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Fuzzy Reliability, Bayesian Estimation and Goodness of Fit Test for A Novel One Parameter Model with Simulation Study and Application

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Abstract: This study investigates the goodness-of-fit test, fuzzy reliability analysis, and Bayesian estimation for a novel one-parameter probability distribution. Specifically, we introduce and analyze the exponential-Lindley and exponential-X-Lindley distributions as extensions of the proposed model. Using comprehensive analytical techniques, several key statistical properties of the distribution are derived and thoroughly examined. To assess the model's behavior under uncertainty, fuzzy reliability measures are developed, demonstrating its robustness and practical applicability in scenarios involving imprecise or vague data. Furthermore, a variety of parameter estimation methods—including classical and Bayesian approaches—are explored to assess the flexibility and precision of the proposed model. A simulation study is conducted using randomly generated datasets to evaluate the performance of the estimation techniques and to gain deeper insights into the model's adaptability across different conditions. Finally, the model's adequacy is validated using a goodness-of-fit test, confirming its potential usefulness in reliability and lifetime data analysis.

Keywords: Exponential XLindley distribution, goodness-of-fit test, Fuzzy reliability function, numerical simulations.

1. Introduction

A finite mixed distribution is created when a finite number of probability distributions are combined with a mixed proportion, resulting in new probability models. These distributions are particularly useful for simulating random events, especially when accounting for undetected data heterogeneity. The mixed distribution model is widely recognized in statistical data modeling, as data sets can often be thought of as mixed populations. Consequently, many researchers are highly interested in studying mixtures of distributions. Notable contributions to this area of research include works by Bouchahed and Zeghdoudi (2018), Bousseba et al. (2024), Bouhadjar et al. (2022), Khodja et al. (2023), Saaidia et al. (2024), and Chouia and Zeghdoudi (2021).

One key area of engineering research is reliability analysis. However, accurately reporting the number of survivors can be challenging in some unforeseen circumstances. For example, human error may lead to incorrect failure recordings during testing, or components may not fail completely. In such cases, survival probabilities are better described as fuzzy real numbers rather than precise values.

Similarly, when failure rates cannot be determined with precision, they should be treated as fuzzy real numbers.

The probability reliability method has become a popular approach to addressing uncertain problems. As science and technology have advanced, researchers have recognized that uncertainty in engineering arises not only from randomness but also from fuzziness (Cremona et al., 1997; Song et al., 2012; Reuter et al., 2011). Fuzzy uncertainty is complex and diverse, and its mathematical expression differs from that of random uncertainty, making traditional probability reliability approaches insufficient to handle such issues.

One-parameter distributions have been extensively studied due to their balance between mathematical simplicity and practical applicability. With fewer parameters, these models are easier to interpret, estimate, and apply, particularly in scenarios involving limited or noisy data. They are especially useful in reliability analysis and survival studies, where collecting large datasets can be difficult. Moreover, one-parameter models can serve as building blocks for more complex distributions or as baseline comparisons when evaluating model performance. Their simplicity also reduces computational burden, making them attractive for both theoretical exploration and real-time applications.

Modeling lifetime data is a fundamental aspect of reliability engineering, survival analysis, and risk assessment. Classical lifetime models, such as the exponential, Weibull, and gamma distributions, have been extensively studied and applied. However, these models often fail to capture the diverse hazard rate behaviors observed in real-world scenarios (Gupta et al., 1998; Murthy et al., 2004). This limitation has prompted the development of new one-parameter distributions that offer both analytical tractability and flexibility in modeling real-world phenomena.

Uncertainty in observed data—stemming from incomplete information, linguistic vagueness, or measurement imprecision—poses a significant challenge to conventional statistical methods. Traditional probabilistic approaches are often inadequate in capturing such ambiguity. To address this, fuzzy set theory, introduced by Zadeh (1965), has increasingly been integrated into reliability modeling to provide more realistic representations of system performance under uncertainty (Bai & Wang, 1993; Li & Pham, 2005). Fuzzy reliability functions enable the analysis of systems when failure data is imprecise or based on expert judgment, making them applicable in fields such as mechanical systems, medical diagnostics, and software reliability.

Furthermore, Bayesian estimation methods have gained prominence in lifetime data analysis due to their ability to incorporate prior knowledge and generate full posterior distributions, particularly beneficial in small-sample or uncertain environments (Box & Tiao, 1973; Bernardo & Smith, 1994). Unlike frequentist methods, Bayesian inference offers a coherent framework for updating beliefs in the presence of new evidence, making it especially suitable for reliability and risk applications where prior expertise is often available.

Motivated by the need for a more adaptable and uncertainty-aware framework, this study proposes a novel one-parameter probability model with desirable statistical properties and practical relevance. The model is analyzed from multiple perspectives:

1. Fuzzy reliability analysis is developed to handle vague or imprecise failure times.

2. Bayesian estimation procedures are formulated under different loss functions and prior structures.

3. A goodness-of-fit test is proposed to validate the empirical adequacy of the model using real and simulated data.

4. A simulation study is conducted to evaluate the performance of the estimators and compare the model's flexibility against classical distributions.

The proposed framework aims to bridge the gap between traditional reliability analysis and modern uncertainty modeling, contributing significantly to applied statistics and engineering fields.

In this work, we primarily investigate the fuzzy reliability of one-parameter models, specifically the exponential XLindley distribution suggested by Grabsia and Grine (2025) and the exponential Lindley distribution presented by Belhamra et al. (2022). Lastly, we examine the one-parameter model (OPM) as a particular instance of the two-parameter family. The density is

$$p(x;\omega) = \frac{\omega}{3}(1+2\omega x)exp^{-\omega x}$$

The corresponding cumulative distribution function (CDF), the survival function (SF) and hazard rate function (HRF) are given by

$$P(x;\omega) = 1 - e^{-\omega x}\left(1+\tfrac{2}{3}\omega^2 x\right), x, \omega \succ 0.$$

$$S(t;\omega) = e^{-\omega x}\left(1+\tfrac{2}{3}\omega^2 x\right), x, \omega \succ 0.$$

$$h(t;\omega) = \frac{\omega^2(1+2\omega x)}{2\omega^2 x+\frac{1}{3}}$$

This is how the remainder of the paper is structured. The new model is explained in the second section, and its statistical characteristics are shown in the third. The comparative study around the fuzzy reliability of three new models in Section 4. The goodness-of-fit test and the numerical simulation of the new model are presented in Section 5. Ultimately, we examine the Bayesian estimators under different loss functions in Section 6.

2. A general theoretical result

Asymptotic behaviour

The shape properties of the PDF and HRF of OPM are covered in this subsection in (ref:PDF) and (ref:HRF), respectively. The OPM behavior at $at\ x = 0\ and\ x = \infty$, respectively, is provided by

$$\lim_{x\to 0} p(x;\omega) = \frac{\frac{1}{3}\omega^2}{\omega\frac{1}{3} + (\frac{2}{3})\omega} = \frac{1}{3}\omega$$

$$\lim_{x\to\infty} p(x;\omega) = 0.$$

The behavior of $h(x;\omega)\ at\ x = 0\ and\ x = \infty$, respectively, are given by

$$\lim_{x\to 0} h(x;\omega) = \frac{\frac{1}{3}\omega^2}{\omega\frac{1}{3} + (\frac{2}{3})\omega} = \frac{1}{3}\omega$$

$$\lim_{x\to\infty} h(x;\omega) = \omega.$$

According to the following claim, the range of the parameters ω for the PDF of the one-parameter polynomial exponential distribution.

Moments and related measures of OPM

Let $X \sim OPM$, Then the i th moment of X is determined as follows

$$E(X^i) = \frac{\Gamma(1+i)}{\omega^{2+i}}[\frac{1}{3}\omega + \frac{2}{3}\omega(1+i)]$$

Hence, the first four moments of the OPM random variable can be found by substituting i=1,2,3,4, respectively, in Equation (ref:MM). They are used to determine variance, Skewness, Kurtosis and coefficient of variation of OPM, respectively, as follows

$$Var(X) = E(X^2) - E(X) = \frac{2\frac{1}{3}\omega^3 + 6(\frac{2}{3})\omega^3 - (\frac{2}{3})^2\omega^2 - 4\frac{1}{3}(\frac{2}{3})\omega^2 - 4(\frac{2}{3})^2\omega^2}{\omega^6}$$

$$= \frac{2(6\omega - 4) + (2\omega - \frac{1}{3})}{3\omega^4},$$

$$Skewness = \sqrt{\beta^1} = \frac{E(X^3)}{Var(X)^{\frac{3}{2}}} = \frac{6\omega^4\left(\left(\frac{8}{3}\right)\omega + \frac{1}{3}\omega\right)}{\left(\frac{2}{3}\omega^3 + 6\left(\frac{2}{3}\right)\omega^3 - \left(\frac{1}{3}\right)^2\omega^2 - \frac{4}{3}\left(\frac{2}{3}\right)\omega^2 - 4\left(\frac{2}{3}\right)^2\omega^2\right)^{\frac{3}{2}}} = \frac{18\omega^5}{\left(\frac{14}{9}\omega^3 - \frac{25}{9}\omega^2\right)^{\frac{3}{2}}}$$

$$Kurtosis = \beta^2 = \frac{E(X^4)}{\left(Var(X)\right)^2} = \frac{(24\omega^6(5(\frac{2}{3})\omega + \omega\left(\frac{1}{3}\right))}{\left(2\left(\frac{1}{3}\right)\omega^3 + 6\left(\frac{2}{3}\right)\omega^3 - \left(\frac{1}{3}\right)^2\omega^2 - 4\left(\frac{1}{3}\right)\left(\frac{2}{3}\omega^2 - 4\left(\frac{2}{3}\right)^2\omega^2\right)^2\right)}, =$$

$$\frac{88\omega^7}{\left(\frac{14}{3}\omega^3 - \frac{25}{9}\omega^2\right)^2}$$

$$C.V = K = \frac{\sqrt{Var(X)}}{E(X)} = \frac{\sqrt{2\frac{1}{3}\omega^3 + 6\left(\frac{2}{3}\right)\omega^2 - \left(\frac{2}{3}\right)^2\omega^2 - 4\frac{1}{3}\left(\frac{2}{3}\right)\omega^2 - 4\left(\frac{2}{3}\right)^2\omega^2}}{2(\left(\frac{2}{3}\right))\omega + \omega\left(\frac{1}{3}\right))}$$

$$= \frac{3}{5\omega}\sqrt{\frac{2}{3}\omega^3 + \frac{11}{9}\omega^2}$$

The moment generating function of the OPM is determined as follows

$$M(s) = \int e^{sx}\, p(x)dx = \frac{\omega^2\left(\left(\frac{2}{3}\right)\omega + \omega\frac{1}{3} - s\frac{1}{3}\right)}{\left(\left(\frac{2}{3}\right)\omega + \frac{1}{3}\omega\right)(\omega - s)^2}, s < \omega$$

$$= \frac{\omega(\omega - s\frac{1}{3})}{(\omega - s)^2}$$

its characteristic function is obtained by replacing t with it in the last equation.

The i-th incomplete moments of OPM is determined as follows

$$T_i(s) = \int_0^s t^i\, p(x)dx = \frac{\left(\frac{2}{3}\right)\omega\Gamma(2+i) + \omega\frac{1}{3}\Gamma(1+i) - \left(\frac{2}{3}\right)\omega\Gamma\left(2+i, s\omega\frac{1}{3}\right) - \frac{1}{3}\omega\Gamma\left(1+i, s\omega\frac{1}{3}\right)}{\omega^i(\frac{2}{3})\omega + \frac{1}{3}\omega^{i+1}}$$

where $\Gamma(\alpha, x) = \int_{-x}^{\infty} t^{\alpha-1}\, e^{-t}dt$. We have first incomplete moments $T_1(s)$ in equation (ref:IM) when $i = 1$ which used to calculate the mean residual life and the mean waiting time which are, respectively, defined as follows

$$\Psi(s) = \frac{1 - T_1(s)}{S(x;\omega) - 1}$$

$$M_1(s) = \frac{1 - T_1(s)}{F(x;\omega)}$$

Another uses of $T_1(s)$ is to calculate Bonferroni and Lorenz curves which are, respectively, defined as follows

$L(p) = \frac{T_1(x)}{E(X)}$

$B(p) = \frac{T_1(x_p)}{pE(X)}$,

Where(x_p) is the quantile function of OPM.

Stochastic orders

Stochastic Order is an order of -largeness -on random variables. More broadly, stochastic orders are orders that are used to compare random variables, or probability distributions or measurements.

Now, we consider 2 random variables V and W . Then V is said smaller than W in the :

- Likelihood ratio order $(V <_{lr} W), if\ \frac{p_v(x)}{p_w(x)}$ is decreasing in x

-Hazard rate order $(V \leq_{hr} W), if\ h_v(x) \geq h_w(x), \forall x$

- Stochastic order $(V <_S W\ ), if\ F_v(x) < F_w(x), \forall x$

- Convex order $(V \leq_{cx} W)$, if for all convex functions $\phi, E[\phi(V)] \leq E[\phi(W)]$(expectation exist).

Theorem 1. Let $V, W \sim OPM$ be two random variables. If

$(\frac{2}{3})\omega_1\frac{1}{3} \leq (\frac{2}{3})\omega_2\frac{1}{3}$ $, and\ \omega_1 \geq \omega_2$ then:$V <_{lr} W;\ V <_{hr} W;\ V <_s W\ and\ V \leq_{cx} W$

Proof. We have:

$$\frac{p_v(x)}{p_w(x)} = \frac{\frac{\omega_1{}^2(\frac{1}{3}+(\frac{2}{3})\omega_1 t)exp(-\omega_1 t))}{\frac{1}{3}+(\frac{2}{3_1})\omega_1 t}}{\frac{\omega_2{}^2(\frac{1}{3}+(\frac{2}{3})\omega_2 t)exp(-\omega_2 t)}{\frac{1}{3}+(\frac{2}{3})\omega_2 t}}$$

To keep it simple, we use $ln\frac{p_v(x)}{p_w(x)}$, which we find after derivation:

$$\frac{d}{dx}ln\frac{p_v(x)}{p_w(x)} = \frac{2(\omega_1-\omega_2)}{3(1+2\omega_1 t)+(1+2\omega_2 t)} - (\omega_1 - \omega_2)$$

In this regard, if $\omega_1\frac{1}{3} \leq \omega_2\frac{1}{3}$ and $\omega_1 \geq \omega_2$, we have $\frac{d}{dx}\ln(\frac{p_v(x)}{p_w(x)}) \leq 0$. It means that $V <_{lr} W$. Moreover, we know that $V <_{lr} W \Rightarrow V <_{hr} W \Rightarrow V <_s W$ and $V \leq_{cx} W \Leftrightarrow V <_S W (if\ E[V] = E[W])$, which the Theorem 1is proved.

Entropies

There is general agreement that entropy and information can be used to calculate the degree of uncertainty in a probability distribution. However, many correlations have been generated from the characteristics of entropy.

The entropy of a random variable X is a measurement of the variability of uncertainty. The entropy of Rényi is defined as:

$$I_R(s) = \frac{1}{(1-s)} log\int_0^\infty p^s\,(x)dx$$

Were s(integer)>0 et s≠1. For the OPM, we have:

$$I_R(s) = \frac{1}{(1-s)}\log\left(\int_0^\infty \left(\frac{\omega^2(\frac{1}{3}+(\frac{2}{3})\omega x)\exp\,(-\omega x))}{\frac{1}{3}\omega+\left(\frac{2}{3}\right)\omega}\right)^s dx\right)$$

$$= \frac{1}{(1-s)}\log\left(\int_0^\infty \frac{\omega^{s2}}{\left(\frac{1}{3}\omega+\left(\frac{2}{3}\right)\omega\right)^s}(\frac{1}{3}+(\frac{2}{3})\omega x)^s e^{-\omega xs}dx\right)$$

We observe that

$$\int_0^\infty \frac{\omega^{s2}}{\left(\frac{1}{3}\omega+\left(\frac{2}{3}\right)\omega\right)^s}(\frac{1}{3}+(\frac{2}{3})\omega x)^s e^{-\omega xs}dx = \frac{\omega^{s2}}{\left(\frac{1}{3}\omega+\left(\frac{2}{3}\right)\omega\right)^s}\sum_{i=0}^{n}\frac{n!\,(\frac{1}{3})^i((\frac{2}{3})\omega)^{n-i}}{(n-i)!\,i!}\int_0^\infty x^{n-i}\,e^{-\omega xs}dx$$

where

$$\int_0^\infty x^{n-i}\,e^{-\omega xs}dx = \frac{-1}{s\omega}\Gamma(n+1-i, s\omega x)(s\omega)^{i-n}$$

Now, the Rényi entropy observes as

$$I_R(s)=\frac{1}{(1-s)}log\left(\frac{\omega^{s2}}{\left(\frac{1}{3}\omega+\left(\frac{2}{3}\right)\omega\right)^s}\sum_{i=0}^{n}\frac{n!(\frac{1}{3})^i((\frac{2}{3})\omega)^{n-i}}{(n-i)!i!}\frac{(s\omega)^{i-n}\Gamma(n-i+1)}{(s\omega)^{n-i}}\right).$$

3. Goodness-of-fit test

As $t > 0$ the OPM considered as a lifetime model. In this section we develop the modified Chi-squared Goodness-of-Fit Test based on the Nikulin-Rao-Robson (NRR)(see Nikulin, M. (1973a,1973b) statistic for the OPM. A simulation study is conducted to generate a wide range of sample sizes and for various values of the parameter θ.

Consider the problem of testing the hypothesis H_0 according to which the distribution of n independent identically distributed random variables $T_1, T_2, \ldots, T_n$ belongs to the family of OPM

$$H_0: P(T_i \le t) = F(t) = 1 - e^{-\omega t}\left(1+\frac{2}{3}\omega^2 t\right), t, \omega \succ 0.$$

We divide the positive real line into r sub-intervals $I_1, I_2, \ldots, I$ by the points

$$0 = a_0 < a_1 < \cdots a_r = +\infty, I =]a_{i-1}, a_i], I_i \cap I_j = \emptyset, i, j = 1,2, \ldots, r; \bigcup_{i=1}^{r} I_i = R_+^*$$

And we group the sample $T_1, T_2, \ldots, T_n$ over these sub-intervals, we obtain the vector of frequencies $v = (v_1, v_2, \ldots, v_n\ )^T$ and the probability vector

Where $p_i(\omega) = \int_{a_{i-1}}^{a_i} f(t)dt = F(a_{i-1}) - F(a_i), i = 1,2,1, \ldots, r.$

To test the hypothesis H_0, Pearson proposed a test based on the so-called Pearson's χ^2 of the form

$$\chi_n^2(\omega) = \chi_n^T(\omega)\chi_n(\omega) = \sum_{i=1}^{r}\frac{(v_i - np_i(\omega))^2}{np_i(\omega)}$$

Where

$$\chi_n(\omega) = \left(\frac{v_1 - np_1(\omega)}{\sqrt{np_1(\omega)}}, \frac{v_2 - np_2(\omega)}{\sqrt{np_2(\omega)}}, \ldots, \frac{v_r - np_r(\omega)}{\sqrt{np_r(\omega)}}\right)^T$$

Under H_0 if ω is known, it was shown by Pearson (1900) that

$$\lim_{x\to\infty} P(\chi_n^2(\omega) \le t) = P(\chi_{r-1}^2 \le t)$$

The hypothesis H_0 must be rejected at a significance level α whenever $\chi_n^2(\omega) > C_\alpha$ where C_α is the critical value of the Pearson's test, $C_\alpha = \chi_{r-1,1-\alpha}^2$ is the upper α- quantile of the χ^2 distribution with $r-1$ degrees of freedom.

But generally ω is unknown and must be estimated using the sample $T_1, T_2, \ldots, T_n$. If we replace ω in Equation 1 by any consistent estimate ω_n^* , the limit distribution of Equation 2 will not χ_{r-1}^2 and changes dramatically, it depends both on the method of estimation of ω and the proprieties of the estimator ω_n^* .

Smith (1973 a, b) proposed to modify the standard Chi-squared Pearson's test (1) for continuous distribution with shift and scale parameters, also Rao *et al*. (1974) had obtained the same result for exponential family, the test is well known as the Nikulin-Rao-Robson (NRR) test Van Der Vaart (1998), and Drost (1988) and can be written as (Greenwood *et al*., 1996):

$$Y_n^2(\widehat{\omega}) = X_n^2(\widehat{\omega}) + X_n^T\left(\widehat{\theta\omega}\right)B(\widehat{\omega})(I(\widehat{\omega}) - J(\widehat{\omega}))^{-1}B^T(\widehat{\omega})X_n(\widehat{\omega})$$

With

$$B(\omega) = (p_1(\omega), p_2(\omega), \dots, p_r(\omega))^T, b_i(\omega) = \frac{1}{\sqrt{p_i(\omega)}}\frac{\partial p_i(\omega)}{\partial \omega}, i = 1, 2, \dots, r$$

And $nJ(\omega) = nB^T(\omega)B(\omega)$ is the Fisher's information matrix of the vector of frequencies, and I is the the Fisher's information matrix of T_i, such that

$$nI = -E\left(\frac{\partial^2 l(\omega)}{\partial \omega^2}\right),$$

Where

$$\frac{\partial^2 l(\omega)}{\partial \omega^2} = -\frac{n}{\omega^2} - \sum_{i=1}^{n} \frac{2{x_i}^2}{(1 + 2\omega x_i)^2}$$

The asymptotic behavior of the statistics $Y_n^2(\widehat{\omega})$ (Smith, 1973 a, b) is given by the following:

$$\lim_{x \to \infty} P(Y_n^2(\widehat{\omega}) \le t) = P(\chi_{r-1}^2 \le t)$$

Notice that Hsuan *et al*. (1976) (Mirvaliev,., 2001; Voinov *et al*., 2008; Voinov *et al*., 2009) have proposed a modification of the standard Pearson's test (1) by using the method of moments.

Simulation Study

For study empirically the behavior of the statistic Y_n^2, we generate from OPM samples size with n=20, 30, 50, 80, 100, 150, 200, 300, 500 and 1000, and for various values of the parameter ω=(0.01, 0.5, 1.5, 5, 10).

In this study, we choose the significance level α=0.05. Consider the case of equiprobability. Each sample (n, θ) is repeated 5000 times. For each operation, we compute the NRR statistic Y_n^2, then we calculate the empirical confidence level *E.L.* Which counts the number of times where $Y_n^2 \le C_\alpha = \chi_{r-1,1-\alpha}^2$ divided by 5000.

For the significance level α=0.05 , theoretically we have E.L.=1- α=0.95 . The results are grouped in Table 2 and Figure 2.

Quick reading of Table 1 indicates that the values of the empirical confidence level (*E.L.*) are very close to the theoretical value 0.95, which confirms the theorem (**3**) established by Smith (1973 a, b).

Table 1 - Empirical confidence level for ω=(0.01, 0.5, 1.5, 5, 10).

n	ω=0.01	ω=0.5	ω=1.5	ω=5	ω=10
20	0.9615	0.9498	0.9525	0.9509	0.9541
30	0.9535	0.9501	0.9512	0.9517	0.9558
50	0.9508	0.9584	0.9533	0.9596	0.9406
80	0.9596	0.9522	0.9580	0.9489	0.9541
100	0.9521	0.9557	0.9511	0.9602	0.9524
150	0.9408	0.9509	0.9522	0.9432	0.9525
200	0.9517	0.9523	0.9602	0.9577	0.9578
300	0.9608	0.9532	0.9589	0.9552	0.9516
500	0.9602	0.9511	0.9507	0.9502	0.9518
1000	0.9589	0.9589	0.9571	0.9578	0.9519

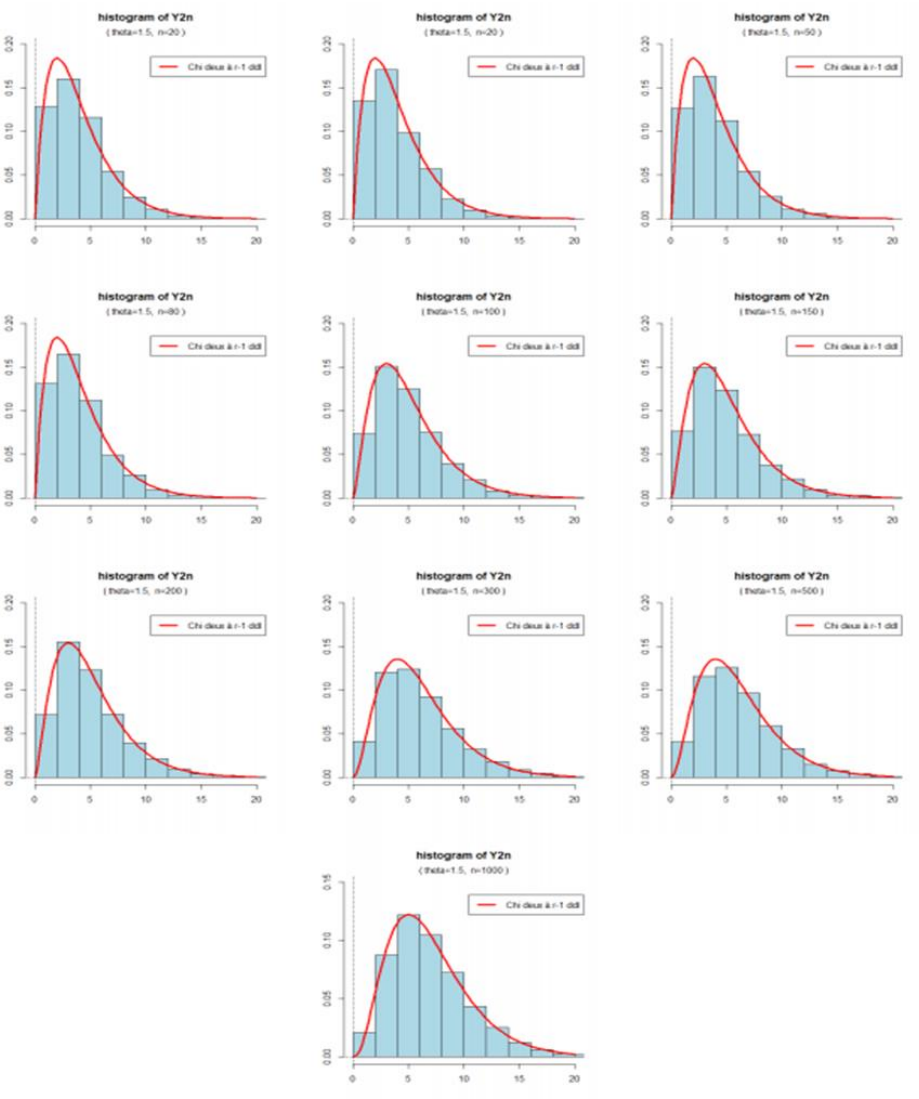

Figure 2. Histograms of $Y^2{}_n$ for ω = 1.5, and n = 20, 30, 50, 80, 100, 150, 200, 300, 500, 1000

4. Fuzzy reliability

In this section, we give a Fuzzy reliability with simulations for three new distributions such as:
- **New Compound Exponential-Lindley Distribution (Belhamra et al. (2023)**
- **Compound Exponential New XLindley Distribution (Grabsia and Grine (2025)**
- **Our new model (OPM)**

Let X be a continuous random variable that represents a system's failure time (component). The fuzzy dependability can then be calculated using the fuzzy probability in formula (see Chen et al. (2001)).

$$R_F(t) = P(T > t) = \int_t^{\infty} \mu(x) p_{OPM}(x)dx, 0 \leq t \leq x < \infty,$$

where *μ(x)* is a membership function that describes the degree to which each element of a given universe belongs to a fuzzy set. Now, assume that *μ(x)* is

$$\mu(x) = \begin{cases} 0 & , \quad x \leq t_1 \\ \dfrac{(x - t_1)}{(t_2 - t_1)} & , 0 \leq t_1 < x < t \\ 1 & , x \geq t_2 \end{cases}$$

For $\mu(x)$, by the computational analysis of the function of fuzzy numbers, the lifetime $x(\alpha)$ can be obtained corresponds to a certain value of $\alpha - Cut, \alpha \in [0,1]$, can by obtained as: $\mu(x) = \alpha \to \frac{x-t_1}{t_2-t_1} = \alpha$, then

$$\begin{cases} x(\alpha) \leq t_1 & , \quad \alpha = 0 \\ x(\alpha) = t_1 + \alpha(t_2 - t_1) & , 0 < \alpha < 1 \\ x(\alpha) \geq t_2 & , \alpha = 1 \end{cases}$$

As a result, the fuzzy reliability values may be determined for all α values. The fuzzy dependability of the *OPM* is determined by the fuzzy reliability definition. The fuzzy reliability of the *OPM* can be define as,

$$R_F(t) = (1 + (\frac{2}{3})\omega t_1)e^{-\omega t_1} - (1 + (\frac{2}{3})\omega x(\alpha))e^{-\alpha x(\alpha)}$$

Then $R_F(t)_{\alpha=0} = 0$.

Numerical values of fuzzy reliability

In this subsection, we obtained comparison between traditional reliability (see Finkelstein (2008)) and Fuzzy reliability, where the traditional reliability is a survival function as

$$R(x) = (1 + (\frac{2}{3})\omega x)e^{-\omega x}$$

Table 2 discussed the comparison. The following observations are based on findings:

- When the $\alpha - Cut$ is increased, the Fuzzy reliability increases.
- When the t_2 of interval of member ship function is increased the Fuzzy reliability increases.
- When the t_1 is decreased the fuzzy reliability increases, and vice versa.

- The traditional reliability with t_2 is lower than the traditional reliability with t_1.

The fuzzy estimation algorithm produces a series of draws from *OPM* as in algorithm 1.

Algorithm 1: fuzzy estimation algorithm

- **Input**: initial values of ω,interval time (t_1, t_2) and α where $0 < \alpha < 1$.
- **Calculate**: $x(\alpha) = t_1 + \alpha(t_2 - t_1)$.
- **For** each method do

Set: i=1.

Estimate parameter as $\widehat{\omega}$.

Calculate

$$\hat{R}_F(t) = (1 + (\frac{2}{3})\omega t_1)e^{-\omega t_1} - (1 + (\frac{2}{3})\omega x(\alpha))e^{-\omega x(\alpha)}$$

- **End**

Table2: Traditional and fuzzy reliability with different values.

					$\hat{R}_F(t) = R(t_1) - R(x)$			
ω	t_1	t_2	$R(t_1)$	$R(t_2)$	0.15	0.55	0.75	0.95
0.1	0.02	1	0.99933	0.96516	4.945×10^{-3}	1.8458×10^{-2}	2.5380×10^{-2}	3.2402×10^{-2}
0.25	0.04	1.5	0.99665	0.85911	1.8888×10^{-2}	7.2872×10^{-2}	0.10126	0.13023
0.5	0.5	2	0.9086	0.61313	4.4482×10^{-2}	0.16533	0.22449	0.28159
0.88	0.6	4	0.79739	9.9059×10^{-2}	0.17569	0.51876	0.61928	0.68577
0.95	0.13	1.75	0.95659	0.39988	8.9212×10^{-2}	0.33235	0.44055	0.53524
1.5	0.4	2.5	0.76834	0.082312	0.18155	0.52037	0.61456	0.67488
2	0.3	3	0.76834	1.2394×10^{-2}	0.29470	0.67317	0.72913	0.75269
2.5	0.1	2.75	0.9086	5.7692×10^{-3}	0.38125	0.83536	0.88435	0.90088
2.75	0.2	1.25	0.78850	0.10581	0.16914	0.5026	0.60255	0.66983

3	0.8	3.5	0.23587	2.2029×10^{-4}	0.14408	0.22999	0.23448	0.23555
3.5	0.75	1.5	0.19921	2.3614×10^{-2}	5.2012×10^{-2}	0.13573	0.15812	0.17281
4	0.95	3.25	7.9043×10^{-2}	2.1850×10^{-5}	0.05398	7.8063×10^{-2}	0.07886	7.9010×10^{-2}
5.25	0.66	4.75	0.10351	2.6056×10^{-10}	9.6701×10^{-2}	0.10351	0.10351	0.10351
6	0.9	0.66	2.0776×10^{-2}	6.9390×10^{-2}	0.65051	0.11803	4.4436×10^{-2}	1.6061×10^{-2}
7.5	0.7	0.2	2.3614×10^{-2}	0.44626	0.56814	6.0613×10^{-2}	1.7131×10^{-2}	4.6272×10^{-3}
8	0.07	1.6	0.78446	2.6319×10^{-5}	0.54215	4.8291×10^{-2}	1.2394×10^{-2}	3.0361×10^{-3}
8.5	0.34	7.5	0.16265	8.944×10^{-3}	0.51695	0.03839	8.944×10^{-3}	1.9866×10^{-3}
9.25	0.52	1.45	3.4275×10^{-2}	1.4876×10^{-5}	2.2915×10^{-2}	3.3747×10^{-2}	3.4166×10^{-2}	3.4253×10^{-2}

Fuzzy reliability of A New Compound Exponential-Lindley Distribution

$$\hat{R}_F(t) = \frac{(\beta+t_1)^2(1+\beta) - t_1(1+\beta)(\beta+t_1) - \beta t_1}{(\beta+t_1)^2(1+\beta)} - \frac{(\beta+x(\gamma))^2(1+\beta) - x(\gamma)(1+\beta)\big(\beta+x(\gamma)\big) - \beta x(\gamma)}{(\beta+x(\gamma))^2(1+\beta)}$$

Table 3: Traditional and fuzzy reliability for Exponential-Lindley Distribution with different values.

					$\hat{R}_F(t) = R(t_1) - R(x)$			
β	t_1	t_2	$R(t_1)$	$R(t_2)$	0.15	0.55	0.75	0.95
0.1	0.02	1	0.70707	1.5778×10^{-2}	0.5455	0.67234	0.684	0.69019
0.25	0.04	1.5	0.76694	4.4898×10^{-2}	0.47572	0.67935	0.70478	0.71935
0.5	0.5	2	0.33333	9.3333×10^{-2}	0.086214	0.19197	0.21799	0.23626
0.88	0.6	4	0.46638	0.10171	0.15537	0.30671	0.3388	0.36035

0.95	0.13	1.75	0.82533	0.2349	0.21109	0.47138	0.53580	0.58114
1.5	0.4	2.5	0.72299	0.28125	0.13323	0.33196	0.38947	0.4326
2	0.3	3	0.83176	0.32	0.15662	0.38642	0.45219	0.50136
2.5	0.1	2.75	0.95097	0.40492	0.15649	0.4024	0.47686	0.53384
2.75	0.2	1.25	0.91535	0.63021	0.057543	0.18158	0.23141	0.27506
3	0.8	3.5	0.74792	0.39941	0.085597	0.24163	0.29534	0.33889
3.5	0.75	1.5	0.79123	0.65333	0.0 24191	0.0 82156	0.10804	0.13213
4	0.95	3.25	0.77706	0.50226	5.8585×10^{-2}	0.17933	0.22584	0.26571
5.25	0.66	4.75	0.87245	0.4851	9.2807×10^{-2}	0.26574	0.32651	0.37628
6	0.9	0.66	0.85336	0.88815	-5.0452×10^{-3}	-1.8791×10^{-2}	-2.5829×10^{-2}	-3.2979×10^{-2}
7.5	0.7	0.2	0.90545	0.97105	-0.009274	-3.4953×10^{-2}	-4.8335×10^{-2}	-6.2099×10^{-2}
8	0.07	1.6	0.99037	0.8179	2.8396×10^{-2}	9.6785×10^{-2}	0.12749	0.15616
8.5	0.34	7.5	0.95765	0.50504	0.11314	0.31588	0.38478	0.44035
9.25	0.52	1.45	0.94186	0.85306	1.4473×10^{-2}	5.0991×10^{-2}	8.4760×10^{-2}	6.8200×10^{-2}

Fuzzy reliability of a Compound Exponential New XLindley

$$\hat{R}_F(t) = \frac{\beta(2\beta + t_1)}{2(t_1 + \beta)^2} - \frac{\beta(2\beta + x(\gamma))}{2(x(\gamma) + \beta)^2}$$

Table 4: Traditional and fuzzy reliability of Exponential New XLindley with different values.

					$\hat{R}_F(t) = R(t_1) - R(x)$			
β	t_1	t_2	$R(t_1)$	$R(t_2)$	0.15	0.55	0.75	0.95
0.1	0.02	1	0.76389	4.9587×10^{-2}	0.50649	0.6765	0.69857	0.71179
0.25	0.04	1.5	0.80262	8.1633×10^{-2}	0.43642	0.66209	0.69607	0.71697
0.5	0.5	2	0.375	0.12	8.7620×10^{-2}	0.20048	0.22967	0.25065
0.88	0.6	4	0.47407	0.10642	0.15519	0.30822	0.34105	0.36320
0.95	0.13	1.75	0.82669	0.23783	0.20985	0.46954	0.53404	0.57953
1.5	0.4	2.5	0.70637	0.25781	0.13847	0.34033	0.39738	0.43966
2	0.3	3	0.81285	0.28	0.16983	0.40905	0.47472	0.5228
2.5	0.1	2.75	0.94305	0.35147	0.17823	0.44516	0.52208	0.57946
2.75	0.2	1.25	0.9006	0.58008	6.6552×10^{-2}	0.20693	0.26202	0.30963
3	0.8	3.5	0.70637	0.33728	9.5157×10^{-2}	0.26144	0.31624	0.35963
3.5	0.75	1.5	0.75087	0.595	2.7882×10^{-2}	9.3777×10^{-2}	0.12277	0.1495
4	0.95	3.25	0.73054	0.42806	6.7486×10^{-2}	0.20162	0.25136	0.29308
5.25	0.66	4.75	0.83872	0.40031	0.11249	0.31002	0.37533	0.42712
6	0.9	0.66	0.81285	0.85626	-6.2565×10^{-3}	-2.3371×10^{-2}	-3.2171×10^{-2}	-4.1139×10^{-2}
7.5	0.7	0.2	0.87559	0.96138	-1.1979×10^{-2}	-4.5401×10^{-2}	-6.2967×10^{-2}	-8.1140×10^{-2}

8	0.07	1.6	0.98703	0.76389	3.7949×10^{-2}	0.12741	0.16667	0.20281
8.5	0.34	7.5	0.94305	0.88167	0.14682	0.38920	0.46506	0.52370
9.25	0.52	1.45	0.92158	0.80591	2.2915×10^{-2}	3.3747×10^{-2}	3.4166×10^{-2}	3.4253×10^{-2}

5. Bayesian Estimation under different loss functions

The loss function is one of the measures of accuracy in the Bayesian estimation process, the loss function is defined as the amount of loss resulting under Bayes's decision around unknown parameters. It is a measure of the difference between the estimated value and the real value of this parameter $(\widehat{\omega} - \omega)$, it should have a real non-negative value and it is usually symbolized by $L(\widehat{\omega} - \omega)$.

a. Prior and posterior distributions.

In the Bayesian approach, the unknown parameter is considered as random variable (r.v.) instead of fixed constants. From this point, the variation in the parameter can be incorporated by assuming prior distributions of the unknown parameter. As a prior distribution, we assume the parameter θ follow the Gamma distribution as a prior:

$$\pi(\omega) = \frac{a^b \omega^{b-1}}{\Gamma(b)} e^{-a\omega}, \qquad \omega > 0, a, b > 0$$

The posterior density is then:

$$\pi(\omega|X) = \frac{\pi(\omega)L(x,\omega)}{\int_0^\infty \pi(\omega)L(x,\omega)d\omega}$$

So

$$\pi(\omega|X) = K\omega^{n+b-1}\prod_{i=1}^{n}(2\omega x_i + 1)e^{-\omega(a+\sum_{i=1}^{n} x_i)}$$

where K is a normalizing constant.

b. Bayesian Estimation under unbalanced different loss functions and their posterior risks.

Loss functions are generally divided into two categories based on symmetry criteria: symmetric loss functions, which assume that the amount of loss incurred in one direction is equal to the amount of loss incurred in the other; one of the most popular types of symmetric loss functions is the quadratic loss function.

The Entropy loss function is one of the loss functions in the second category, known as the Asymmetric loss function. In this type of loss function, we assume that the amount of loss in both the positive and negative directions under the Bayes decision is not required to be equal.

We will now locate Bayes estimators under unbalanced loss functions in the following sequential order:

1. The generalized quadratic loss function

The generalized quadratic loss function is defined as $L(\widehat{\omega}, \omega) = \tau(\omega)(\widehat{\omega} - \omega)^2$ where

$\tau(\omega) = \omega^{a-1}$. In the case of the generalized quadratic loss function, the Bayes estimators are given by the formulas:

$$\widehat{\omega}_{GQ} = \frac{\int_0^\infty \omega^{a+n+b-1} \prod_{i=1}^{n}(2\omega x_i + 1)e^{-\omega\left(a+\sum_{i=1}^{n} x_i\right)}\, d\omega}{\int_0^\infty \omega^{a+n+b-2} \prod_{i=1}^{n}(2\omega x_i + 1)e^{-\omega\left(a+\sum_{i=1}^{n} x_i\right)}\, d\omega}$$

The corresponding posterior risks are then

$$PR\left(\widehat{\omega}_{GQ}\right) = \mathbb{E}_\pi(\omega^{a+1}) - 2\widehat{\omega}_{GQ}\mathbb{E}_\pi(\omega^{a}) + \widehat{\omega}_{GQ}^2\mathbb{E}_\pi(\omega^{a-1})$$

2. The Entropy loss function

The Entropy loss function is defined as $L(\widehat{\omega}, \omega) = \left(\frac{\widehat{\omega}}{\omega}\right)^p - p\ln\left(\frac{\widehat{\omega}}{\omega}\right) - 1$

Under the entropy loss function, we obtain the following estimators:

$$\widehat{\omega}_E = \left[K\int_0^\infty \omega \prod_{i=1}^{n}(2\omega x_i + 1)e^{-\omega\left(a+\sum_{i=1}^{n} x_i\right)} d\omega\right]$$

The corresponding posterior risks are then

$$PR(\widehat{\omega}_E) = p\mathbb{E}_\pi[\ln(\omega) - \ln(\widehat{\omega}_E)]$$

c. Bayesian Estimation under Balanced different loss functions and their posterior risks.

The symmetry criterion previously discussed is not the only standard used to classify loss functions; Zellner (1994) proposed a more thorough standard known as the balanced criterion, also known as the equilibrium criterion. Increasing accuracy and conformance in the estimation process is the goal of reaching equilibrium in the loss function. An unbalanced loss function is one of the loss functions covered above.

In accordance with Zellner's formula, the loss function can be balanced in addition to meeting the symmetry criterion as follows:

$$L_{\varphi,c,\omega_0}(\widehat{\omega}, \omega) = c\varphi\,(\widehat{\omega}, \omega_0) + (1-c)\varphi(\widehat{\omega}, \omega)$$

where

L_{φ,c,ω_0}: Balanced loss function.

$\varphi(\widehat{\omega}, \omega)$: Unbalanced loss function.

ω_0 : Primary estimator for the parameter ω depends on the observations.

ω weighted coefficient, $c \in (0, 1)$, now we will find Bayes estimators under unbalanced loss functions sequentially as follows:

1. The generalized quadratic loss function

The Bayes estimators under the balanced generalized quadratic loss function are given by the formula:

$$\widehat{\omega}_{GQB} = \frac{c\left[\widehat{\omega}_{GQ}\right]^{a} + (1-c)\mathbb{E}_{\pi}(\omega^{a})}{c\left[\widehat{\omega}_{GQ}\right]^{a-1} + (1-c)\mathbb{E}_{\pi}(\omega^{a-1})}$$

and the corresponding posterior risks are

$$PR\left(\widehat{\omega}_{GQB}\right) = \mathbb{E}_{\pi^*}\left[\omega^{a-1}\left(\omega - \widehat{\omega}_{GQB}\right)^{2}\right]$$

2. The Entropy loss function

The Bayes estimator under the balanced Entropy loss function is given by the formula:

$$\widehat{\omega}_{EB} = \left[\frac{c}{(\widehat{\omega}_{E})^{p}} + (1-c)\mathbb{E}_{\pi}\left(\frac{1}{\omega^{p}}\right)\right]^{-\frac{1}{p}}$$

and the corresponding posterior risks are

$$PR(\widehat{\omega}_{EB}) = p\left[\mathbb{E}_{\pi^*}[\ln(\omega) - \ln(\widehat{\omega}_{EB})]\right]$$

NUMERICAL SIMULATION

We are going to compare the performance of the proposed Bayes estimators under the balanced loss function with the MLE estimators, for that purpose we perform a MCMC simulation method, we assume that $\omega = 0.8$ and $a = b = 1$, then $\omega = 0.2$ and $a = 0.5$ $b = 0.1$ we use different sample sizes $n = 30, 50, 100, 200, 500$ respectively, thus we obtain the following results.

Bayesian Estimation

(A). The balanced generalized quadratic loss function

Using the balanced generalized quadratic loss function

Table 5. The Bayes estimators under the balanced generalized quadratic loss function and PR with *ω = 0.8* and *a = b = 1* (in brackets).

N	Parameter	α							
		-2.5	-2	-1	-0.5	0,5	1	1.5	2
30	ω	0,6267 (0,0047)	0,8961 (0,0054)	0,9233 (0,0021)	0,9723 (0,0009)	0,8494 (0,0084)	0,7609 (0,0086)	0,8351 (0,0057)	0,1276 (0,0067)
50	ω	0,6490 (0,0089)	0,7990 (0,0087)	0,9181 (0,0005)	0,9798 (0,0008)	0,7510 (0,0085)	0,7575 (0,0087)	0,6743 (0,0073)	0,1099 (0,0088)

100	ω	0,8745 (0,0076)	0,8950 (0,0075)	0,8199 (0,0015)	0,9998 (0,0006)	0,8598 (0,0054)	0,8976 (0,0043)	0,8654 (0,0034)	0,1087 (0,0054)
200	ω	0,7825 (0,0041)	0,0825 (0,0061)	0,9739 (0,0001)	0,9878 (0,0569)	0,7926 (0,0077)	0,0977 (0,0078)	0,5632 (0,0081)	0,0990 (0,0081)
500	ω	0,6432 (0,0011)	1,2127 (0,0006)	2,0018 (0,0031)	0,9796 (0,0003)	1,0839 (0,0020)	1,3240 (0,0021)	1,1243 (0,0032)	1,1841 (0,0012)

We remark that the value *α* = *- 0, 5* gives the best posterior risk. Also, we obtain the smallest suitable posterior risk when n is high.

Table 6. The Bayes estimators under the balanced generalized quadratic loss function and PR with *ω = 0.2* and *a = 0.5 b = 0.1* (in brackets).

N	Parameter	*α*							
		-2.5	-2	-1	- 0.5	0.5	1	1.5	2
30	ω	0,07231 (0,0047)	0,05961 (0,0054)	0,09221 (0,0021)	0,09823 (0,0019)	0,08494 (0,0084)	0,07609 (0,0086)	0,08351 (0,0057)	0,01276 (0,0067)
50	ω	0,07490 (0,0089)	0,0710 (0,0087)	0,09181 (0,0005)	0,9998 (0,0008)	0,07510 (0,0085)	0,07575 (0,0087)	0,06743 (0,0073)	0,01099 (0,0088)
100	ω	0,06792 (0,0076)	0,0850 (0,0075)	0,08199 (0,0015)	0,09991 (0,0006)	0,08598 (0,0054)	0,08976 (0,0043)	0,08654 (0,0034)	0,01087 (0,0054)
200	ω	0,2825 (0,0041)	0,0825 (0,0061)	0,0731 (0,0001)	0,9869 (0,0729)	0,7926 (0,0077)	0,0977 (0,0078)	0,5632 (0,0081)	0,0990 (0,0081)
500	ω	0,0453 (0,0016)	0,0127 (0,0016)	0,0932 (0,0001)	0,09878 (0,0001)	0,1839 (0,0020)	0,1841 (0,0031)	0,1232 (0,0042)	0,0741 (0,0042)

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(B). The balanced Entropy loss function

Using the balanced Entropy loss function

Table 7. The Bayes estimators under the balanced entropy loss function and PR with *ω = 0.8* and *a = b = 1* (in brackets).

n	Parameter	P							
		-2	*-1.5*	-1	-0.5	0.5	1	1.5	2
30	*ω*	1,0995 (0,0046)	1,1181 (0,0003)	0,7209 (0,0171)	0,8835 (0,0336)	0,6834 (0,0433)	1,1188 (0,0060)	0,8765 (0,034)	0,9076 (0,1153)
50	*ω*	1,0945 (0,009)	1,1067 (0,0091)	1,1041 (0,0009)	1,7981 (0,0039)	1,6981 (0,0038)	1,1194 (0,0081)	1,7053 (0,0035)	1,7697 (0,0099)
100	*ω*	1,3894 (*0*,102)	1,2387 (0,0033)	0,7645 (0,0101)	0,8966 (0,0344)	0,5623 (0,0653)	1,0987 (0,0080)	0,6759 (0,0002)	0,8765 (0,1003)
200	*ω*	1,3994 (0,1646)	1,2188 (0,1443)	0,6205 (0,0171)	0,7832 (0,0735)	0,4830 (0,0733)	1,1088 (0,0070)	0,6701 (0,0667)	0,7654 (0,1173)
500	*ω*	1,2148 (0,002)	1,2179 (0,001ç)	1,2167 (0,0001)	1,2149 (0,0008)	1,2148 (0,0009)	1,1038 (0,001è)	1,0969 (0,0008)	1,0886 (0,0028)

Table 8. The Bayes estimators under the balanced entropy loss function and PR with *ω = 0.2* and *a = 0.5 b = 0.1* (in brackets).

n	Parameter	P							
		- 2	- *1.5*	- 1	- 0.5	0.5	1	1.5	2
30	*ω*	0,06911 (0, 0004)	0,4189 (0, 0013)	0,1409 (0, 0071)	0,1830 (0, 003)	0,0831 (0, 0033)	00982 (0, 0003)	0,8465 (0, 004)	0,1076 (0, 1003)
50	*ω*	0,0942 (0, 082)	0,1067 (0, 0001)	0,1111 (0, 0019)	0,1981 (0, 0008)	0,0582 (0, 0038)	0,1194 (0, 0001)	1,7053 (0, 0015)	0,7697 (0, 0009)

100	ω	0,3892 (0, 100)	02387 (0, 0033)	0,1045 (0, 0101)	0,8455 (0, 0343)	0,2623 (0, 0653)	0,0994 (0, 0009)	0,6759 (0, 0002)	0,1765 (0, 1003)
200	ω	0,1943 (0, 1004)	0,2188 (0, 1401)	0,1005 (0, 0101)	0,7830 (0, 070)	0,4830 (0, 0033)	0,1022 (0, 0020)	0,6701 (0, 021)	0,114 (0, 0123)
500	ω	0,2144 (0, 0011)	0,2179 (0, 0012)	0,1167 (0, 0009)	0,2148 (0, 0005)	0,2148 (0, 0003)	0,1002 (0, 0011)	0,0969 (0, 0009)	0,0886 (0, 0031)

we obtain the following table, where we can remark that the values *p = 1*, *n = 500* provide the best posterior risk .

Comparison of the estimators using Pitman's closeness criterion.

Table 9. Bays estimators and PR (in brackets) under the Two loss function with *θ = 0.8* and *a = b = 1*

n	Parameter	Entropy (*P* = 1)	*GQ* (λ= 0, 5)
30	ω	1,1188 (0, 0060)	0,9723 (0, 0009)
50	ω	1,1194 (0, 0081)	0,9798 (0, 0008)
100	ω	1,0987 (0, 0080)	0,9898 (0, 0006)
200	ω	1,1088 (0, 0070)	0,9895 (0, 0729)
500	ω	1,1038 (0, 0018)	0,9899 (0, 0001)

Table 10. Bays estimators and PR (in brackets) under the Two loss function with *ω = 0.2* and *a = 0.5 b = 0.1*

n	Parameter	Entropy (*P* = 1)	*GQ* (*λ*= 0, 5)
30	ω	00982 (0, 0003)	0,09823 (0, 0019)
50	ω	0,1194 (0, 0001)	0,9998 (0, 0008)
100	ω	0,0994 (0, 0009)	0,09991 (0, 0006)
200	ω	0,1022 (0, 0020)	0,9869 (0, 0729)
500	ω	0,1002 (0, 0011)	0,09878 (0, 0001)

COMPARISON OF THE ESTIMATION METHODS

We suggest contrasting the maximum likelihood estimators with the best Bayesian estimators found above in this subsection. We suggest using the following criteria for this. The closeness of Pitman (Pitman 1937). Jozani et al. 2012; Fuller, 1982) as well as the integrated mean square error (IMSE), which is defined as follows:

Definition 2.

An estimator ω_1 of a parameter ω dominates in the sense of Pitman closeness criterions another estimator ω_2, if for all $\omega \in \Theta$:

$$P_\omega[|\omega_1 - \omega| < |\omega_2 - \omega|] > \frac{1}{2}$$

Consider the estimators ω_i, $i \in \{1, ..., N\}$ obtained with N samples of the model. In the following, we present the values of the Pitman probabilities which allow us to compare the Bayesian estimators with the MLE under the tow loss function.

The table 5 where should as follows, when the probability is greater than *-0, 5*, the Bayesian estimators is better than the MLE estimators. Then we notice that, according to this criterion: According to Pitman's criterion, the Bayesian estimators of the parameters are better than the MLE, Also the generalized quadratic loss function has the best values in comparison with the other loss functions.

Table 11. Pitman comparison of the estimators with $\omega = 0.8$ and $a = b = 1$.

n	*Entropy* (*p* = 1)	*GQ* (*α* = - 0, 5)
30	0, 287	0, 366
50	0, 796	0, 598
100	0, 688	0, 579
200	0, 698	0, 632
500	0, 706	0, 702

Table 12. Pitman comparison of the estimators with $\omega = 0.2$ and $a = 0.5$ $b = 0.1$.

n	*Entropy* (*p* = 1)	*GQ* (*α* = - 0, 5)
30	0, 498	0, 456
50	0, 699	0, 634
100	0, 787	0, 556
200	0, 755	0, 603
500	0, 722	0, 611

6. Application and comparison

In this section, the real-life applicability of the new one-parameter model is demonstrated by voltage data. The data set represents the failure and running times of a sample of devices from a larger system field-tracking research. The data studied by Meeker et al.(2022) , and the data are given by: *275, 13, 147, 23, 181, 30, 65, 10, 300, 173, 106, 300, 300, 212, 300, 300, 300, 2, 261, 293, 88, 247, 28, 143, 300, 23, 300, 80, 245, and 266.*

We evaluate the OPM from gamma (Ga), exponential (Exp), Lindley, XLindley , new XLindley and Shanker distributions for this data. Information of p.d.f about competitor models is provided as follows:

$$g_{Ga}(x) = \frac{1}{\theta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} exp\left(\frac{-x}{\theta}\right), \qquad x, \alpha, \theta > 0$$

$$g_{Exp}(x) = \theta exp(-\theta x), \qquad x, \theta > 0$$

$$g_{Lindley}(x) = \frac{\theta^2}{(1+\theta)^2}(1+x)\, exp(-\theta x), \qquad x, \theta > 0$$

$$g_{Shanker}(x) = \frac{\theta^2}{\theta^2+1}(\theta+x)\, exp(-\theta x), \qquad x, \theta > 0$$

$$g_{NXLindley}(x) = \frac{\theta}{2}(1+\theta x)\, exp(-\theta x), \qquad x, \theta > 0$$

$$g_{XLindley}(x) = \frac{\theta^2}{(1+\theta)^2}(2+\theta+x)\, exp(-\theta x), \qquad x, \theta > 0$$

Table 13. Goodness of fit statistics of OPM

Model	par	AIC	BIC	$-L$	$ACIC$	KS (KS p-value)
$Lindley$	0.0112	376.9248	378.3260	187.4624	377.0677	0.229(0.085)
XXLindley	0.0014	376.1434	377.5446	187.0717	376.2862	0.228(0.087)
$new\ XLindley$	0.0087	370.2716	371.6728	184.1358	370.4144	0.215(0.124)
Gamma	0.0067 1.1896	374.0413	376.8437	185.0207	374.4858	0.217(0.118)
exp	1.1896	372.5803	373.9815	185.2901	372.7231	0.216(0.121)
Shanker	0.0113	377.9493	379.3505	187.9747	378.0922	0.230(0.083)
OPM	**0.0094**	**370.1214**	**371.5225**	**368.1214**	**370.2642**	**0.210(0.128)**

The values of $AIC, BIC, -2logL, K\text{-}S$ statistics in Table 13, indicate that OPM is a strong competitor to the other distributions commonly used in literature for fitting lifetime data, moreover the best fit measured the previous goodness of fit statistics.

Conclusion and Perspectives

In this study, we introduced and thoroughly analyzed a novel one-parameter probability distribution tailored for applications in reliability analysis and lifetime modeling. Through rigorous mathematical derivations, we established several statistical properties of the proposed model, confirming its flexibility and suitability for voltage data set.

The integration of fuzzy reliability analysis provided a powerful framework for modeling uncertainty in failure data, offering more realistic and adaptable insights compared to traditional crisp reliability measures. This is particularly beneficial in engineering and medical applications where imprecise or expert-based information is common.

Furthermore, the application of Bayesian estimation techniques under different prior assumptions and loss functions demonstrated the robustness of the model's parameter inference. The Bayesian framework not only accommodated prior knowledge but also yielded more informative posterior distributions, especially in small-sample or noisy environments.

A comprehensive goodness-of-fit analysis and simulation study confirmed the model's empirical validity and estimation accuracy. The results indicated superior performance compared to benchmark distributions in terms of fitting capability and parameter estimation precision.

Looking ahead, several avenues for future research can be considered. First, extending the model to a two-parameter or composite form may enhance its flexibility to capture more complex data behaviors. Second, incorporating real-world datasets from diverse fields such as biostatistics, reliability engineering, and environmental science could further validate its applicability. Lastly, the development of computational tools and software packages for implementing fuzzy reliability and Bayesian estimation methods would make the proposed framework more accessible to practitioners and researchers alike.

AUTHORS CONTRIBUTIONS

Razika Grine: Investigation; formal analysis; methodology, writing—original draft; simulation; interpretation of results; writing—review and editing.

Meriem Bouhadjar: methodology, writing—original draft ; formal analysis; validation, supervision.

Imene Grabsia: Methodology ; Software; application, comparison and interpretation of results.

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