $g > 0$, the body may interpenetrate into the fondation, but the penetration is limited that is $u_\nu \leq g$. In this case of penetration (*i.e.* $u_\nu \geq 0$), when $0 \leq u_\nu < g$ then $-\sigma_\nu = p(u_\nu)$ which means that the reaction of the foundation is uniquely determined by the normal displacement and $\sigma_\nu \leq 0$. Since p is an increasing function then the reaction is increasing with the penetration. When $u_\nu = g$ then $-\sigma_\nu \geq p\ (g)$ and σ_ν is not uniquely determined. When $g > 0$ and $p = 0$, condition (3.5) become the Signorini's contact conditions with a gap and adhesion

$$u_\nu \leq g, \quad \sigma_\nu - c_\nu\beta^2 R_\nu(u_\nu) \leq 0, \ (\sigma_\nu - c_\nu\beta^2 R_\nu(u_\nu))(u_\nu - g) = 0.$$

When $g = 0$, the conditions (3.5) combined with hypothese (3.23) lead to the Signorini contact conditions with adhesion, with zero gap, given by

$$u_\nu \leq 0, \quad \sigma_\nu - c_\nu\beta^2 R_\nu(u_\nu) \leq 0, (\sigma_\nu - c_\nu\beta^2 R_\nu(u_\nu))u_\nu = 0.$$

These contact conditions were used in [20]. It follows from (3.5) that there is no penetration between the body and the foundation, since $u_\nu \leq 0$ during the process. Also, note that when the bonding field vanishes, then the contact conditions (3.5) become the classical Signorini contact conditions with zero gap, that is,

$$u_\nu \leq 0, \quad \sigma_\nu \leq 0, \quad \sigma_\nu u_\nu = 0.$$

Condition (3.6) represent Couloub's law of dry friction with adhesion where μ denotes the coefficient of friction. Equation (3.7) represents the ordinary differentail equation which describes the evolution of the bonding field and it was already used in [20, 21] . Since $\beta \leq 0$ on $\Gamma_3 \times [0, T]$, once debonding occurs bonding cannot be reestablished, indeed, the adhesion process is irreversible. Also from [19] it must be pointed out clearly that condition (3.7) does not allow for complete debonding in finite time. In Equation (3.8) β_0 denotes the initial bonding. Finally, in equation (3.9) u_0 is the initial displacement and u_1 the initial velocity.

4. Variational formulation

For a weak formulation of problem P_1, let V be the closed subspace of H_1 defined by

$$V = \{\upsilon \in H_1 : \upsilon = 0 \text{ on } \Gamma_1\}.$$

And the convex subset of admissible displacement given by

$$K = \{\upsilon \in V : \upsilon_\nu \leq g \ \text{ a.e on } \Gamma_3\}.$$

Since *meas* $\Gamma_1 > 0$, the following Korn's inequality holds [8]

$$\|\varepsilon(\upsilon)\|_{\mathcal{H}} \geq c_\Omega \|\upsilon\|_{H_1} \quad \forall \upsilon \in V. \quad (4.1)$$

Where $c_\Omega > 0$ is a constant which depends only on Ω and Γ_1. We equip V with the inner product

$$(u, \upsilon)_V = \langle \varepsilon(u), \varepsilon(\upsilon)\rangle_{\mathcal{H}}.$$

And $\|.\|_V$ is the associated norm. It follows from Korn's inequality (4.1) that the norms $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent on V. Then $(V, \|.\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which only depends on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(\mathbb{L}^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \quad \forall v \in V. \tag{4.2}$$

The body forces and surface tractions have the regularity

$$\varphi_1 \in C([0,T];H), \qquad \varphi_2 \in C\left([0,T];\left(\mathbb{L}^2(\Gamma_3)\right)^d\right) \tag{4.3}$$

The function $f : [0.T] \to V$ defined by

$$(f(t),v)_V = \int_\Omega \varphi_1(t) v dx + \int_{\Gamma_2} \varphi_2(t) da \quad \forall v \in V, \quad t \in [0,T], \tag{4.4}$$

And that (4.3) and (4.4) imply

$f \in C([0.T]; V)$

In the study of the mechanical problem P_1 (3.1)-(3.9) let the following assumptions: The elasticity operator $\mathcal{A}$ satisfies

$$\begin{cases} (a)\ \mathcal{A}: \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d, \\ (b)\ there\ exists\ M > 0\ such\ that:\ |\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)| \leq M|\varepsilon_1 - \varepsilon_2| \\ \qquad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, a.e.\ x \in \Omega \\ (c)\ there\ exists\ m > 0\ such\ that: \\ \qquad \left(\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)\right).(\varepsilon_1 - \varepsilon_2) \geq m|\varepsilon_1 - \varepsilon_2|^2 \\ \qquad \forall\ \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, a.e.\ x \in \Omega, \\ (d)\ the\ mapping\ x \longrightarrow \mathcal{A}(x,\varepsilon)\ is\ lebesgue\ measurable\ in\ \Omega\ for\ all \\ \qquad \varepsilon \in \mathbb{S}^d \\ (e)\ the\ mapping\ x \longrightarrow \mathcal{A}(x,0) \in \mathcal{H}. \end{cases} \tag{4.5}$$

The viscosity operator $\mathcal{G}$ satisfies

$$\begin{cases} (a)\ \mathcal{G}: \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d, \\ (b)\ there\ exists\ L_\mathcal{G} > 0\ such\ that: |\mathcal{G}(x,\xi_1) - \mathcal{G}(x,\xi_2)| \leq L_\mathcal{G}|\xi_1 - \xi_2| \\ \qquad \forall \xi_1, \xi_2 \in \mathbb{S}^d, \\ (c)\ the\ mapping\ x \longrightarrow \mathcal{G}(x,\xi)\ is\ lebesgue\ measurable\ in\ \Omega\ for\ all \\ \qquad \xi \in \mathbb{S}^d, \\ (d)\ the\ mapping\ x \longrightarrow \mathcal{G}(x,0) \in \mathcal{H}. \end{cases} \tag{4.6}$$

The mass density satisfies

$$\rho \in \mathbb{L}^\infty(\Omega),\ \text{there exists}\ \rho^* > 0\ \text{such that}\ \rho(x) \geq \rho^*,\ \text{a.e.}\ x \in \Omega. \tag{4.7}$$

The space of the tensors of fourth order defined by

$\mathcal{H}_\infty = \{\varepsilon = (\varepsilon_{ijkl}): \varepsilon_{ijkl} = \varepsilon_{jikl} = \varepsilon_{klij} \in \mathbb{L}^\infty(\Omega), 1 \leq i,j,k,l \leq d\}.$

Which is the real Banach space with the norm

$\|\varepsilon\|_{\mathcal{H}_\infty} = \max_{1 \leq i,j,k,l \leq d} \|\varepsilon_{ijkl}\|_{\mathbb{L}^\infty(\Omega)}.$

The tensor of relaxation $\mathcal{F}$ satisfies

$$\mathcal{F} \in C([0.T]; \mathcal{H}_\infty) \tag{4.8}$$

The adhesion coefficients c_ν, c_τ and ε_a satisfy

$c_\nu, c_\tau \in \mathbb{L}^\infty(\Gamma_3), \varepsilon_a \in \mathbb{L}^2(\Gamma_3)$ and $c_\nu, c_\tau, \varepsilon_a > 0$ a.e. on Γ_3 . (4.9)

That the initial bonding field satisfies

$\beta_0 \in \mathbb{L}^2(\Gamma_3), \ 0 \leq \beta_0 \leq 1$ a.e on Γ_3. (4.10)

Finally, the initial data satisfy

$u_0 \in V$ and $u_1 \in H$. (4.11)

A modified inner product on the Hilbert space $H = \mathbb{L}^2(\Omega)^d$ given by

$((u, v))_H = (\rho u, v)_H \ \forall u, v \in H$,

that, it is weighted with ρ and let $\|.\|_H$ be the associated norm i.e.

$$\|v\|_H = (\rho v, v)_H^{\frac{1}{2}} \quad \forall v \in H.$$

It follows from assumptions (4.7) that $\|.\|_H$ and $|.|_H$ are equivalent norms on H and also the inclusion mapping of $(V, |.|_V)$ into $(H, |.|_H)$ is continuons and dense. Let V' be the dual space of V. Identifying H with its own dual, it can write the Gelfand triple

$V \subset H \subset V'$;

The notation $(.,.)_{V' \times V}$ represents the duality pairing between V' and V recall that

$(u, v)_{V' \times V} = \left((u, v)\right)_H \quad \forall u \in H, \quad \forall v \in V.$

Next, the subset W of H_1 are defined as

$W = \{v \in H_1 : \operatorname{div}(v) \in H\}$,

and let $j_c : V \times V \to$ R, $j_f : (V \cap W) \times V \to \mathbb{R}$ be the functionals given by

$$j_c(u, v) = \int_{\Gamma_3} p(u_\nu) v_\nu da \quad \forall (u, v) \in V \times V,$$

$$j_f(u, v) = \int_{\Gamma_3} \mu |R\sigma_\nu(u)| |v_\tau| da \quad \forall (u, v) \in (V \cap W) \times V,$$

Where

$R: H^{\frac{1}{2}}(\Gamma) \longrightarrow \mathbb{L}^2(\Gamma_3)$ is a linear and continuous mapping (*see* [7]). (4.12)

The coefficient of friction μ is assumed to satisfy

$\mu \in \mathbb{L}^\infty(\Gamma_3)$ and $\mu \geq 0$ a.e on Γ_3 (4.13)

Next, let

$j = j_c + j_f$

The functional h is defined by

$h : \mathbb{L}^2(\Gamma_3) \times V \times V \longrightarrow \mathbb{R},$

$$h(\beta, u, v) = \int_{\Gamma_3} (-c_\nu \beta^2 R_\nu(u_\nu) v_\nu + c_\tau \beta^2 R_\tau(u_\tau) v_\tau) da, \forall(\beta, u, v) \in \mathbb{L}^2(\Gamma_3) \times V \times V$$

where the normal compliance function p satisfies:

$$\begin{cases} (a)\ p: \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+, \\ (b)\ \exists L_p \ such\ that:\ |p(x - r_1) - p(x - r_2)| \leq L_p |r_1 - r_2| \\ \qquad \forall\, r_1, r_2 \in \mathbb{R},\ a.e.\ x \in \Gamma_3, \\ (c)\ \big(p(x, r_1) - p(x, r_2)\big)(r_1 - r_2) \geq 0, \\ \qquad \forall\, r_1, r_2 \in \mathbb{R},\ a.e.\ x \in \Gamma_3, \\ (d)\ the\ mapping\ x \longrightarrow p(x, r)\ is\ Lebesgue\ measurable\ on\ \Gamma_3, \\ \qquad for\ every\ r \in \mathbb{R}, \\ (e)\ p(x, 0) = 0,\ a.e.\ x \in \Gamma_3. \end{cases} \tag{4.14}$$

Finaly, the following set of the bonding field:

$B = \{\theta: [0, T] \longrightarrow \mathbb{L}^2(\Gamma_3): 0 \leq \theta(t) \leq 1, \forall \in [0, T], a.e. on\ \Gamma_3\}.$

By a standard procedure based on Green's formula the following variational formulation of problem P_1 is derived in terms of displacement and bonding field.

Problem PV. Find a displacement field u: $\Omega \times [0.T] \longrightarrow \mathbb{R}^d$, a stress field σ: $\Omega \times [0.T] \longrightarrow \mathbb{S}^d$ and a bonding field β: $\Gamma_3 \times [0.T] \longrightarrow [0.1]$ such that $u(t) \in K \cap W$,

$$\sigma(t) = \mathcal{A}\varepsilon\big(u(t)\big) + \mathcal{G}\varepsilon\big(\dot{u}(t)\big) + \int_0^1 \mathcal{F}(t - s)\varepsilon\big(u(s)\big) ds \quad a.e.\ t \in [0, T]$$

$$\big((\ddot{u},\ \omega - \dot{u})\big)_H + \big(\mathcal{A}\varepsilon(u), \varepsilon(\omega) - \varepsilon(\dot{u})\big)_{\mathcal{H}} + \big(\mathcal{G}\varepsilon(\dot{u}), \varepsilon(\omega) - \varepsilon(\dot{u})\big)_{\mathcal{H}}$$

$$+ \left(\int_0^t \mathcal{F}(t - s)\varepsilon\big(u(s)\big) ds, \varepsilon(\omega) - \varepsilon(\dot{u})\right)_{\mathcal{H}} + h(\beta(t), u(t), \omega - \dot{u}) + j_c(u(t), \omega)$$

$$-j_c(u(t), \dot{u}) + j_f(u(t), \omega) - j_f(u(t), \dot{u}) \geq \big(f(t), \omega(t) - \dot{u}(t)\big)_V \tag{4.15}$$

$$\dot{\beta}(t) = -\left[\beta(t)\left(c_\nu \left(R_\nu\big(u_\nu(t)\big)\right)^2 + c_\tau \big|R_\tau\big(u_\tau(t)\big)\big|^2 - \varepsilon_a\right)\right]_+ \quad a.e.\ t \in [0, T] \tag{4.16}$$

$$u(0) = u_0,\ \ \dot{u}(0) = u_1 = v_0,\ \ \beta(0) = \beta_0. \tag{4.17}$$

5. Existence and uniqueness result

The main result in this section is the following existence and uniqueness result.

Theorem 5.1: Let the assumptions (4.3)-(4.13) hold. Then, there exists a constant $\mu_0 > 0$ such that problem PV has a unique solution (u, σ, β) which satisfies if

$$\|\mu\|_{\mathbb{L}^\infty(\Gamma_3)} < \mu_0.$$

The proof of theorem 5.1 is carried out several steps. In the first step, the closed subset Z of the space $C([0.T] ; \mathbb{L}^2(\Gamma_3))$ is defined as

$$Z = \{\theta \in C([0,T];\ \mathbb{L}^2(\Gamma_3)) \cap B: \theta(0) = \beta_0\},$$

where the Banach space $C([0.T] ; \mathbb{L}^2(\Gamma_3))$ is endowed with the norm

$$\|B\|_k = \max_{t\in[0,T]}[exp(-kt)\|\beta(t)\|_{\mathbb{L}^2(\Gamma_3)}],\ k > 0.$$

Next for a given $\xi \in Z$, let the following variational problem.

Problem P1ξ. Find $u_\xi \in C$ ([0.T] ; V) such that $u_\xi \in$ K ∩ W

$$\left(\left(\ddot{u}_\xi, \omega - \dot{u}_\xi\right)\right) + \left(\mathcal{A}\varepsilon(u_\xi), \varepsilon(\omega) - \varepsilon(\dot{u}_\xi)\right)_{\mathcal{H}} + \left(\mathcal{G}\varepsilon(\dot{u}_\xi), \varepsilon(\omega) - \varepsilon(\dot{u}_\xi)\right)_{\mathcal{H}}$$

$$+\left(\int_0^t \mathcal{F}(t-s)\,\varepsilon\left(u_\xi(s)\right)ds, \varepsilon(\omega) - \varepsilon(\dot{u}_\xi)\right)_{\mathcal{H}} + h(\beta(t), u_\xi(t), \omega - \dot{u}_\xi) + j_c(u_\xi(t), \omega) -$$

$$j_c(u_\xi(t), \dot{u}_\xi) + j_f(u_\xi(t), \omega) - j_f(u_\xi(t), \dot{u}_\xi) \geq \left(f(t), \omega(t) - \dot{u}_\xi(t)\right)_V\ \forall\ \omega \in K,\ t \in [0,T] \quad (5.1)$$

We have the following results

Theorem 5.2. There exists a constant $\mu_1 > 0$ such that problem P_ξ has a unique solution if

$\|\mu\|_{\mathbb{L}^\infty(\Gamma_3)} < \mu 1$

Let $\eta \in C$ ([0.T] ; V) be given the following intermediate problem is introduced by:

Problem $P_{\xi\eta}$. Find $u_{\xi\eta} \in C([0,T];V)$ such that $u_{\xi\eta} \in K \cap W$

$$\left(\left(\ddot{u}_{\xi\eta}, \omega - \dot{u}_{\xi\eta}\right)\right) + \left(\mathcal{G}\varepsilon(\dot{u}_{\xi\eta}), \varepsilon(\omega) - \varepsilon(\dot{u}_{\xi\eta})\right)_{\mathcal{H}} + \left(\eta(t), \varepsilon(\omega) - \varepsilon(\dot{u}_{\xi\eta})\right)_{\mathcal{H}}$$

$$+j_f(u_{\xi\eta}(t), \omega) - j_f(u_{\xi\eta}(t), \dot{u}_{\xi\eta}) \geq \left(f(t), \omega(t) - \dot{u}_{\xi\eta}(t)\right)_V$$

$$u_{\xi\eta}(0) = u_0,\ \dot{u}_{\xi\eta}(0) = u_1 \quad \forall\ \omega \in\ K,\ t \in [0,T] \quad (5.2)$$

Since Riesz's representation theorem representation theorem implies that there exists an element $f_\eta \in C$ ([0.T] ; V) such that

$$\left(f_\eta(t), \omega\right)_V = (f(t), \omega)_V - \left(\eta(t), \varepsilon(\omega)\right)_{\mathcal{H}},$$

the problem $P_{\xi\eta}$ is equivalent to the following problem.

Problem $P_{2\xi\eta}$. Find $u_{\xi\eta} \in C$ ([0.T] ; V) such that

$u_{\xi\eta} \in K \cap W$,

$$\left(\left(\ddot{u}_{\xi\eta}, \omega - \dot{u}_{\xi\eta}\right)\right) + \left(\mathcal{G}\varepsilon(\dot{u}_{\xi\eta}), \varepsilon(\omega) - \varepsilon(\dot{u}_{\xi\eta})\right)_{\mathcal{H}} + j_f(u_{\xi\eta}(t), \omega) - j_f(u_{\xi\eta}(t), \dot{u}_{\xi\eta})$$

$$\geq \left(f_\eta(t), \omega - u_{\xi\eta}\right)_V \quad \forall \omega \in K, t \in [0,T] \quad (5.3)$$

Lemma 5.3 There exists a constant $\mu_1 > 0$ such that problem $\mathbf{P}_{2\xi\eta}$ has a unique solution if $\|\mu\|_{\mathbb{L}^\infty(\Gamma\ )} < \mu 1$.

The proof is based on several step by using arguments on Banach fixed point theorem. Indeed, let $q \in C_+$ where C_+ is a non-empty closed subset of $\mathbb{L}^2(\Gamma_3)$

defined as $C_+ = \{s \in \mathbb{L}^2(\Gamma_3); s \geq 0\ a.e.on\Gamma_3\}$

And let the functional $j_q: V \longrightarrow \mathbb{R}$ given by

$$j_q(v) = \int_{\Gamma_3} \mu q|v_\tau|da \quad \forall v \in V.$$

We consider the following auxiliary problem.

Problem $\boldsymbol{P_{\xi\eta q}}$. Find $u_{\xi\eta q} \in C([0.T]; \mathrm{V})$ such that

$$u_{\xi\eta q} \in K, \Big(\big(\ddot{u}_{\xi\eta q}, \omega - \dot{u}_{\xi\eta q}\big)\Big) + \Big(\mathcal{G}\varepsilon\big(\dot{u}_{\xi\eta q}\big), \varepsilon(\omega) - \varepsilon\big(\dot{u}_{\xi\eta q}\big)\Big)_{\mathcal{H}} + j_q(\omega)$$
$$-j_q\big(u_{\xi\eta q}(t), \dot{u}_{\xi\eta q}\big) \geq \big(f_\eta(t), \omega - u_{\xi\eta q}\big)_V;\ \forall \omega \in K, t \in [0,T] \tag{5.4}$$

Lemma 5.4. Problem $\boldsymbol{P_{\xi\eta q}}$ has a unique solution with the regularity $v_{\xi\eta q} \in C(0,T;H) \cap \mathbb{L}^2(0,T;V) \cap W^{1,2}(0,T;V')$.

Proof: The continuous injection of V into $\mathbb{L}^2(\Gamma_3)^d$ implies that $\boldsymbol{j}$ is continuous

and convex. We define the sequence:

$$j_\varepsilon(v) = \int_{\Gamma_3} uq\sqrt{|v_\tau|^2 + \varepsilon^2}\, ds$$

$\forall\, v \in V, \forall \varepsilon > 0$

Its derivative of Fréchet is given by :

$$j_\varepsilon'(v).\omega = \int_{\Gamma_3} uq \frac{(v_\tau, \omega_\tau)}{\sqrt{|v_\tau|^2 + \varepsilon^2}} ds, \quad \forall v \in V, \forall \varepsilon > 0.$$

Then $\boldsymbol{j_\varepsilon}$ is of class C^1. Direct algebraic calculations show that $\forall\, \alpha \geq 0$,

$\beta \geq 0$ such that $\alpha + \beta = 1$ and for any real x and y, $n \geq 1$:

$$\sqrt{(\alpha x + \beta y)^2 + \frac{1}{n}} \leq \alpha\sqrt{x^2 + \frac{1}{n}} + \beta\sqrt{y^2 + \frac{1}{n}}$$

So $\boldsymbol{j_\varepsilon}$ is convex $\forall \varepsilon > 0$. also:

$$\exists C > 0, \forall \omega \in V, |j_\varepsilon'(\omega)|_{V'} \leq C|g|_{\mathbb{L}^2(\Gamma 3)} \tag{5.5}$$

The hypothesis (4.5) (a) implies that $G : V \rightarrow V'$is a continuous Lipschitz operator. Since j_ε' is continuous then $G + j_\varepsilon'$ is a continuous and therefore

hemicontinuous operator. Now, according to (4.5)(b) and the monotony of j_ε' we find :

$$\langle (G + j_\varepsilon')u - (G + j_\varepsilon')v, u - v\rangle_{V' \times V} \geq m_{\mathcal{G}}|u - v|_V^2 \qquad \forall u, v \in V \tag{5.6}$$

Then $G+j'_\varepsilon$ is a monotone operator. By taking $\upsilon = 0_V$ on (5.6) and using the Inequality $ab \leq \frac{m_{\mathcal{G}}}{2}a^2 + \frac{1}{2m_{\mathcal{G}}}b^2$,

it results $\forall u, \upsilon \in V$:

$$\langle (G+j'_\varepsilon)u, u\rangle_{V'\times V} \geq m_{\mathcal{G}}|u|_V^2 - |G0_V|_{V'}.$$

And $|u|_V \geq \frac{1}{2}m_{\mathcal{G}}|u|_V^2 - \frac{1}{2m_{\mathcal{G}}}|G0_V|_{V'}^2$, For $\omega = \frac{1}{2}m_{\mathcal{G}}, C = \frac{1}{2m_{\mathcal{G}}}|G0_V|_{V'}^2 \in \mathbb{R}$.

Next, by using (4.5)(a) and (5.5) then :

$$|(G+j'_\varepsilon)u - (G+j'_\varepsilon)\upsilon|_{V'} \leq L_{\mathcal{G}}|u-\upsilon|_V + C.$$

Choosing $\upsilon = 0_V$ it result:

$$|(G+j'_\varepsilon)u|_{V'} \leq C(|u-\upsilon|_V + 1),\ \forall u \in V$$

Finally, by using (4.11) that there exists $\upsilon_\eta^\varepsilon \in \mathbb{L}^2(0,T;V) \cap C([0,T];H)$ and $\dot{\upsilon}_\eta^\varepsilon \in \mathbb{L}^2([0,T];V')$, such that :

$$\begin{cases} \dot{\upsilon}_\eta^\varepsilon(t) + G\upsilon_\eta^\varepsilon(t) + j_\eta^\varepsilon(\upsilon_\eta^\varepsilon) = f_\eta(t) \quad in\ V'a.e.t \in [0,T], \\ \upsilon_\eta^\varepsilon(0) = u_1. \end{cases} \tag{5.7}$$

Then $\upsilon_\eta^\varepsilon \in \mathbb{L}^2([0,T];V) \cap W^{1,2}(0,T;V')$ which satisfies:

$$\left(\dot{\upsilon}_\eta^\varepsilon(t), \omega - \upsilon_\eta^\varepsilon(t)\right)_{V'\times V} + \left(\mathcal{G}\upsilon_\eta^\varepsilon(t), \omega - \upsilon_\eta^\varepsilon(t)\right)_{V'\times V} + j_\varepsilon(\omega) - j_\varepsilon(\upsilon_\eta^\varepsilon) \geq \left(f_\eta(t), \omega - \upsilon_\eta^\varepsilon(t)\right)_{V'\times V}$$

$$\upsilon_\eta(0) = u_1 \tag{5.8}$$

Using (5.7) to obtain:

$$\left(\dot{\upsilon}_\eta^\varepsilon(t), \upsilon_\eta^\varepsilon(t)\right)_{V'\times V} + (\mathcal{G}\upsilon_\eta^\varepsilon(t), \upsilon_\eta^\varepsilon(t))_{V'\times V} + \left(j'_\varepsilon(\upsilon_\eta^\varepsilon), \upsilon_\eta^\varepsilon\right)_{V'\times V}$$

$$= \left(f_\eta(t), \upsilon_\eta^\varepsilon(t)\right)_{V'\times V}$$

$$\upsilon_\eta(0) = u_1 \tag{5.9}$$

By using (4.8), the monotony of j'_ε and (5.3) it comes that :

$$\exists\, C > 0, \forall t \in [0,T], \left|\upsilon_\eta^\varepsilon(t)\right|_H \leq C, \int_0^T \left|\upsilon_\eta^\varepsilon(t)\right|_H^2 dt \leq C,\ \int_0^T \left|\dot{\upsilon}_\eta^\varepsilon(t)\right|_H^2 dt \leq C. \tag{5.10}$$

So there is a sub-sequence (v_η) such that :

$\upsilon_\eta^\varepsilon \longrightarrow \upsilon_\eta$ weakly in $\mathbb{L}^2(0,T;V)$ and weakly star in $\mathbb{L}^\infty(0,T;H)$.

$$\dot{\upsilon}_\eta^\varepsilon \longrightarrow \dot{v}_\eta \text{ and weakly star in } \mathbb{L}^2(0,T;V) \tag{5.11}$$

It comes that :

$$\upsilon_\eta \in C(0,T;H) \text{ and } \upsilon_\eta^\varepsilon(t) \longrightarrow \upsilon_\eta(t) \text{ weakly in } H, \forall t \in [0,T]. \tag{5.12}$$

By integration of (5.8),then $\forall\omega \in \mathbb{L}^2(0,T;V)$:

$$\int_0^T \left(\dot{\upsilon}_\eta^\varepsilon(t),\omega\right)_{V'\times V}dt + \int_0^T \left(\mathcal{G}\upsilon_\eta^\varepsilon(t),\omega\right)_{V'\times V}dt + \int_0^T j_\varepsilon(\omega)dt$$

$$\geq \int_0^T \left(\dot{\upsilon}_\eta^\varepsilon(t),\upsilon_\eta^\varepsilon(t)\right)_{V'\times V}dt + \int_0^T \left(\mathcal{G}\upsilon_\eta^\varepsilon(t),\upsilon_\eta^\varepsilon(t)\right)_{V'\times V}dt$$

$$+\int_0^T j_\varepsilon(\upsilon_\eta^\varepsilon)dt + \int_0^T \left(f_\eta(t),\omega-\upsilon_\eta^\varepsilon(t)\right)_{V'\times V}dt$$

$$\geq \frac{1}{2}\left|\upsilon_\eta^\varepsilon(T)\right|_H^2 - \frac{1}{2}\left|\upsilon_\eta^\varepsilon(0)\right|_H^2 + \int_0^T \left(\mathcal{G}\upsilon_\eta^\varepsilon(t),\upsilon_\eta^\varepsilon(t)\right)_{V'\times V}$$

$$+\int_0^T j_\varepsilon(\upsilon_\eta^\varepsilon)dt + \int_0^T \left(f_\eta(t),\omega-\upsilon_\eta^\varepsilon(t)\right)_{V'\times V}dt. \tag{5.13}$$

By (5.11), (5.12) and the weak semi-continuity below it result that:

$\forall\omega \in \mathbb{L}^2(0,T;V)$,

$$\int_0^T \left(\dot{\upsilon}_\eta(t),\omega-\upsilon_\eta(t)\right)_{V'\times V}dt + \int_0^T \left(\mathcal{G}\upsilon_\eta(t),\omega-\upsilon_\eta(t)\right)_{V'\times V}dt + \int_0^T \left(j_q(\omega)-j_q(\upsilon_\eta)\right) \geq \int_0^T \left(f_\eta(t),\omega-\upsilon_\eta(t)\right)_{V'\times V}dt$$

Which implies that:

$$\left(\dot{\upsilon}_\eta(t),\omega-\upsilon_\eta(t)\right)_{V'\times V} + \left(\mathcal{G}\upsilon_\eta(t),\omega-\upsilon_\eta(t)\right)_{V'\times V} + j_q(\omega) - j_q(\upsilon_\eta)$$

$$\geq \left(f_\eta(t),\omega-\upsilon_\eta(t)\right)_{V'\times V}, \ \forall\omega\in V, \forall t\in[0,T].$$

So the problem $P_{\eta q}$ has a solution $\upsilon_\eta \in C(0,T;H)\cap \mathbb{L}^2(0,T;V)\cap W^{1,2}(0,T;V')$. For uniqueness, let $\upsilon_\eta^1,\upsilon_\eta^2$ be two solutions of $P_{\eta q}$. Then for all $t\in[0,T]$,

$$\left(\dot{\upsilon}_\eta^2(t)-\dot{\upsilon}_\eta^1(t),\upsilon_\eta^2(t)-\upsilon_\eta^1(t)\right)_{V'\times V} + \left(\mathcal{G}\upsilon_\eta^2(t)-\mathcal{G}\upsilon_\eta^1(t),\upsilon_\eta^2(t)-\upsilon_\eta^1(t)\right)_{V'\times V} \leq 0.$$

by integrating the previous inequation and using (4.5) then :

$$\frac{1}{2}\left|\upsilon_\eta^2(t)-\upsilon_\eta^1(t)\right|_V^2 + m_{\mathcal{G}}\int_0^T\left|\upsilon_\eta^2(s)-\upsilon_\eta^1(s)\right|_V^2 ds \leq 0, \ \forall t\in[0,T].$$

It implies $\upsilon_\eta^2 = \upsilon_\eta^1$.

In the study of the problem $P_{\eta q}$ we have the following result :

Lemma 5.5 : The problem $P\upsilon_{\eta q}$ has a unique solution $u_{\eta q} \in W^{1,2}(0,T;V)\cap C^1(0,T;H)\cap W^{2,2}(0,T;V')$. Moreover, if u_1,u_2 two solutions of the problem $P\upsilon_{\eta q}$ corresponding to the data $\eta_1,\eta_2 \in \mathbb{L}^2(0,T;V')$ and $q_1,q_2 \in C_+$ then there exists $c>0$ such that :

$$\left|\dot{u}_{\eta_1 q_1}(t)-\dot{u}_{\eta_2 q_2}(t)\right|_V^2 \leq c\int_0^t|\eta_1(s)-\eta_2(s)|_{V'}^2 ds + \int_0^t|q_1(s)-q_2(s)|_{V'}^2 ds$$

$$\left|u_{\eta_1 q_1}(t)-u_{\eta_2 q_2}(t)\right|_V^2 \leq c\int_0^t|\eta_1(s)-\eta_2(s)|_{V'}^2 ds + \int_0^t|q_1(s)-q_2(s)|_{V'}^2 ds. \tag{5.14}$$

Proof: The proof is a consequence of the lemma (5.4) and the relation (4.5). For proving the inequality (5.14), let $u_{\eta_1,}u_{\eta_2}$ be two solutions of problems Pv_{η_1} and Pv_{η_2} respectively, then:

$$\left(\ddot{u}_{\eta_i q_i}, \dot{u}_{\eta_j q_j} - \dot{u}_{\eta_i q_i}\right) + \left(\mathcal{G}\dot{u}_{\eta_i q_i}, \dot{u}_{\eta_j q_j} - \dot{u}_{\eta_i q_i}\right) + j_q\left(\dot{u}_{\eta_j q_j}\right)$$

$$-j_q\left(\dot{u}_{\eta_i q_i}\right) \geq \left(f - \eta_i, \dot{u}_{\eta_j q_j} - \dot{u}_{\eta_i q_i}\right).$$

where $i = 1$ if $j = 2$ and $i = 2$ if $j = 1.$

Doing the addition so we have

$$\left(\ddot{u}_{\eta_1 q_1} - \ddot{u}_{\eta_2 q_2}, \dot{u}_{\eta_2 q_2} - \dot{u}_{\eta_1 q_1}\right) + \left(\mathcal{G}\dot{u}_{\eta_1 q_1} - \mathcal{G}\dot{u}_{\eta_2 q_2}, \dot{u}_{\eta_2 q_2} - \dot{u}_{\eta_1 q_1}\right)$$

$$+j_{q_1}\left(\dot{u}_{\eta_2 q_2}\right) - j_{q_1}\left(\dot{u}_{\eta_1 q_1}\right) + j_{q_2}\left(\dot{u}_{\eta_1 q_1}\right) - j_{q_2}\left(\dot{u}_{\eta_2 q_2}\right) \geq \left(\eta_1 - \eta_2, \dot{u}_{\eta_1 q_1} - \dot{u}_{\eta_2 q_2}\right) \qquad (5.15)$$

Using (4.6) and integring, the inequation (5.15) becomes :

$$\int_0^t \left(\ddot{u}_{\eta_1 q_1}(s) - \ddot{u}_{\eta_2 q_2}(s), \dot{u}_{\eta_2 q_2}(s) - \dot{u}_{\eta_1 q_1}(s)\right) ds$$

$$+ \int_0^t \left(\mathcal{G}\dot{u}_{\eta_1 q_1}(s) - \mathcal{G}\dot{u}_{\eta_2 q_2}(s), \dot{u}_{\eta_2 q_2}(s) - \dot{u}_{\eta_1 q_1}(s)\right) ds + \int_0^t q_1 \left|\dot{u}_{\eta_2 q_2}(s)\right| ds$$

$$+ \int_0^t q_1 \left|\dot{u}_{\eta_1 q_1}(s)\right| ds + \int_0^t q_2 \left|\dot{u}_{\eta_1 q_1}(s)\right| ds - \int_0^t q_2 \left|\dot{u}_{\eta_2 q_2}(s)\right| ds - \geq \int_0^t |\eta_1(s) - \eta_2(s)|_{V'} \left|\dot{u}_{\eta_1 q_1}(s) - \dot{u}_{\eta_2 q_2}(s)\right|_V ds.$$

And we have $\langle \ddot{u}, \dot{u} \rangle = \frac{1}{2}\langle \dot{u}, \dot{u} \rangle' = \frac{1}{2}\frac{d}{dt}|\dot{u}|'$

According to this equation we find

$$\frac{1}{2}\frac{d}{dt}\left|\dot{u}_{\eta_1 q_1} - \dot{u}_{\eta_2 q_2}\right|_V^2 + m_{\mathcal{G}}\left|\dot{u}_{\eta_1 q_1} - \dot{u}_{\eta_2 q_2}\right|_V^2$$

$$\leq |\eta_1 - \eta_2|\left|\dot{u}_{\eta_1 q_1} - \dot{u}_{\eta_2 q_2}\right| + A.$$

With $A = j_{q_1}\left(\dot{u}_{\eta_2 q_2}\right) - j_{q_1}\left(\dot{u}_{\eta_1 q_1}\right) + j_{q_2}\left(\dot{u}_{\eta_1 q_1}\right) - j_{q_2}\left(\dot{u}_{\eta_2 q_2}\right).$

$$A = \int_{\Gamma_3} \mu\left[(q_2 - q_1)\left|\dot{u}_{\eta_1 q_1}(s)\right| + (q_1 - q_2)\left|\dot{u}_{\eta_2 q_2}(s)\right|\right] ds.$$

$$A \leq \mu_1 |q_2 - q_1|_{\mathbb{L}^2(\Gamma_3)}\left|\dot{u}_{\eta_1 q_1} - \dot{u}_{\eta_2 q_2}\right|_{\mathbb{L}^2(\Omega)}.$$

Then: $\frac{1}{2}\left|\dot{u}_{\eta_1 q_1} - \dot{u}_{\eta_2 q_2}\right|_V^2 + \frac{m_{\mathcal{G}}}{2}\int_0^t \left|\dot{u}_{\eta_1 q_1}(s) - \dot{u}_{\eta_2 q_2}(s)\right|_V^2 ds$

$$\leq \int_0^t |\eta_1 - \eta_2|\left|\dot{u}_{\eta_1 q_1}(s) - \dot{u}_{\eta_2 q_2}(s)\right| ds + \int_0^t A ds.$$

So

$$\frac{1}{2}\left|\dot{u}_{\eta_1 q_1} - \dot{u}_{\eta_2 q_2}\right|_V^2 + m_{\mathcal{G}} \int_0^t \left|\dot{u}_{\eta_1 q_1}(s) - \dot{u}_{\eta_2 q_2}(s)\right|_V^2 ds$$

$$\leq \int_0^t \left(\frac{m_{\mathcal{G}}}{2}|\eta_1(s) - \eta_2(s)|^2 + \frac{1}{2m_{\mathcal{G}}}\left|\dot{u}_{\eta_1 q_1}(s) - \dot{u}_{\eta_2 q_2}(s)\right|^2\right) ds$$

$$+ \int_0^t \mu_1 |q_2 - q_1|_{\mathbb{L}^2(\Gamma_3)} \left|\dot{u}_{\eta_1 q_1}(s) - \dot{u}_{\eta_2 q_2}(s)\right|_{\mathbb{L}^2(\Omega)} ds. \quad (5.16)$$

But $\left(ab \leq \frac{m_{\mathcal{G}}}{2} a^2 + \frac{1}{2m_{\mathcal{G}}} b^2\right)$ then:

$$\frac{1}{2}\left|\dot{u}_{\eta_1 q_1} - \dot{u}_{\eta_2 q_2}\right|_V^2 + m_{\mathcal{G}} \int_0^1 \left|\dot{u}_{\eta_1 q_1}(s) - \dot{u}_{\eta_2 q_2}(s)\right|_V^2 ds$$

$$\leq \int_0^t \left(\frac{m_{\mathcal{G}}}{2}\left|\dot{u}_{\eta_1 q_2}(s) - \dot{u}_{\eta_2 q_2}\right|_V^2 + \frac{1}{2m_{\mathcal{G}}}|\eta_1(s) - \eta_2(s)|_{V'}^2\right) ds. \quad (5.17)$$

From which:

$$\frac{1}{2}\left|\dot{u}_{\eta_1 q_1} - \dot{u}_{\eta_2 q_2}\right|_V^2 + \frac{m_{\mathcal{G}}}{2}\int_0^t \left|\dot{u}_{\eta_1 q_1}(s) - \dot{u}_{\eta_2 q_2}(s)\right|_V^2 ds \leq \frac{1}{2m_{\mathcal{G}}}\int_0^t |\eta_1(s) - \eta_2(s)|_{V'}^2 ds$$
(5.18)

It comes

$$\begin{cases} \left|\dot{u}_{\eta_1 q_1} - \dot{u}_{\eta_2 q_2}\right|_V^2 \leq c \int_0^1 |\eta_1(s) - \eta_2(s)|_{V'}^2 ds \\ \int_0^t \left|\dot{u}_{\eta_1 q_1}(s) - \dot{u}_{\eta_2 q_2}(s)\right|_V^2 ds \leq c \int_0^t |\eta_1(s) - \eta_2(s)|_{V'}^2 ds \end{cases} \quad (5.19)$$

On the other part of $u_1(0) = u_2(0) = u_0$ then:

$$\left|u_{\eta_1 q_1}(s) - u_{\eta_2 q_2}(s)\right|_V^2 \leq \int_0^t \left|\dot{u}_{\eta_1 q_1}(s) - \dot{u}_{\eta_2 q_2}(s)\right|_V ds \quad (5.20)$$

And:

$$\left|u_{\eta_1 q_1}(s) - u_{\eta_2 q_2}(s)\right|_V^2 \leq \int_0^t \left|\dot{u}_{\eta_1 q_1}(s) - \dot{u}_{\eta_2 q_2}(s)\right|_V^2 ds \quad (5.21)$$

And so:

$$\left|u_{\eta_1 q_1}(s) - u_{\eta_2 q_2}(s)\right|_V^2 \leq c \int_0^t |\eta_1(s) - \eta_2(s)|_{V'}^2 ds \quad (5.22\ )$$

Hence it result (5.14).

Now, let the map $\psi_t : C_+ \longrightarrow C_+$ be defined by

$$\psi_t(q) = \left|R\sigma_\nu\left(u_{\xi\eta q}(t)\right)\right|.$$

Lemma 5.6 There exists a constant $\mu_1 > 0$ such that the mapping ψ_t has a unique fixed point q^* and $u_{\xi\eta q^*}$ (t) is a unique solution of the inequality (5.3) if $\|\mu\|_{\mathbb{L}^\infty(\Gamma_3)} < \mu_1$

Proof. Let q_1, $q_2 \in C_+$. Using (4.16), it follows that there exists a constant $c_0 > 0$ such that

$$\|\psi_t(q_1)-\psi_t(q_2)\|_{\mathbb{L}^2(\Gamma_3)} \le c_0 \left\|\sigma_\nu\left(u_{\xi\eta q_1}(t)\right)-\sigma_\nu\left(u_{\xi\eta q_2}(t)\right)\right\|_{H^{-\frac{1}{2}}(\Gamma)}. \quad (5.23)$$

Moreover using (4.5) (b) yields

$$\left\|\sigma_\nu\left(u_{\xi\eta q_1}(t)\right)-\sigma_\nu\left(u_{\xi\eta q_2}(t)\right)\right\|_{H^{-\frac{1}{2}}(\Gamma)} \le M\left\|u_{\xi\eta q_1}(t)-u_{\xi\eta q_2}(t)\right\|_V . \quad (5.24)$$

Using (4.2), (4.5) (c), (4.13) (c) and the properties of R_ν and R_τ to find after some calculus algebra that

$$\left\|u_{\xi\eta q_1}(t)-u_{\xi\eta q_2}(t)\right\|_V \le \frac{\|\mu\|_{\mathbb{L}^\infty(\Gamma_3)} d\Omega}{m}\|q_1-q_2\|_{\mathbb{L}^2(\Gamma_3)}. \quad (5.25)$$

Hence, taking into account (4.12) ,combining (5.23) , (5.24) and (5.25) to deduce that

$$\|\psi_t(q_1)-\psi_t(q_2)\|_{\mathbb{L}^2(\Gamma_3)} \le \|\mu\|_{\mathbb{L}^\infty(\Gamma_3)}\frac{c_0 M d\Omega}{m}\|q_1-q_2\|_{\mathbb{L}^2(\Gamma_3)}.$$

Take $\mu_1 = m/c_0 M d\Omega$, then this inequality shows that if $\|\mu\|_{\mathbb{L}^\infty(\Gamma_3)} < \mu_1$,

ψ is a contraction; thus it has unique fixed point q^* and $u_{\eta q^*}(t)$ is a unique solution of (5.3). Denote $u_{\xi\eta q^*} = u_\eta$.Now shall see that $u_{\xi\eta} \in C([0.T]; V)$.

Indeed, let $t_1, t_2 \in [0, T]$. Taking $v = u_{\xi\eta}(t_2)$ in (5.3) written for $t = t_1$ and then

$v = u_{\xi\eta}(t_1)$ in the same inequality written for $t = t_2$ Using (4.5) (c) , (4.11) , (4.13) (c) and the properties of R_ν and R_τ , and adding the resulting inequalities, it follows that there exists a constant $c_1 > 0$ such that

$$\left\|u_{\xi\eta}(t_2)-u_{\xi\eta}(t_1)\right\|_V \le \frac{c_1}{m-\|\mu\|_{\mathbb{L}^\infty(\Gamma_3)}}(\|\xi(t_2)-\xi(t_1)\|_{\mathbb{L}^2(\Gamma_3)} + \|\eta(t_2)-\eta(t_1)\|_{\mathcal{H}}$$

$$+\|f(t_2)-f(t_1)\|_V).$$

Then, as $\xi \in C\left([0,T];\mathbb{L}^2(\Gamma_3)\right)$, $\eta \in C([0,T];\mathcal{H})$ and$f \in ([0,T];V)$,it immediately concludes that $u_{\xi\eta}(t) \in W, \forall t \in [0,T]$. Indeed, for each $t \in [0, T]$,

denote $\sigma\left(u_{\xi\eta}(t)\right) = \mathcal{A}\varepsilon\left(u_{\xi\eta}(t)\right) + \eta(t)$, take $v = u_{\xi\eta}(t) \pm \varphi$ in inequality (5.3) where $\varphi \in \left(C_0^\infty(\Omega)\right)^d$and use Green's formula with regularity $\varphi_1(t) \in H$

leads to div$\sigma\left(u_{\xi\eta}(t)\right) \in H$ and then $u_{\xi\eta}(t) \in W$.

Now introducing the operator $\Lambda_\xi : C([0,T];\mathcal{H}) \longrightarrow C([0,T];\mathcal{H})$ with $\eta \longrightarrow \Lambda_\xi$

defined by (5.26)

$$\langle \Lambda_\xi \eta, \omega\rangle = \langle \mathcal{A}\varepsilon(u_{\xi\eta}), \varepsilon(\omega)\rangle_{\mathcal{H}} + h(u_{\beta\eta}, \omega) + j_c(u_{\xi\eta}) + \int_0^t \mathcal{F}(t-s)\varepsilon(u_{\xi\eta})ds$$

Lemma 5.7. The operator Λ_ξ has a unique fixed point η_ξ.

Proof. Let $\eta_1, \eta_2 \in C([0,T];\mathcal{H})$. Using (5.4) , (5.26) and (4.8) we obtain for

$\|\mu\|_{\mathbb{L}^\infty(\Gamma_3)} < \mu_1$at follow that

$\left\|\Lambda_\xi\eta_1(t) - \Lambda_\xi\eta_2(t)\right\|_{\mathcal{H}} \leq c_2 \int_0^t \|\eta_1(t) - \eta_2(t)\|_{\mathcal{H}} ds.\ t \in [0,T]$

where $c_2 > 0$. Reiterating this inequality n times, yields

$\left\|\Lambda_\xi^n\eta_1(t) - \Lambda_\xi^n\eta_2(t)\right\|_{C([O,T];\mathcal{H})} \leq \frac{(c_2T)^n}{n!}\|\eta_1 - \eta_2\|_{C([0,T];\mathcal{H})}$

As $\lim_{n\to+\infty}\frac{(c_2T)^n}{n!} = 0$, it follows that for a positive integer n sufficiently large, Λ_ξ^n is a contraction; then, by using the Banach fixed point theorem, it has a unique fixed point η_ξ which is also a unique fixed of Λ_ξ i.e.,

$$\Lambda_\xi\eta_\xi = \eta_\xi(t),\ \forall t \in [0,T] \tag{5.27}$$

Then by (4.3) and (4.27) we conclude that $u_{\xi\eta_\xi}$ is the unique solution of

Problem $P_{1\xi}$. In the second step stating the following problem.

Problem P_{ad}. Find β^* : [0, T] ⟶ $\mathbb{L}^2(\Gamma_3)$such that

$$\dot{\beta}^*(t) = -\left[\beta^*(t)\left(c_\nu\left(R_\nu\left(u_{\beta^*\nu}(t)\right)\right)^2 + c_\tau\left|R_\tau\left(u_{\beta^*\tau}(t)\right)\right|^2 - \varepsilon_a\right)\right]_+ \tag{5.28}$$

$$\beta^*(0) = \beta_0 \tag{5.29}$$

Let obtain the following result be given

Proposition 5.8. Problem P$_{ad}$ has a unique solution β^*which satisfies $\beta^* \in W^{1,\infty}\left(0,T;\mathbb{L}^2(\Gamma_3)\right) \cap B$.

Proof. Let $t \in [0,T]$ and consider the mapping ϕ: Z → Z defined by

$$\phi\beta(t) = \beta_0 - \int_0^t\left[\beta(s)\left(c_\nu\left(R_\nu\left(u_\beta(s)\right)\right)\right)^2 + c_\tau\left|R_\tau\left(u_{\beta\tau}(s)\right)\right|^2 - \varepsilon_a\right]_+ ds,$$

where u_β is the solution of Problem $P_{1\beta}$. For$\beta_1, \beta_2 \in B$, there exists a constant $c_3 > 0$ such that

$$\|\phi\beta_1(t) - \phi\beta_2(t)\|_{\mathbb{L}^2(\Gamma_3)}$$

$$\leq c_3\int_0^t\left\|\beta_1(s)\left(R_\nu\left(u_{\beta_1\nu}(s)\right)\right)^2 - \beta_2(s)\left(R_\nu\left(u_{\beta_2\nu}(s)\right)\right)^2\right\|_{\mathbb{L}^2(\Gamma_3)} ds$$

$$+c_3\int_0^t\left\|\beta_1(s)\left(R_\tau\left(u_{\beta_1\tau}(s)\right)\right)^2 - \beta_2(s)\left(R_\tau\left(u_{\beta_2}(s)\right)\right)^2\right\|_{\mathbb{L}^2(\Gamma_3)} ds$$

As in [20, 21] it deduces

$$\|\phi\beta_1(t) - \phi\beta_2(t)\|_{\mathbb{L}^2(\Gamma_3)} \leq c_4\left(\int_0^t\|\beta_1(s) - \beta_2(s)\|_{\mathbb{L}^2(\Gamma_3)} ds + \int_0^t\left\|u_{\beta_1}(s) - u_{\beta_2}(s)\right\|_V ds\right).$$

(5.30)

For some constant $c_4 > 0$. Now to continue the proof it has needed to prove the following lemma.

Lemma 5.9. There exists a constant $\mu_0 > 0$ such that:

$\left\|u_{\beta_1}(t) - u_{\beta_2}(t)\right\|_V \leq c\|\beta_1(t) - \beta_2(t)\|_{\mathbb{L}^2(\Gamma_3)}\ \forall t \in [0, T]$,

Proof. Let $t \in [0, T]$. Take $u_{\beta_2}(t)$ in (4.1) satisfied by $u_{\beta_1}(t)$, then take $u_{\beta_1}(t)$ in the same inequality satisfied by $u_{\beta_2}(t)$; by addding the resulting inequalities

$$\langle \ddot{u}_1 - \ddot{u}_2, \dot{u}_1 - \dot{u}_2\rangle + \langle \mathcal{A}\varepsilon\left(u_{\beta 1}(t)\right) - \mathcal{A}\varepsilon\left(u_{\beta 2}(t)\right), \varepsilon\left(\dot{u}_{\beta 1}(t)\right) - \varepsilon\left(\dot{u}_{\beta 2}(t)\right)\rangle_{\mathcal{H}}$$
$$\leq \langle \int_0^t \mathcal{F}(t-s)\left(\varepsilon\left(u_{\beta 1}(s)\right) - \varepsilon\left(u_{\beta 2}(s)\right)\right)ds, \varepsilon\left(u_{\beta 2}(t)\right) - \varepsilon\left(u_{\beta 1}(t)\right)\rangle_{\mathcal{H}}$$
$$+ \langle \mathcal{G}\varepsilon\left(\dot{u}_{\beta 1}(t)\right) - \mathcal{G}\varepsilon\left(\dot{u}_{\beta 2}(t)\right), \varepsilon\left(\dot{u}_{\beta 2}(t)\right) - \varepsilon\left(\dot{u}_{\beta 1}(t)\right)\rangle_{\mathcal{H}}$$
$$+h\left(\beta_1(t), u_{\beta 1}(t), u_{\beta 2}(t) - u_{\beta 1}(t)\right) + j_c\left(u_{\beta 1}(t), u_{\beta 2}(t)\right)$$
$$+h\left(\beta_2(t), u_{\beta 2}(t), u_{\beta 1}(t) - u_{\beta 2}(t)\right) - j_c\left(u_{\beta 2}(t), u_{\beta 2}(t)\right)$$
$$+j_c\left(u_{\beta 2}(t), u_{\beta 1}(t)\right) - j_c\left(u_{\beta 2}(t), u_{\beta 2}(t)\right) + j_f\left(u_{\beta 1}(t), u_{\beta 2}(t)\right)$$
$$-j_f\left(u_{\beta 1}(t), u_{\beta 1}(t)\right) + j_f\left(u_{\beta 2}(t), u_{\beta 1}(t)\right) - j_f\left(u_{\beta 2}(t), u_{\beta 2}(t)\right).$$

and using (4.4) (b) then we obtain

$$\frac{1}{2}\frac{d}{dt}|\dot{u}_1 - \dot{u}_2|^2 + m\left\|u_{\beta_1}(t) - u_{\beta_2}(t)\right\|_V^2$$
$$\leq \langle \int_0^t \mathcal{F}(t-s)\left(\varepsilon\left(u_{\beta_1}(s)\right) - \varepsilon\left(u_{\beta_2}(s)\right)\right)ds, \varepsilon\left(u_{\beta_2}(t) - u_{\beta_1}(t)\right)\rangle_{\mathcal{H}}$$
$$+ \langle \mathcal{G}\varepsilon(\dot{u}_{\beta_1}) - \mathcal{G}\varepsilon(\dot{u}_{\beta_2}), \varepsilon\left(\dot{u}_{\beta_2}(t)\right) - \varepsilon\left(\dot{u}_{\beta_1}(t)\right)\rangle_{\mathcal{H}}$$
$$+h\left(\beta_1(t), u_{\beta_1}(t), u_{\beta_2}(t) - u_{\beta_1}(t)\right) + h\left(\beta_2(t), u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t)\right)$$
$$+j_c\left(u_{\beta_1}(t), u_{\beta_2}(t)\right) - j_c\left(u_{\beta_1}(t), u_{\beta_1}(t)\right) + j_c\left(u_{\beta_2}(t), u_{\beta_1}(t)\right)$$
$$-j_c\left(u_{\beta_2}(t), u_{\beta_2}(t)\right) + j_f\left(u_{\beta_1}(t), u_{\beta_2}(t)\right) - j_f\left(u_{\beta_1}(t), u_{\beta_1}(t)\right)$$
$$+j_f\left(u_{\beta_2}(t), u_{\beta 1}(t)\right) - j_f\left(u_{\beta 2}(t), u_{\beta 2}(t)\right) \tag{5.31}$$

We have

$$\langle \int_0^t \mathcal{F}(t-s)\left(\varepsilon\left(u_{\beta 1}(s) - \varepsilon(u_{\beta 2}(s)\right)ds, \varepsilon\left(\dot{u}_{\beta 2}(t)\right) - \varepsilon\left(\dot{u}_{\beta 1}(t)\right)\right)\rangle_{\mathcal{H}}$$

$$\leq c_5 \left(\int_0^t \left\|u_{\beta 1}(s)-u_{\beta 2}(s)\right\|_V ds\right)\left\|u_{\beta 1}(t)-u_{\beta 2}(t)\right\|_V$$

for some positive constant c_5.Using Young's inequality, it finds

$$\langle \int_0^t \mathcal{F}(t-s)\left(\varepsilon\left(u_{\beta 1}(s)\right)-\varepsilon(u_{\beta 2}(s))\right)ds, \varepsilon\left(\dot{u}_{\beta 1}(t)\right)-\varepsilon(\dot{u}_{\beta 2}(t))\rangle_{\mathcal{H}}$$

$$\leq \frac{c_5^2}{2m}\left(\int_0^t \left\|u_{\beta 1}(s)-u_{\beta 2}(s)\right\|_V ds\right)^2+\frac{m}{2}\left\|u_{\beta 1}(t)-u_{\beta 2}(t)\right\|_V^2 \quad (5.32)$$

Using the properties of R_ν and R_τ (see [1, 20,21]), we have

$$h\left(\beta_1(t), u_{\beta 1}(t), u_{\beta 2}(t)-u_{\beta 1}(t)\right)+h\left(\beta_2(t), u_{\beta 2}(t), u_{\beta 1}(t)-u_{\beta 2}(t)\right)$$

$$\leq c_6\|\beta_1(t)-\beta_2(t)\|_{\mathbb{L}^2(\Gamma_3)}\left\|u_{\beta 1}(t)-u_{\beta 2}(t)\right\|_V$$

where $c_6>0$. Using also (4.2) , (4.13) and (4.14) (c) yields

And using Young's inequality it results:

$$c_6\|\beta_1(t)-\beta_2(t)\|_{\mathbb{L}^2(\Gamma_3)}\left\|u_{\beta 1}(t)-u_{\beta 2}(t)\right\|_V$$

$$\leq c_7\|\beta_1(t)-\beta_2(t)\|_{\mathbb{L}^2(\Gamma_3)}^2+\frac{m}{4}\left\|u_{\beta 1}(t)-u_{\beta 2}(t)\right\|_V^2 \quad (5.33)$$

for some contant $c_7>0$. Then (5.33) implies that

$$\frac{1}{2}\frac{d}{dt}|\dot{u}_1-\dot{u}_2|^2+\frac{m}{4}\left\|u_{\beta 1}(t)-u_{\beta 2}(t)\right\|_V^2$$

$$\leq c_0 M d\Omega\|\mu\|_{\mathbb{L}^\infty(\Gamma_3)}\left\|u_{\beta 1}(t)-u_{\beta 2}(t)\right\|_V^2+\frac{c_5^2}{2m}\left(\int_0^t \left\|u_{\beta 1}(s)-u_{\beta 2}(s)\right\|_V ds\right)^2$$

$$+c_7\|\beta_1(t)-\beta_2(t)\|_{\mathbb{L}^2(\Gamma_3)}^2+\frac{m_{\mathcal{G}}}{2}|\dot{u}_1-\dot{u}_2|^2$$

Let now $\mu_0=\frac{\mu_1}{4}$, then if $\|\mu\|_{\mathbb{L}^2(\Gamma_3)}<\mu_0$, it deduces that there exists a constant

$c_8>0$ such that

$$\int_0^s \frac{1}{2}\frac{d}{dt}|\dot{u}_1-\dot{u}_2|^2 dt+\int_0^s \left\|u_{\beta 1}(t)-u_{\beta 2}(t)\right\|_V^2 dt$$

$$\leq c_8\int_0^s\left(\int_0^t \left\|u_{\beta 1}(s)-u_{\beta 2}(s)\right\|_V^2 ds+\|\beta_1(t)-\beta_2(t)\|_{\mathbb{L}^2(\Gamma_3)}^2\right)dt \;+\frac{M_{\mathcal{G}}}{2}|\dot{u}_1-\dot{u}_2|^2.$$

Then using Gronwall's argument, it follows that there exists a constant $c>0$

Such that

$$\frac{1}{2}|\dot{u}_1-\dot{u}_2|^2+\int_0^s \left\|u_{\beta 1}(t)-u_{\beta 2}(t)\right\|_V^2 dt$$

$$\int_0^s\left(\int_0^t \left\|u_{\beta 1}(s)-u_{\beta 2}(s)\right\|_V^2 ds+\|\beta_1(s)-\beta_2(s)\|_{\mathbb{L}^2(\Gamma_3)}^2\right)dt+\frac{M_{\mathcal{G}}}{2}|\dot{u}_1-\dot{u}_2|^2.$$

(5.34)

Now to end the proof of Proposition 5.8 using (5.30) and (5.34) to deduce

$$\|\phi\beta_1(t) - \phi\beta_2(t)\|_{\mathbb{L}^2(\Gamma_3)} \leq c_9 \int_0^t \|\beta_1(s) - \beta_2(s)\|_{\mathbb{L}^2(\Gamma_3)} ds \quad \forall t \in [0, T],$$

where $c_9 > 0$ and then we obtain

And reiterating this inequality n times, yield

$$\|\phi\beta_1 - \phi\beta_2\|_{\mathbb{L}^2(\Gamma_3)} \leq \frac{c_9}{k} \|\beta_1 - \beta_2\|_{\mathbb{L}^2(\Gamma_3)}.$$

$$\|\phi^n\beta_1(t) - \phi^n\beta_2(t)\|_{\mathbb{L}^2(\Gamma_3)} \leq \left(\frac{c_9 T}{k}\right)^n \frac{1}{n!} \|\beta_1 - \beta_2\|_{\mathbb{L}^2(\Gamma_3)}.$$

As $\lim_{n\to+\infty} \left(\frac{c_9 T}{k}\right)^n \frac{1}{n!} = 0,$

It follows that for a position integer n sufficiently large, ϕ^n is a contraction; then, by using the Banach fixed point theorem, it has a unique fixed point β^* which satisfies (5.28) and (5.29). Now we have all ingredients to prove Theorem 5.1.

Proof of theorem 5.1. Existence. Let $\beta = \xi^*$ and let u_{ξ^*} the solution

of problem P_1. We conclude by (5.1), (5.28) and (5.29) that (u_{ξ^*}, ξ^*) is a solution of problem PV.

Uniqueness. Suppose that (u, β) is a solution of problem PV which satisfies (4.15), (4.16) and (4.17) . It follows from (4.15) that u is a solution to problem $P_{1\xi}$ and from Theorem 5.2 that $u = u_\beta$.Take $u = u_\beta$in (4.15) and use the initial condition (4.17), we deduce that β is a solution to problem$P_{ad.}$.

Therefor, we obtain from Proposition 5.8 that $\beta = \beta^*$ and then we conclude that (u_{β^*}, β^*) is a unique solution to problem PV.

Let now σ^* be the function defined by (3.1) which corresponds to the function $u_\beta*$. Then, it results from (4.5), (4.6) and (4.8) that$\sigma^* \in C([0, T]; \mathcal{H})$.Using also a standard argument, it follows from the inequality (4.15) that

$$Div\sigma^*(t) + \varphi_1(t) = \rho\ddot{u} \ \ in\ \Omega, \text{ for all t} \in [0, T].$$

Therefor, using the regularity $\varphi_1 \in C([0, T]; H)$, we deduce that $div\ \sigma^* \in C([0, T]; H)$ which implies that $\sigma^* \in C([0, T]; \mathcal{H}_1)$. The triple $(u_\beta*, \sigma^*, \beta^*)$ which satisfies (3.1) and (4.15)−(4.17) is called a weak solution of problem P_1.Moreover, the regularity of the weak solution is $u_\beta* \in C([0, T]; V)$, $\sigma^* \in C([0, T]; \mathcal{H}_1)$ and $\beta^* \in W^{1,\infty}([0, T]; \mathbb{L}^2(\Gamma_3)) \cap B$.

6. CONCLUDING REMARK

Scientific study and contemporary publication in mechanics focus on two primary components: one pertaining to the laws of behaviour and other concerning the boundary conditions imposed in the body.

Numerous publications have employed constitutive laws incorporating internal variables to represent the influence of internal variable on the behaviour of materials such as metals, rocks and polymers, wherein the rate of deformation is contingent upon these internal variables. Our model is obtained by comnining the viscoelastic constitutive law with friction, long memory and internal state variable β, which describes the pointwise fractional density of active bonds on the contact surface and is sometimes referred to as the intensity of adhesion.

Mathematically, the idea is to reduce the second order nonlinear evolution inequality of the system to the first order evolution inequality. After this, we use classical results on first order evolution nonlinear inequalities, differential equations and the fixed point arguments.

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