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Nonlocal Initial Value Problems for Hybrid Caputo Fractional Integro-Differential Equations

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Abstract: This paper investigates nonlocal initial value problems for hybrid Caputo fractional integro-differential equations. By employing a fixed-point theorem due to Dhage, we establish the existence of solutions to these problems. The theoretical findings are illustrated through a concrete example, showcasing the applicability of our results.

Keywords: The fractional Caputo derivative, The fractional integral, hybrid, Dhage fixed point.

Introduction

Fractional calculus, an extension of classical calculus, has garnered significant attention in recent years due to its ability to describe complex phenomena that integer-order derivatives and integrals fail to capture accurately. This branch of mathematics has proven its utility in various domains, including physics, engineering, biology, and economics. ([10],[13],[14],[16],[4],[12]).

Nonlocal initial value problems (NIVPs) represent a category of problems in which the initial conditions depend on the values of the unknown function at multiple points rather than a single point. Such problems naturally arise in numerous real-world applications, such as heat transfer, viscoelastic material behavior, and control systems.

Hybrid differential equations, which combine differential and integral operators, have emerged as a powerful tool for modeling complex systems. In recent years, there has been growing interest in studying hybrid fractional differential equations, which incorporate fractional derivatives and integrals into the hybrid structure. ([10],[13],[14],[16],[4],[12]).

Lakshmikanthan and Dhage [7] They initiated the study of hybrid equations by introducing a novel class of nonlinear differential equations known as ordinary hybrid differential equations.

$$\begin{cases} \frac{d}{dt}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)),\ a.e.\ t \in I_0, \\ x(t_0) = x_0 \in \mathbb{R} \end{cases}$$

They formulated essential hybrid differential inequalities that serve as key tools for proving the existence of extremal solutions.

Zhao et al. [18] extended Dhage's work to the fractional-order case by examining boundary value problems involving fractional hybrid differential equations.

$$\begin{cases} D_{0^+}^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \ \ t \in [0,T], \\ x(t_0) = 0, \end{cases}$$

where $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order $0 < \alpha < 1$.

Hybrid fractional differential equations and inclusions have been the focus of considerable research in recent years. Prior to proceeding, we present a brief overview of some relevant contributions in this field. Ahmad et al. [2] examined the existence of solutions for a hybrid inclusion problem involving nonlocal boundary conditions.

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}\left(\frac{x(t)-\sum_{i=1}^{m} I_{0^+}^{\beta_i} h_i(t,x(t))}{g(t,x(t))}\right) \in \mathcal{G}(t,x(t)), \ a.e. \ t \in [0,1], \\ x(0) = \mu(\xi), \ \ x(1) = a \in \mathbb{R}. \end{cases}$$

where ${}^{C}D_{0^+}^{\alpha}$ denotes the Caputo fractional derivative of order $1 < \alpha \leq 2$ and $I_{0^+}^{\beta_i}$ is the Riemann–Liouville fractional integral of order $\beta_i > 0$ with $i \in \{1,2,3\ldots,m\}$. In [6], Derbazi et al. confirmed the existence and uniqueness of solutions for a fractional hybrid boundary value problem.

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}\left(\frac{x(t)-h(t,x(t))}{g(t,x(t))}\right) = \Theta(t,x(t)), \ a.e. \ t \in [0,T], \\ a_1\left(\frac{x(t)-h(t,x(t))}{g(t,x(t))}\right)|_{t=0} + b_1\left(\frac{x(t)-h(t,x(t))}{g(t,x(t))}\right)|_{t=T} = \lambda_1, \\ a_2^{C}D_{0^+}^{\beta}\left(\frac{x(t)-h(t,x(t))}{g(t,x(t))}\right)|_{t=\eta} + b_2^{C}D_{0^+}^{\beta}\left(\frac{x(t)-h(t,x(t))}{g(t,x(t))}\right)|_{t=T} = \lambda_2. \end{cases}$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1, \eta \in [0,T]$ and $a_1, a_2, b_1, b_2, \lambda_1, \lambda_2$ are real constants. . Baleanu et al. [3] they employed a generalized version of Dhage's hybrid fixed point theorem for the sum of three fractional operators to examine the existence of solutions to a fractional hybrid integro-differential equation subject to mixed hybrid integral boundary conditions.

$$\begin{cases} {}^{C}D_{0^+}^{\omega}\left(\frac{x(t)-h(t,x(t),I_{0^+}^{\gamma_1}x(t),I_{0^+}^{\gamma_2}x(t),\ldots,I_{0^+}^{\gamma_n}x(t))}{g(t,x(t),I_{0^+}^{\mu_1}x(t),I_{0^+}^{\mu_2}x(t),\ldots,I_{0^+}^{\mu_m}x(t))}\right) = \Upsilon(t,x(t)), \ a.e. \ t \in [0,1], \\ \lambda_1 \int_0^1 \ {}^{C}D_{0^+}^{\beta_1}\left(\frac{x(s)-h(s,x(s),I^{\gamma_1}x(s),I^{\gamma_2}x(s),\ldots,I^{\gamma_n}x(s))}{g(s,x(s),I^{\mu_1}x(s),I^{\mu_2}x(s),\ldots,I^{\mu_m}x(s))}\right) ds \\ +\lambda_2 \ {}^{C}D_{0^+}^{\alpha_1}\left(\frac{x(t)-h(t,x(t),I_{0^+}^{\gamma_1}x(t),I_{0^+}^{\gamma_2}x(t),\ldots,I_{0^+}^{\gamma_n}x(t))}{g(t,x(t),I_{0^+}^{\mu_1}x(t),I_{0^+}^{\mu_2}x(t),\ldots,I_{0^+}^{\mu_m}x(t))}\right)|_{t=1} \\ +\lambda_3\left(\frac{x(t)-h(t,x(t),I_{0^+}^{\gamma_1}x(t),I_{0^+}^{\gamma_2}x(t),\ldots,I_{0^+}^{\gamma_n}x(t))}{g(t,x(t),I_{0^+}^{\mu_1}x(t),I_{0^+}^{\mu_2}x(t),\ldots,I_{0^+}^{\mu_m}x(t))}\right)|_{t=0} = 0 \\ \lambda_4 \int_0^1 \ {}^{C}D_{0^+}^{\beta_2}\left(\frac{x(s)-h(s,x(s),I^{\gamma_1}x(s),I^{\gamma_2}x(s),\ldots,I^{\gamma_n}x(s))}{g(s,x(s),I^{\mu_1}x(s),I^{\mu_2}x(s),\ldots,I^{\mu_m}x(s))}\right) ds \\ +\lambda_5 \ {}^{C}D_{0^+}^{\alpha_2}\left(\frac{x(t)-h(t,x(t),I_{0^+}^{\gamma_1}x(t),I_{0^+}^{\gamma_2}x(t),\ldots,I_{0^+}^{\gamma_n}x(t))}{g(t,x(t),I_{0^+}^{\mu_1}x(t),I_{0^+}^{\mu_2}x(t),\ldots,I_{0^+}^{\mu_m}x(t))}\right)|_{t=1} \\ +\lambda_6\left(\frac{x(t)-h(t,x(t),I_{0^+}^{\gamma_1}x(t),I_{0^+}^{\gamma_2}x(t),\ldots,I_{0^+}^{\gamma_n}x(t))}{g(t,x(t),I_{0^+}^{\mu_1}x(t),I_{0^+}^{\mu_2}x(t),\ldots,I_{0^+}^{\mu_m}x(t))}\right)|_{t=0} = 0. \end{cases}$$

where $1 < \omega \leq 2, \beta_1, \beta_2 \in (0,1], \alpha_1, \alpha_2 \in (0,1], \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \mathbb{R}^+$ and $\gamma_i > 0, \mu_j > 0$ with $i \in \{1,2, \dots n\}$ and $j \in \{1,2, \dots m\}$.

Building upon the previous works, we establish an existence result for a class of fractional hybrid integro-differential problem.

$$^{C}D_{0^+}^{\alpha}\left(\frac{u(t)}{f(t,u(t))}\right) + g(t, I_{0^+}^{\mu_1}u(t), I_{0^+}^{\mu_2}u(t), \dots, I_{0^+}^{\mu_n}u(t)) + \int_0^t K\,(t,s,u(s)ds = 0, t \in I \qquad (1)$$

$$u(0) = h(u), \;\; aD\left(\frac{u(t)}{f(t,u(t))}\right)|_{t=0} + b\,{}^{C}D_{0^+}^{\alpha}\left(\frac{u(t)}{f(t,u(t))}\right)|_{t=1} = 0. \qquad (2)$$

where $\alpha \in (1,2], a, b \in \mathbb{R}, I := [0,1]$. Also, ${}^{C}D_{0^+}^{\alpha}$ denotes the fractional Caputo derivative of order $\alpha, I_{0^+}^{\mu_i}$ denotes the fractional Riemann–Liouville integral of order $\mu_i > 0$ for all $i \in \{1,2, \dots, n\}$, and the maps $f: I \times \mathbb{R} \to \mathbb{R}^*$, $g: I \times \mathbb{R}^{n+1} \to \mathbb{R}$ and $K: I \times I \times \mathbb{R} \to \mathbb{R}$ are continuous.

This paper proceeds as follows: Some fundamental preliminaries are revisited in Section 2. In Section 3, we present the equivalent fractional integral equation corresponding to the linear part of the hybrid fractional differential equation ([1],[2]), and we prove the main existence result of this paper.One example is given in Section 4 to support the established findings.

Preliminaries

In this section, we present definitions and properties of fractional integration and differentiation, as well as the fixed point theorem employed in this work. For further details, the reader may refer to references [[10], [13], [14] , [8]].

Definition 1. *Let g be a real function defined on $[0,1]$ and $\alpha > 0$. Then the left and right Riemann-Liouville fractional integrals of order α of g are defined respectively by* $I_{0^+}^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{g(s)}{(t-s)^{1-\alpha}}ds$

$I_{1^-}^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)}\int_t^1 \frac{g(s)}{(s-t)^{1-\alpha}}ds$

Definition 2. *The left and the right Caputo fractional derivative of order $\alpha > 0$, of a function g are, respectively* ${}^{C}D_{0^+}^{\alpha}g(t) = (I_{0^+}^{n-\alpha}\frac{d^n}{dt^n}g(t))$ ${}^{C}D_{1^-}^{\alpha}g(t) = (-1)^n(I_{1^-}^{n-\alpha}\frac{d^n}{dt^n}g(t))$ *where* $n-1 < \alpha < n$.

Proposition 3. *Let $n-1 < \alpha < n$ and $f \in L_1[0\,,1]$. Then* (1) $I_{0^+}^{\alpha}\,{}^{C}D_{0^+}^{\alpha}f(t) = f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}t^k$

(2) $I_{1^-}^{\alpha}\,{}^{C}D_{1^-}^{\alpha}f(t) = f(t) - \sum_{k=0}^{n-1}\frac{(-1)^k f^{(k)}(1)}{k!}(1-t)^k$

Theorem 4. *(Dhage fixed point theorem**)[**8]) Let M be a closed, bounded, convex and nonempty subset of a Banach algebra $(E, \|.\|)$, and let $A: E \to E$ and $B: M \to E$ be two operators such that* ***(i)*** *A is Lipschitzian with Lipschitz constant λ,*

(ii) *B is completely continuous,,*

(iii) $x = AxBz \Rightarrow x \in M\ for\ all\ z \in M$,

(iv) $\lambda L < 1$, *where* $L = \| B(M) \| = sup\{\| B(x) \| : x \in M\}$.

Then the operator equation $AyBy = y$ has a solution in M.

Main results

Lemma 5. *Let* $y \in AC([0,1], \mathbb{R})$ *Then* u *is a solution of the hybrid fractional integrodifferential problem*

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}\left(\frac{u(t)}{f(t,u(t))}\right) + y(t) = 0, \quad t \in I := [0,1], \\ u(0) = h(u), \;\; aD\left(\frac{u(t)}{f(t,u(t))}\right)|_{t=0} + b\,{}^{C}D_{0^+}^{\alpha}\left(\frac{u(t)}{f(t,u(t))}\right)|_{t=1} = 0. \end{cases} \quad (3)$$

if and only if u *is a solution for the integral equation*

$$u(t) = f(t,u(t))\left[-\int_0^t \frac{(t-\tau)^{\alpha}-1}{\Gamma(\alpha)} y(\tau)d\tau + \frac{\Gamma(3-\alpha)bt}{a\Gamma(3-\alpha)+b}\int_0^1 y(\tau)d\tau + \frac{h(u)}{f(0,h(u))}\right] \quad (4)$$

Proof. we apply the right-hand side fractional integral $I_{0^+}^{\alpha}$ to equation ([3]).We get

$$\frac{u(t)}{f(t,u(t))} = -\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)d\tau + c_1 + c_2 t. \quad (5)$$

Using the conditions nonlocal $u(0) = h(u)$, so $c_1 = \frac{h(u)}{f(0,h(u))}$.and we have

$${}^{C}D_{0^+}^{\alpha-1}\left(\frac{u(t)}{f(t,u(t))}\right) = -\int_0^t y(\tau)d\tau + c_2\frac{t^{2-\alpha}}{\Gamma(3-\alpha)},$$

and

$$D\frac{u(t)}{f(t,u(t))} = -\int_0^t \frac{(t-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} y(\tau)d\tau + c_2,$$

so

$$aD\left(\frac{u(t)}{f(t,u(t))}\right)|_{t=0} + b\,{}^{C}D_{0^+}^{\alpha}\left(\frac{u(t)}{f(t,u(t))}\right)|_{t=1} = ac_2 + b\left(-\int_0^1 y(\tau)d\tau + \frac{c_2}{\Gamma(3-\alpha)}\right)$$

then we get

$$c_2 = \frac{\Gamma(3-\alpha)b}{a\Gamma(3-\alpha)+b}\int_0^1 y(\tau)d\tau$$

Substituting the values of c_1, c_2 in ([5]), we get solution ([4]). The converse follows by direct computation. This completes the proof.

In the sequel, we need the following assumptions.

(H1) The function $f: I \times \mathbb{R} \to \mathbb{R} - \{0\}$ is a continuous function satisfying the Lipschitz condition for a constant λ_f

$$|\, f(t,u) - f(t,v) \,| \leq \lambda_f \,|\, u - v \,|.$$

(H2) The function $h: \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists a constant $M_0 > 0$ such that : $|\frac{h(u)}{f(0,h(u))}| \leq M_0$.

(H3) The function $g: I \times \mathbb{R}^{n+1} \to \mathbb{R}$ is a continuous function and there exists a bounded mapping $\theta: I \times \to \mathbb{R}^{+}$ such that for all $u_i, v_i \in E$,

$\mid g(t,u_1(t),u_2(t),\ldots,u_{n+1}(t)) - g(t,v_1(t),v_2(t),\ldots,v_{n+1}(t)) \mid \leq \theta(t)\sum_{i=1}^{n+1} \mid u_i(t) - v_i(t) \mid,$

(H4) The function $K: I \times I \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function and there exists a constant $M_1 > 0$such that :

$$Max\{K(t,s,u(s)):\ t,s \in I;\ \mid u(s) \mid \leq R\} \leq M_1$$

where $R = \frac{M_f \Upsilon}{1-\lambda_f \Upsilon}$ and

$$\Upsilon = \left[\frac{1}{\Gamma(\alpha+1)} + \frac{b\Gamma(3-\alpha)}{a\Gamma(3-\alpha)+b}\right][\theta^* \xi R + G^* + M_1] + M_0$$

where $\theta^* = sup_{t\in I}\theta(t),\ \ G^* = sup_{t\in I} \mid g(t,0,0,\ldots,0) \mid,\ \ M_f = sup_{t\in I} f(t,0),\ \ and\ \ \xi = 1 + \sum_{i=1}^{n} \frac{1}{\Gamma(1+\mu_i)}$

Theorem 6. *Assume that conditions* ***(H1)–(H4)*** *hold.and if* $\lambda_f \Upsilon < 1$.*then the problem ([1]-[2]) has at least one solution in* $E = C([0,1])$

Proof. We consider a subset Ω of E given by $\Omega = \{u \in E: \parallel u \parallel_E \leq R\}$ and we define the operators $A: E \to E$ and $B: \Omega \to E$ as follows:

$$Au(t) = f(t,u(t)). \qquad t \in I,$$

$$\begin{aligned} Bu(t) = &-\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left[g(\tau, I_{0^+}^{\mu_1}u(\tau), I_{0^+}^{\mu_2}u(\tau),\ldots, I_{0^+}^{\mu_n}u(\tau)) + \int_0^\tau K(\tau,s,u(s)ds\right]d\tau \\ &+ \frac{b\Gamma(3-\alpha)t}{a\Gamma(3-\alpha)+b}\int_0^1\left[g(\tau, I_{0^+}^{\mu_1}u(\tau), I_{0^+}^{\mu_2}u(\tau),\ldots, I_{0^+}^{\mu_n}u(\tau)) + \int_0^\tau K(\tau,s,u(s)ds\right]d\tau \\ &+ \frac{h(u)}{f(0,h(u))}. \end{aligned}$$

Obviously, problem ([1]-[2]) has a solution if and only if $AxBx$ has a fixed point. Now, we show that the operators A and B satisfy all the conditions of theorem 4 in a series of steps.

claim 1 A is a Lipschitz on E Let $u,v \in E$ for all $t \in I$.

Then in view of condition **(H1)**, we get

$$\mid Au(t) - Av(t) \mid = \mid f(t,u(t)) - f(t,v(t)) \mid \leq \lambda_f \mid u - v \mid$$

Then, for each $t \in I$ we obtain $\parallel Au - Av \parallel_E \leq \lambda_f \parallel u - v \parallel_E$

claim 2 B is completely continuous on Ω. We firstly show that$B$ is uniformly bounded for any $u \in \Omega$, by **(H2)-(H5)**, we have.

$$
\begin{aligned}
|Bu(t)| \leq & \int_0^t \frac{|t-\tau|^{\alpha-1}}{\Gamma(\alpha)} \Big[| g(\tau, u(\tau), I_{0^+}^{\mu_1} u(\tau), I_{0^+}^{\mu_2} u(\tau), \ldots, I_{0^+}^{\mu_n} u(\tau)) | + \int_0^\tau | K(\tau, s, u(s) ds | \Big] d\tau \\
& + \frac{b\Gamma(3-\alpha)t}{a\Gamma(3-\alpha)+b} \int_0^1 \Big[| g(\tau, u(\tau), I_{0^+}^{\mu_1} u(\tau), I_{0^+}^{\mu_2} u(\tau), \ldots, I_{0^+}^{\mu_n} u(\tau)) | + \int_0^\tau | K(\tau, s, u(s) | ds \Big] d\tau \\
& + \left| \frac{h(u)}{f(0, h(u))} \right|, \\
\leq & \int_0^t \frac{|t-\tau|^{\alpha-1}}{\Gamma(\alpha)} [| g(\tau, u(\tau), I_{0^+}^{\mu_1} u(\tau), I_{0^+}^{\mu_2} u(\tau), \ldots, I_{0^+}^{\mu_n} u(\tau)) - g(t, 0, \ldots, 0) | \\
& + | g(t, 0, \ldots, 0) | + \int_0^\tau | K(\tau, s, u(s) | ds] d\tau \\
& + \frac{b\Gamma(3-\alpha)t}{a\Gamma(3-\alpha)+b} \int_0^1 [| g(\tau, u(\tau), I_{0^+}^{\mu_1} u(\tau), I_{0^+}^{\mu_2} u(\tau), \ldots, I_{0^+}^{\mu_n} u(\tau)) - g(t, 0, \ldots, 0) | \\
& + | g(t, 0, \ldots, 0) | + \int_0^\tau | K(\tau, s, u(s) | ds] d\tau \\
& + \left| \frac{h(u)}{f(0, h(u))} \right|, \\
\leq & \frac{1}{\Gamma(\alpha+1)} \left[\theta^* \left(1 + \frac{1}{\Gamma(\mu_1+1)} + \frac{1}{\Gamma(\mu_2+1)} + \ldots + \frac{1}{\Gamma(\mu_n+1)} \right) | u | + G^* + M_1 \right] \\
& + \frac{b\Gamma(3-\alpha)}{a\Gamma(3-\alpha)+b} \left[\theta^* \left(1 + \frac{1}{\Gamma(\mu_1+1)} + \frac{1}{\Gamma(\mu_2+1)} + \ldots + \frac{1}{\Gamma(\mu_n+1)} \right) | u | + G^* + M_1 \right] + M_0 \\
\leq & \left[\frac{1}{\Gamma(\alpha+1)} + \frac{b\Gamma(3-\alpha)}{a\Gamma(3-\alpha)+b} \right] [\theta^* \xi R + G^* + M_1] + M_0 \\
\leq & \Upsilon
\end{aligned}
$$

Thus, we get $\| Bu \|_E \leq \Upsilon$for all $u \in \Omega$, this proves that B is uniformly bounded in Ω. Next we show that B is continuous on Ω. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in Ω converging to a point $u \in \Omega$.Then by **(H2)-(H4)**, for all $t \in I$, one has

$$
\begin{aligned}
| Bu_n(t) - Bu(t) | \leq & \int_0^t \frac{|t-\tau|^{\alpha-1}}{\Gamma(\alpha)} [| g(\tau, u_n, I_{0^+}^{\mu_1} u_n(\tau), I_{0^+}^{\mu_2} u_n(\tau), \ldots, I_{0^+}^{\mu_n} u_n(\tau)) \\
& - g(\tau, u(\tau), I_{0^+}^{\mu_1} u(\tau), I_{0^+}^{\mu_2} u(\tau), \ldots, I_{0^+}^{\mu_n} u(\tau)) | + \int_0^\tau | K(\tau, s, u_n(s) - K(\tau, s, u(s) \\
& | ds] d\tau \\
& + \frac{b\Gamma(3-\alpha)t}{a\Gamma(3-\alpha)+b} \int_0^1 [| g(\tau, u_n, I_{0^+}^{\mu_1} u_n(\tau), I_{0^+}^{\mu_2} u_n(\tau), \ldots, I_{0^+}^{\mu_n} u_n(\tau)) \\
& - g(\tau, u(\tau), I_{0^+}^{\mu_1} u(\tau), I_{0^+}^{\mu_2} u(\tau), \ldots, I_{0^+}^{\mu_n} u(\tau)) | + \int_0^\tau | K(\tau, s, u_n \\
& (s) - K(\tau, s, u(s) | ds] d\tau \\
& + \left| \frac{h(u_n)}{f(0, h(u_n))} - \frac{h(u_n)}{f(0, h(u_n))} \right|, \\
\leq & \frac{1}{\Gamma(\alpha+1)} [\theta^* \left(1 + \frac{1}{\Gamma(\mu_1+1)} + \frac{1}{\Gamma(\mu_2+1)} + \ldots + \frac{1}{\Gamma(\mu_n+1)} \right) | u_n - u | \\
& + k_0 | u_n - u |] \\
& + \frac{b\Gamma(3-\alpha)}{a\Gamma(3-\alpha)+b} [\theta^* \left(1 + \frac{1}{\Gamma(\mu_1+1)} + \frac{1}{\Gamma(\mu_2+1)} + \ldots + \frac{1}{\Gamma(\mu_n+1)} \right) | u_n - u | \\
& + k_0 | u_n - u |] \\
& + \left| \frac{h(u_n)}{f(0, h(u_n))} - \frac{h(u_n)}{f(0, h(u_n))} \right|, \\
\leq & \left[\frac{1}{\Gamma(\alpha+1)} + \frac{b\Gamma(3-\alpha)}{a\Gamma(3-\alpha)+b} \right] [\theta^* \xi + k_0] \| u_n - u \|_E \\
& + \left| \frac{h(u_n)}{f(0, h(u_n))} - \frac{h(u_n)}{f(0, h(u_n))} \right|,
\end{aligned}
$$

Since that functions h and f are continuous, we deduce

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$$\| Bu_n - Bu \|_E \to 0 \quad as \quad n \to \infty.$$

Then B is continuous.

Next we prove that the operator$B$ equicontinuous. Let $u \in \Omega$ and $t_1, t_2 \in I$ with $t_1 < t_2$ Then we have

$$\begin{aligned}
| Bu(t_2) - Bu(t_1) | \leq & \, | \int_0^{t_2} \frac{(t_2-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left[g(\tau, u(\tau), I_{0^+}^{\mu_1} u(\tau), I_{0^+}^{\mu_2} u(\tau), \ldots, I_{0^+}^{\mu_n} u(\tau)) + \int_0^{\tau} K(\tau, s, u(s) ds \right] d\tau \\
& + \frac{b\Gamma(3-\alpha) t_2}{a\Gamma(3-\alpha)+b} \int_0^1 \left[g(\tau, u(\tau), I_{0^+}^{\mu_1} u(\tau), I_{0^+}^{\mu_2} u(\tau), \ldots, I_{0^+}^{\mu_n} u(\tau)) + \int_0^{\tau} K(\tau, s, u(s) ds \right] d\tau \\
& - \int_0^{t_1} \frac{(t_1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left[g(\tau, u(\tau), I_{0^+}^{\mu_1} u(\tau), I_{0^+}^{\mu_2} u(\tau), \ldots, I_{0^+}^{\mu_n} u(\tau)) + \int_0^{\tau} | K(\tau, s, u(s) ds \right] d\tau \\
& + \frac{b\Gamma(3-\alpha) t_1}{a\Gamma(3-\alpha)+b} \int_0^1 \left[g(\tau, u(\tau), I_{0^+}^{\mu_1} u(\tau), I_{0^+}^{\mu_2} u(\tau), \ldots, I_{0^+}^{\mu_n} u(\tau)) + \int_0^{\tau} K(\tau, s, u(s) ds \right] d\tau |
\end{aligned}$$

$$\begin{aligned}
\leq & \int_0^{t_1} | \frac{(t_2-\tau)^{\alpha-1} - (t_1-\tau)^{\alpha-1}}{\Gamma(\alpha)} | [\quad | g(\tau, u(\tau), I_{0^+}^{\mu_1} u(\tau), I_{0^+}^{\mu_2} u(\tau), \ldots, I_{0^+}^{\mu_n} u(\tau)) + \int_0^{\tau} K(\tau, s, u(s) ds |] d\tau \\
& + \int_{t_1}^{t_2} | \frac{(t_2-\tau)^{\alpha-1}}{\Gamma(\alpha)} | [\quad | g(\tau, u(\tau), I_{0^+}^{\mu_1} u(\tau), I_{0^+}^{\mu_2} u(\tau), \ldots, I_{0^+}^{\mu_n} u(\tau)) + \int_0^{\tau} K(\tau, s, u(s) ds |] d\tau \\
& + \frac{b\Gamma(3-\alpha) | t_2 - t_1 |}{a\Gamma(3-\alpha)+b} \int_0^1 [\quad | g(\tau, u(\tau), I_{0^+}^{\mu_1} u(\tau), I_{0^+}^{\mu_2} u(\tau), \ldots, I_{0^+}^{\mu_n} u(\tau)) | + \int_0^{\tau} | K(\tau, s, u(s) ds |] d\tau \\
\leq & \left[\frac{\theta^* \xi R + G^* + M_1}{\Gamma(\alpha+1)} \right] [2 \mid t_2 - t_1 \mid^{\alpha} + t_2^{\alpha} - t_1^{\alpha}] \\
& + \frac{b\Gamma(3-\alpha) | t_2 - t_1 |}{a\Gamma(3-\alpha)+b} [\theta^* \xi R + G^* + M_1]
\end{aligned}$$

which is independent of $u \in \Omega$. As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. Therefore, it follows from the Arzel´a-Ascoli theorem that B is a completely continuous operator on Ω.

claim 3 Now we show that the (iii) hypothesis of theorem 4 is satisfied. Let $u \in E$ and $v \in \Omega$ such that $u = AuBv$ Then, for $t \in I$ we have

$$\begin{aligned}
| u(t) | \leq & | Au(t) || Bv(t) | \\
\leq & | f(t, u(t)) | [\quad | - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left[g(\tau, I_{0^+}^{\mu_1} u(\tau), I_{0^+}^{\mu_2} u(\tau), \ldots, I_{0^+}^{\mu_n} u(\tau)) + \int_0^{\tau} K(\tau, s, u(s) ds \right] d\tau \\
& + \frac{b\Gamma(3-\alpha) t}{a\Gamma(3-\alpha)+b} \int_0^1 \left[g(\tau, I_{0^+}^{\mu_1} u(\tau), I_{0^+}^{\mu_2} u(\tau), \ldots, I_{0^+}^{\mu_n} u(\tau)) + \int_0^{\tau} K(\tau, s, u(s) ds \right] d\tau \\
& + \frac{h(u)}{f(0, h(u))} |] \\
\leq & [| f(t, u(t)) - f(t, 0) | + | f(t, 0) |] \left(\left[\frac{1}{\Gamma(\alpha+1)} + \frac{b\Gamma(3-\alpha)}{a\Gamma(3-\alpha)+b} \right] [\theta^* \xi R + G^* + M_1] + M_0 \right) \\
\leq & \left[\lambda_f \mid u \mid + M_f \right] \Upsilon.
\end{aligned}$$

Thus, we obtain

$$\| u(t) \|_E \leq \frac{M_f \Upsilon}{1 - \lambda_f \Upsilon} = R$$

Then $u \in \Omega$, thus the (iii) hypothesis of theorem 4 is satisfied. **claim 4** Now, we show that $\lambda_f L < 1$, where $L = \| B(\Omega) \|_E = \sup\{\| Bu \|_E : u \in \Omega\}$ Since $L = \sup_{u\in\Omega}\{\sup_{t\in I} \mid Bu(t) \mid\} \leq \Upsilon$, then $\lambda_f \Upsilon < 1$,

Thus all the conditions of theorem 4 are satisfied and hence the operator equation $u = AuBu$ has a solution in Ω. In consequence, problem ([1]-[2]) has a solution on I. This completes the proof

An Example

This section includes an example that showcases how Theorem 6 can be applied. Let us consider the following boundary value problem:

$$ {}^{C}D_{0^+}^{1.7}\left[\frac{u(t)}{f(t,u(t))}\right] + g(t,u(t),I_{0^+}^{0.7}u(t),I_{0^+}^{0.2}u(t)) + \int_0^t K(t,s,u(s)))ds = 0 \ ,\ t \in I = [0,\ 1]. \quad \mathbf{(6)} $$

$$ u(0) = h(u), \quad 100D\left(\frac{u(t)}{f(t,u(t))}\right)|_{t=0} + {}^{C}D_{0^+}^{0.7}\left(\frac{u(t)}{f(t,u(t))}\right)|_{t=1} = 0. \quad \mathbf{(7)} $$

here $\alpha = 1.7,\ \ \mu_1 = 0.7,\ \ \mu_2 = 0.2,\ \ a = 100$ and $b = 1$.

Where $f(t,u(t)) = \frac{7e^{-3t}}{15(t^2+2)}\frac{|u|+1}{|u|+2}$, and $h(u) = \frac{\sin(u)}{100+u^2}$,

and

$$ g(t,u(t),I_{0^+}^{0.7}u(t),I_{0^+}^{0.2}u(t)) = \frac{3}{100(t^2+1)}\left[u(t) + |\cos\left(I_{0^+}^{0.7}u(t)\right) - ch\left(I_{0^+}^{0.2}u(t)\right)|\right] + \frac{t}{100}, $$

and

$K(t,s,u(s)) = \frac{\cos(u^2)e^{-t}(s^2+1)}{100}$ Note that $M_f = \sup_{t\in I} \mid f(t,0) \mid = \frac{7e^{-3t}}{30(t^2+2)} = \frac{7}{60}$ and $M_0 = \mid \frac{h(u(t)}{f(0,h(u(t))} \mid = \frac{60}{70}$

Setting $M_1 = \frac{2}{100}$, $G^* = \sup_{t\in I} \mid g(t,0,\ldots,0) \mid = 0.01$, and we have

$$ \begin{aligned} \mid f(t,u) - f(t,v) \mid &\leq \frac{7e^{-3t}}{15(t^2+2)} \left| \frac{u+1}{u+2} - \frac{v+1}{v+2} \right| \\ &\leq \frac{7}{30} \left| \frac{|v-u|}{|v+2||u+2|} \right| \leq \frac{7}{30} \mid v-u \mid. \end{aligned} $$

then $\lambda_f = \frac{7}{30}$, for $u, v \in \mathbb{R}$, we have

$$ \begin{aligned} &\mid g(t,v(t),I_{0^+}^{0.7}v(t),I_{0^+}^{0.2}v(t)) - g(t,u(t),I_{0^+}^{0.7}u(t),I_{0^+}^{0.2}u(t)) \mid \\ &\leq \frac{3}{100(t^2+1)}\left(1 + \frac{t^{0.2}}{\Gamma(1.2)} + \frac{t^{0.7}}{\Gamma(1.7)}\right) \mid v-u \mid \\ &\leq \frac{3}{100(t^2+1)}\left(1 + \frac{1}{\Gamma(1.2)} + \frac{1}{\Gamma(1.7)}\right) \mid v-u \mid \end{aligned} $$

Thus, the assumption (A2) holds true with $\theta(t) = \frac{3}{100(t^2+1)}$ and $\theta^* = 0.03$ and $\xi = 3.189672$. By the above data, we get $\Upsilon = 0.939755$ and $\lambda_f \Upsilon = 0.219276 < 1$.. Thus, we can choose

$0.935712 < R < 54.19965$. Accordingly, all the conditions of Theorem 6 are fulfilled, the hybrid fractional problem (6-7) has at least one solution on [0, 1].

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