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Numerical Method for the Solution of Integro-Differential Equations of the Second Kind

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Abstract: The purpose of this work is to search for an approximate solution to the Fredholm and Volterra integro-differential equations using Genocchi polynomials, replacing the initial conditions if necessary, where the integrals can be calculated using numerical methods, in order to obtain a variation problem and reduce it to a linear system, where its solution is to find the coefficients of the function. Unknowns and then solve the equation. The convergence and effectiveness of this method are confirmed by numerical examples that will be presented.

Keywords: Integro-differential equations, Numerical method, Genocchi polynomials.

1. Introduction

Integro-differential equations are considered one of the most important fields in mathematical disciplines, for example pure mathematics and applied mathematics. Integro-differential equations linear and non linear have a very important role in modern science and technology such as heat transfer, diffusion processes, mechanics, biological species, and many other fields. To learn more about the sources in which these types of equations are studied in applications of physics, biology, and engineering, as well as in books on advanced integral equations. References can be found [4, 10, 11, 15].

The numerical solution of second order integro-differential equations with the boundary conditions of the Fredholm and Volterra equations and other equations related to this type of equations has been done by some authors. For example, the authors in [4] discussed the Chebyshev Collocation Method for the Solution of Linear Integro-Differential Equations, which is the compact finite difference method, and the monotonic iterative sequence method for solving the second order Volterra integro-differential equation was implemented in [5, 27]. However, a sequential solution of second order integro-differential equations with boundary conditions of Fredholm and Volterra types by the homotopy analysis method was also considered in [16].

Accordingly, this work aims to find approximate solutions to Fredholm and Volterra linear integro-differential equations of the second type using polynomials of the Genocchi type with the numerique method and then compare the approximate solutions with the exact solutions to see the effectiveness of the method through the examples that we will present.

2. Genocchi polynomial method for I.D.E

Consider the following integro-differential equation

$$y'(x) = f(x) + \int_a^x k(x,t)y(t)\,dt \tag{1}$$

$$y(a) = \propto$$

Where $f(x)$ and $k(x,t)$ are known functions, while $y(x)$ is the unknown function to be determined. The method under consideration employs Genocchi polynomials, as thoroughly discussed in references [12, 13, 14, 22]. These polynomials are used as a basis to approximate the solution over a closed and finite interval. It is assumed that

$$y_n(x) = \sum_{i=0}^{n} \beta_i \, G_i(x) = \sum_{i=0}^{n} \beta_i \, G_i\left(\frac{x-a}{b-a}\right) \tag{2}$$

Where $G_i\left(\frac{x-a}{b-a}\right)$ is shifted Genocchi polynomial at $[a, b]$

Note that when we take the value $x = a$ we get $\frac{x-a}{b-a} = 0$, and when $x = b$ we get $\frac{x-a}{b-a} = 1$.

So we have

$$y'(x) = y_n'(x) = \sum_{i=0}^{n} \left(\frac{1}{b-a}\right) \beta_i G_i'\left(\frac{x-a}{b-a}\right) \tag{3}$$

Substituting (2) and (3) into (1), results in

$$\sum_{i=0}^{n} \left(\frac{1}{b-a}\right) \beta_i G_i'\left(\frac{x-a}{b-a}\right) = f(x) + \int_a^x k(x,t) \sum_{i=0}^{n} \beta_i G_i\left(\frac{t-a}{b-a}\right) dt = f(x) + \sum_{i=0}^{n} \beta_i \int_a^x k(x,t) G_i\left(\frac{t-a}{b-a}\right) dt \tag{4}$$

To determine unknown coefficients β_i, we use the method technique by multiplying equation (4) by $G_j\left(\frac{t-a}{b-a}\right)$ and then integrating with respect to x from 0 to 1. So we have

$$\sum_{i=0}^{n} \left(\frac{1}{b-a}\right) \beta_i \int_0^1 G_i'\left(\frac{x-a}{b-a}\right) G_j\left(\frac{x-a}{b-a}\right) dx = \int_0^1 f(x) \, G_j\left(\frac{x-a}{b-a}\right) dx + \int_0^1 \left[\sum_{i=0}^{n} \beta_i \int_a^x k(x,t) G_i\left(\frac{t-a}{b-a}\right) dt\right] G_j\left(\frac{x-a}{b-a}\right) dx \tag{5}$$

for $= 0,1,\dots,n$, or equivalently

$$\sum_{i=0}^{n} \left(\frac{1}{b-a}\right) \beta_i \int_0^1 G_i'\left(\frac{x-a}{b-a}\right) G_j\left(\frac{x-a}{b-a}\right) dx = \int_0^1 f(x) \, G_j\left(\frac{x-a}{b-a}\right) dx + \sum_{i=0}^{n} \beta_i \int_0^1 \left[\int_a^x k(x,t) G_i\left(\frac{t-a}{b-a}\right) dt\right] G_j\left(\frac{x-a}{b-a}\right) dx \tag{6}$$

If necessary, the integrals can be evaluated using numerical techniques. This process results in a system of linear equations involving the unknown coefficients $\{\beta_0, \beta_1, \dots, \beta_n\}$. In many studies, researchers incorporate the initial condition by directly substituting it into the system.

$$y(a) = \propto \Rightarrow \sum_{i=0}^{n} \beta_i G_i\left(\frac{a-a}{b-a}\right) = \sum_{i=0}^{n} \beta_i G_i\,(0) = \propto \tag{7}$$

Maintaining an equal number of equations in the previously constructed linear system, the unknown coefficients are determined by simultaneously solving equations (6) and (7). Once obtained, these values are substituted into equation (2) to derive an approximate solution to the original equation (1).

3. Genocchi Polynomials and Their Properties

The classical Genocchi polynomial $G_n(x)$ is usually defined by means of the exponential generating functions

$$\frac{2te^{xt}}{e^t+1}=\sum_{n=0}^{+\infty} G_n(x)\frac{t^n}{n!}$$

Where $G_n(x)$ is the Genocchi polynomial of degree n and is given by

$$G_n(x) \ =\sum_{k=0}^{n}\binom{n}{k}G_{n-k}(x)\,x^k$$

G_{n-k}. Is the Genocchi number.

Some of the important properties of these polynomials include

$$\begin{cases} \int_0^1 G_p(x)G_q(x)\,dx=\dfrac{2(-1)^p p!\,q!}{(p+q)!}G_{p+q}, p,q\in\mathbb{N}^* \\ \dfrac{dG_p(x)}{dx}=nG_{n-1}(x),\ \ p\in\mathbb{N}^* \\ G_p(1)+G_q(0)=0,\ \ p\in\mathbb{N}^* \end{cases}$$

4. Numerical Examples

In this section, we intend to show the efficiency of the method for solving Fredholm and Volterra integro-differential equations of the second kind by Genocchi polynomials by presenting six illustrative examples. The absolute error for this formulation is need by $E(x)=|y(x)-y_n(x)|$.

Example 1. *Let us consider the linear integro-differential equation of Volterra*.

$$y'(x)=2-\frac{x^2}{4}+\frac{1}{4}\int_0^x y(t)dt$$

With the initial condition $y(0)=0$

Where the function $y(x)=2x$ *is the exact solution.*

The approximate solution $y_n(x)$ *of* $y(x)$ *is obtained by the Genocchi polynomial method.*

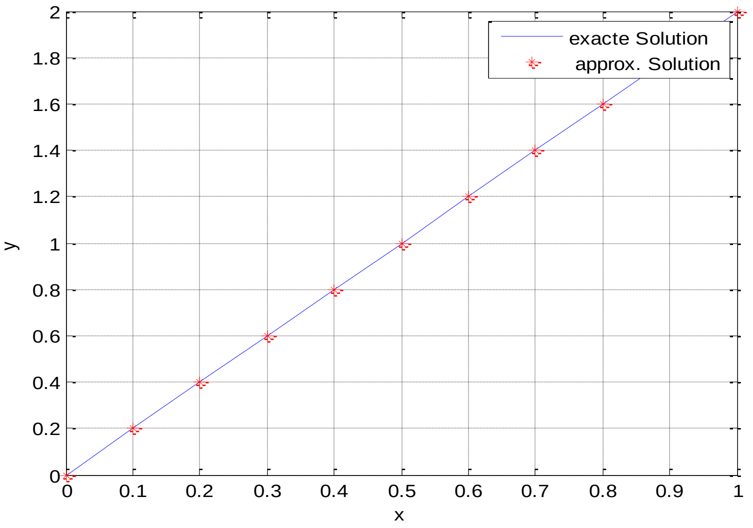

***Figure 1.** Graph for example 1*

Example 2. *Let us consider the linear integro-differential equation of Volterra.*

$$y'(x) = 1 - 2x\sin(x) + \int_0^x y(t)\,dt$$

With the initial condition $y(0) = 0$

The exact solution is given by

$$y(x) = x\cos(x)$$

The approximate solution $y_n(x)$ *of* $y(x)$ *is obtained by the Genocchi polynomial method.*

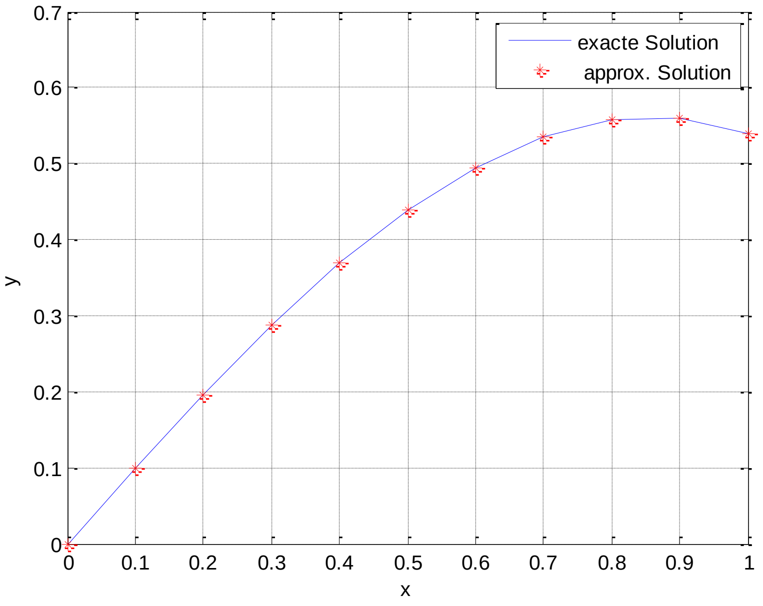

***Figure 2.** Graph for example 2*

Example 3. *Let us consider the linear integro-differential equation of Fredholm*

$$y'(x) = 3e^{3x} - \frac{1}{3}(2e^3 + 1)x + \int_0^1 3xty(t)\,dt$$

With the initial condition $y(0) = 1$

The exact solution is given by

$$y(x) = e^{3x}$$

The approximate solution $y_n(x)$ *of* $y(x)$ *is obtained by the Genocchi polynomial method.*

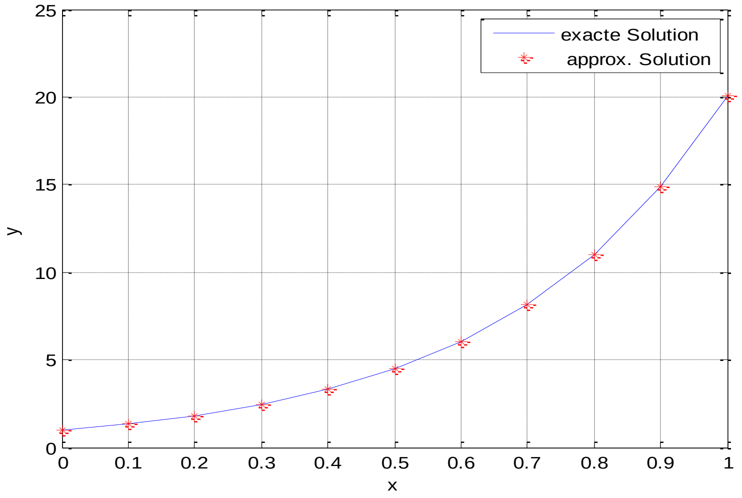

***Figure 3.** Graph for example 3*

Example 4. *Let us consider the linear integro-differential equation of Fredholm.*

$$y'(x) = 3 + 6x + \int_0^1 xt\,y(t)dt$$

With the initial condition $y(0) = 0$

The exact solution is given by

$$y(x) = 3x + 4x^2$$

The approximate solution $y_n(x)$ *of* $y(x)$ *is obtained by the Genocchi polynomial method.*

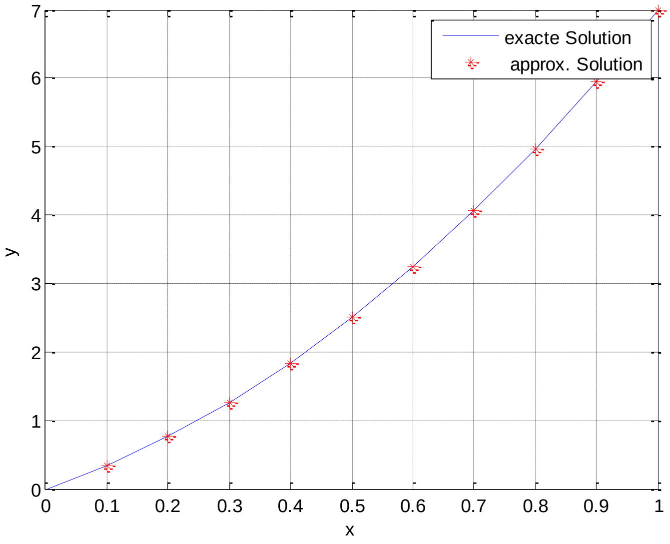

***Figure 4.** Graph for example 4*

Example 5. *Consider linear Volterra integro-differential equation of second kind*

$$y''(x) = x + \int_0^x (x-t)y(t)\,dt$$

With the initial condition $y(0) = 0, y'(0) = 1$

Where the function $y(x) = \sinh(x)$ *is the exact solution*

The approximate solution $y_n(x)$ *of* $y(x)$ *is obtained by the Genocchi polynomial method.*

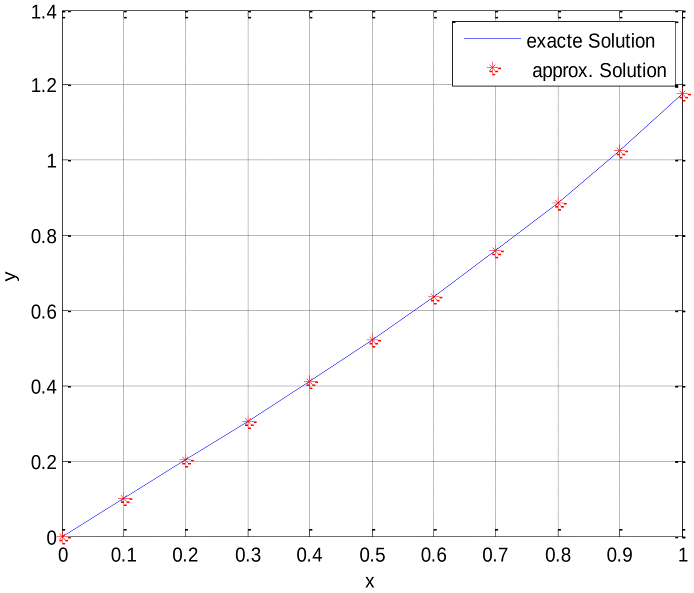

***Figure 5.** Graph for example 5*

Example 6. *Consider the integro-differential equation*

$$y'(x) = -\cos(2\pi x) - 2\pi\sin(2\pi x) - \frac{1}{2}\sin(4\pi x) + \int_{0}^{1} \sin(4\pi x + 2\pi t)\, y(t)\, dt$$

With the initial condition $y(0) = 1$,

Where the function $y(x) = \cos(2\pi x)$ *is the exact solution*

The approximate solution $y_n(x)$ *of* $y(x)$ *is obtained by the Genocchi polynomial method.*

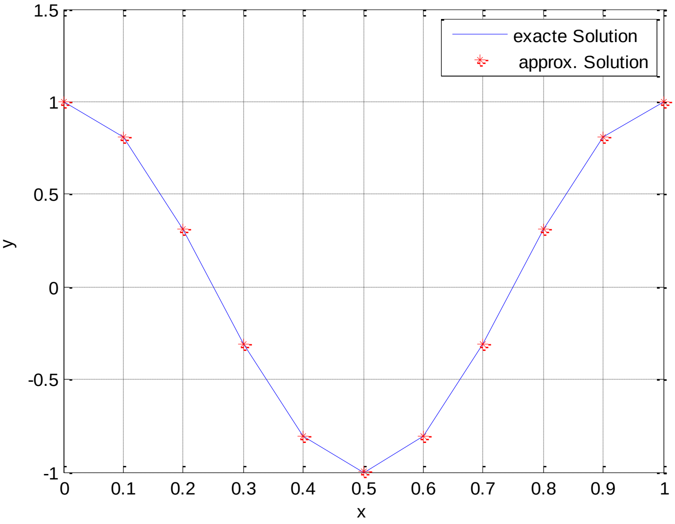

Figure 6. Graph for example 6

Conclusion

This article deals with the numerical solution of first order Fredholm and Volterra integro-differential equations of the second kind, using the method technique by means of Genocchi polynomials. This technique was tested on six examples shown in the obtained figures, and the results were satisfactory and the method was quite effective. In addition, this method can be applied to high order Fredholm and Volterra integro-differential equations of the second kind, where the Matlab program is used to obtain approximate solutions.

This technique will be applied in the future to fractional integro-differential equations and nonlinear integro-differential equations.

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