Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 10s (2025)

Some Results on Partial Cone Metric Spaces with an Application

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Article History:

Received: 12-01-2025

Revised: 15-02-2025

Accepted: 01-03-2025

Abstract:

The objective of this paper is to determine some fixed point theorems for generalized $\alpha - \psi$ contractive mappings in the framework of partial cone metric spaces. In addition, we prove a unique fixed point theorem using a rational contractive condition. Our findings align with previous research in this area. We also show that our result can be applied to the problem of determining the existence of solutions to second-order differential equations.

Keywords: $\alpha - \psi$ contractive mappings, partial cone metric spaces, α admissible Mappings.

1. Introduction

The Banach contraction principle [1] was a foundation for a development of metric fixed point theory which has been generalized by utilizing various contractive conditions in various contexts. In 1906, Frechet [2] introduced the notion of metric spaces. In 2007, Huang and Zhang [7] introduced the concept of cone metric space which is a generalization of metric space. Another generalization of metric spaces is partial metric spaces which was introduced by Matthews [3, 4] in which the self distance need not be equal to zero and proved the partial metric version of Banach fixed point theorem. Partial cone metric spaces have been investigated by Mahlotra et al. [10] and Sonmez. They proved some fixed point theorems in this space. Recently many papers on cone metric spaces and partial cone metric spaces have been appeared e.g. see . [8, 9, 13, 14, 15, 16, 17, 18]. On the other hand, Samet et al. [5] extended and generalized the Banach contraction principle by introducing a new class of contractive type mappings known as $\alpha - \psi$ contractive type mappings. Karapinar and Samet [6] generalized the $\alpha - \psi$ contractive type mappings and established various fixed point theorems.

To begin, we will define partial metric spaces , cone metric spaces, and partial cone metric spaces as well as their properties:

Definition 1.1. (Partial metric space) A partial metric on a non-empty set X is a function

$\rho: X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$ the following hold

1. $x = y \Leftrightarrow \rho(x,x) = \rho(y,y) = \rho(x,y)$;

2. $\rho(x,x) \leq \rho(x,y)$;
3. $\rho(x,y) = \rho(y,x)$;
4. $\rho(x,y) \leq \rho(x,z) + \rho(z,y) - \rho(z,z)$. For all $x, y, z \in X$.

Then the pair (X, ρ) is called a partial metric space. It is clear that if ρ(x, y) = 0, then (1) and (2) imply that x = y. But if x = y, ρ(x, y) may not be 0. A basic example of partial metric space is the pair $(\mathbb{R}^+, \rho)$ where $\rho(x,y) = max\{x,y\}$ for all $x, y \in \mathbb{R}^+$.

Let E be a real Banach space and P a subset of E. P is called a cone if it satisfies the following.

(1) P is closed, non-empty, and $P \neq 0$,

(2) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers $a, b \in \mathbb{R}$,

(3) $P \cap (-P) = \{0\}$.

For a specified cone $P \subset E$, we can establish a partial ordering ≤ on E in relation to P by defining $x \leq y$ if and only if $y - x \in P$. The notation $x < y$ is used to signify that $x \leq y$ and $x \neq y$, while $x \ll y$ indicates that $y - x \in intP$, with $intP$ representing the interior of P . The cone P is termed normal if there exists a constant $\mathrm{K} > 0$ such that for all $\mathrm{x}, \mathrm{y} \in \mathrm{E}$ where $0 \leq \mathrm{x} \leq \mathrm{y}$, it follows that $||\mathrm{x}|| \leq \mathrm{K}||\mathrm{y}||$. The smallest positive value that satisfies this condition is referred to as the normal constant of P.

Let E be a Banach space, P a cone in E with $intP \neq \phi$ and ≤ is partial ordering with respect to P.

Definition 1.2 (Cone metric space) Let X be a non empty set. The mapping $d_c : X \times X \to E$ is said to be a cone metric on X if for all $x, y, z \in X$. The followings hold:

(1) $0 \leq d_c(x,y)$ and $d_c(x,y) = 0$ if and only if $x = y$,

(2) $d_c(x,y) = d_c(y,x)$,

(3) $d_c(x,y) \leq d_c(x,z) + d_c(y,z)$.

and $(\mathrm{X}, \mathrm{d}_c)$ is called a cone metric space.

Mahlotra et al. [10] and Sonmez [11] introduced the notion of partial cone metric space and its topological characterization. We now state the definition of partial cone metric space.

Definition 1.3 (Partial cone metric space) A partial cone metric on a non-empty

set X is a function $\rho_c: X \times X \to E$ such that for all $x, y, z \in X$

(1) $0 \leq \rho_c(x,x) \leq \rho_c(x,y)$,

(2) $x = y$ if and only if $\rho_c(x,x) = \rho_c(x,y) = \rho_c(y,y)$,

(3) $\rho_c(x,y) = \rho_c(y,x)$,

(4) $\rho_c(x,y) \leq \rho_c(x,z) + \rho_c(z,y) - \rho_c(z,z)$.

A partial cone metric space is a pair (X,ρ_c) such that X is a non-empty set and ρ_c is a partial cone metric on X . It is clear that, if $\rho_c(x,y) = 0$, then (1) and (2) imply that x = y. But the converse is not true in general. A cone metric space is a partial cone metric space, but there exist partial cone metric spaces which are not cone metric spaces. we give the following example from [11]

Example 1.4 Consider a Banach space $E = \mathbb{R}^2, P = \{(x,y) \in E : x,y \geq 0\}$

and $X = \mathbb{R}^+$ and $\rho_c : X \times X \to E$ defined $by\ \rho_c(x,y) = (max\{x,y\}, kmax\{x,y\})$

where $k \geq 0$ is a constant. Then (X,ρ_c) is a partial cone metric space which is not a cone metric space.

Remark 1.5 Suppose (X,ρ_c) is a partial cone metric space, then

$$d_c(x,y) = 2\rho_c(x,y) - \rho_c(x,x) - \rho_c(y,y)$$

For all $x,y,z \in X$ defines a cone metric on X.

Theorem1.6 Every partial cone metric space (X,ρ_c) is a topological space.

Following, We give some properties of partial cone metric spaces, for more details see [11].

Definition 1.7 Let (X,ρ_c) be a partial cone metric space. Let $\{x_n\}$ be a sequence in X

and $x \in X$

(1) $\{x_n\}$ is said to be convergent to x and x is called a limit of $\{x_n\}$ if
$$\lim_{n\to\infty} \rho_c(x_n,x) = \lim_{n\to\infty} \rho_c(x_n,x_n) = \rho_c(x,x)$$

(2) $\{x_n\}$ is Cauchy sequence if there is $x \in P$ such that for every $\epsilon > 0$ there is $\mathbb{N}$ such that for all $n,m > \mathbb{N}$, $||\rho_c(x_n,x_m) - x|| < \epsilon$.

(3) (X,ρ_c) is said to be complete if every Cauchy sequence in (X,ρ_c) is convergent in (X,ρ_c).

In 2012, Samet et al. [5] introduced α-admissible mapping as follows:

Definition 1.8 [5] Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. T is said to α-admissible if

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(T\,x,T\,y) \geq 1$$

for all $x,y \in X$.

2. Main Results

[12] Let Ψ be the family of non-decreasing function $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$ where ψ^n is nth iterate of ψ.

Lemma 2.1 [12] For every function $\psi : [0,\infty) \to [0,\infty)$ the following holds:

If ψ is non decreasing, then for each $t > 0$, $\lim_{n\to\infty} \psi^n(t) = 0$ implies $\psi(t) < t$

and $\psi(0) = 0$.

Definition 2.2 Let (X, ρ_c) be a partial cone metric space P is a normal cone with constant K.

Let $T : X \to X$ be a self mapping. Then T is said to be generalized $\alpha - \psi$ contractive mapping if there exists two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ for all $x, y \in X$ we have

$$\alpha(x,y)\rho_c(T\,x, T\,y) \le \psi(M\,(x,y)) \tag{2.1}$$

where

$$M\,(x,y) = max\{\rho_c(x,y), \rho_c(x,T\,x), \rho_c(y,T\,y)\} \tag{2.2}$$

Theorem 2.3 Let $(X, \rho_c\,)$ be a complete partial cone metric space and $T : X \to X$ be self mapping. Suppose $\alpha : X \times X \to [0,\infty)$ be the mappings satisfying the conditions:

(i) T is α admissible;
(ii) T is generalized $\alpha - \psi$ contractive mapping;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, T\,x_0) \ge 1$;
(iv) T is continuous or if $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x$ as $n \to \infty$ then $\alpha(x_n, x) \ge 1$ for all n.

Then T has a fixed point in X.

Proof: Let x_0 be an arbitrary point such that $\alpha(x_0, T\,x_0) \ge 1$. Suppose we have a sequence $\{x_n\}$ in X such that $x_{n+1} = T\,x_n$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T and the existence part of the proof is finished. Suppose $x_n \ne x_{n+1}$for every $n \in \mathbb{N}$ Now, since T is α-admissible, so

$$\alpha(T\,x_0, T\,x_1) = \alpha(x_1, x_2) \ge 1$$

$$\alpha(T\,x_1, T\,x_2) = \alpha(x_2, x_3) \ge 1$$

and using induction we have $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$.

Now, from (2.1) we have

$$\begin{aligned} \rho_c(x_n, x_{n+1}) &= \rho_c(T\,x_{n-1}, T\,x_n) && (2.3) \\ &\le \alpha(x_{n-1}, x_n)\rho_c(T\,x_{n-1}, T\,x_n) \\ &\le \psi(M\,(x_{n-1}, x_n)) && (2.4) \end{aligned}$$

where

$$M\,(x_{n-1}, x_n) = max\{\rho_c(x_{n-1}, x_n), \rho_c(x_{n-1}, T\,x_{n-1}), \rho_c(x_n, T\,x_n)\}$$

$$= max\{\rho_c(x_{n-1}, x_n), \rho_c(x_{n-1}, x_n), \rho_c(x_n, x_{n+1})\} \quad (2.5)$$

Now, if $\rho_c(x_n, x_{n+1}) > \rho_c(x_{n-1}, x_n)$. Then

$$||\rho_c(x_n, x_{n+1}))|| \leq \psi(||\rho_c(x_n, x_{n+1}))||) < ||\rho_c(x_n, x_{n+1}))|| \quad (2.6)$$

This is a contradiction. Thus for all $n \geq 1$ we have

$$max\{\rho_c(x_{n-1}, x_n), \rho_c(x_n, x_{n+1})\} = \rho_c(x_{n-1}, x_n) \quad (2.7)$$

$$\rho_c(x_n, x_{n+1})) \leq \psi(\rho_c(x_{n-1}, x_n)) \quad (2.8)$$

Continuing this process inductively, we obtain

$$\rho_c(x_n, x_{n+1})) \leq \psi^n (\rho_c(x_0, x_1)) \quad (2.9)$$

Now for $m > n$, using (2.9) and triangular inequality , we obtain

$$\rho_c(x_m, x_n) \leq \rho_c(x_m, x_{m-1}) + \rho_c(x_{m-1}, x_{m-2}) \ldots\ldots\ldots \rho_c(x_{n+1}, x_n) - \sum_{k=1}^{m-n-1} \rho_c(x_{m-k}, x_{m-k})$$

$$\leq \rho_c(x_m , x_{m-1}) + \rho_c(x_{m-1}, x_{m-2}) \ldots\ldots\ldots \rho_c(x_{n+1}, x_n)$$

$$\leq (\psi^{m-1} + \psi^{m-2} + \ldots\ldots\ldots \psi^n)\rho_c(x_0, x_1)$$

$$= \frac{\psi^n}{1-\psi}\rho_c(x_0, x_1) \quad (2.10)$$

Since P is normal cone with normal constant K, we find that

$$||\rho_c(x_m, x_n))|| \leq K||\frac{\psi^n}{1-\psi}\rho_c(x_0, x_1)|| \quad (2.11)$$

Which implies that $\rho_c(x_m, x_n) \rightarrow 0$ as $n, m \rightarrow \infty$. Hence $\{x_n\}$ a Cauchy sequence in partial cone metric space which is complete hence it must be convergent in X, let $\lim_{n\rightarrow\infty} x_n = z$ therefore

$$\rho_c(z, z) = \lim_{n\rightarrow\infty} \rho_c(x_n, z) = \lim_{n\rightarrow\infty} \rho_c(x_n, x_n) = 0$$

Case 1. T is continuous, then we have $x_{n+1} = T\,x_n \rightarrow T\,z$ as $n \rightarrow \infty$. By uniqueness of limit $T\,z = z$. Hence z is a fixed point of T .

Case 2 If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \rightarrow z$ as $n \rightarrow \infty$. Then $\alpha(x_n, z) \geq 1$ for all n. Now we show that $||\rho_c(T\,z, z)|| \geq 0$, On contrary, assume $||\rho_c(T\,z, z)|| > 0$ we have

$$\rho_c(T\,z, z) \leq \rho_c(T\,z, T\,x_n) + \rho_c(T\,x_n, z) - \rho_c(T\,x_n, T\,x_n)$$

$$\leq \alpha(x_n, z)\rho_c(T\,z, T\,x_n) + \rho_c(T\,x_n, z) - \rho_c(T\,x_n, T\,x_n)$$

$$\leq \psi(M\,(x_n, z)) + \rho_c(x_{n+1}, z) \quad (2.12)$$

Since P is normal cone with normal constant K, we have

$$||\rho_c(T\,z, z)|| \leq K||\psi(M\,(x_n, z)) + \rho_c(x_{n+1}, z)|| \quad (2.13)$$

where

$$\begin{aligned} M\,(x_n, z) &= max\{\rho_c(x_n, z), \rho_c(x_n, T\,x_n), \rho_c(z, T\,z)\} \\ &= max\{\rho_c(x_n, z), \rho_c(x_n, x_{n+1}), \rho_c(z, T\,z)\} \qquad (2.14) \end{aligned}$$

Taking $n \to \infty$ we get

$$M\,(x_n, z) = \rho_c(z, T\,z) \qquad (2.15)$$

Now, Taking $n \to \infty$ in (2.12) we get that

$$||\rho_c(T\,z, z)|| \leq K||\psi(\rho_c(z, T\,z)\big|| \leq K\big|\rho_c(z, T\,z)|| \qquad (2.16)$$

which is not true for all $K > 0$. So we get a contradiction. $Therefore\ ||\rho_c(T\,z, z)|| \to 0$ as $n \to \infty$. It implies that $T\,z = z$ and hence z is a fixed point of T .This completes the proof.

Example 2.4 $Let\ X = [0, \infty)$ and $\rho_c(x, y) = (max\{x, y\}, k\,max\{x, y\})$. Then (X, ρ_c) is a complete partial cone metric space. Consider the mapping $T : X \to X$defined by

$$T(x) = \begin{cases} x - \frac{2}{3} & x > 1 \\ \frac{x}{3} & 0 \leq x \leq 1 \end{cases} \qquad (2.17)$$

and let $\psi : [0, \infty) \to [0, \infty)$ be such that $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. If we define the functions $\alpha, \beta : X \times X \to [0, \infty)$ as

$$\alpha(x, y) = \begin{cases} \frac{3}{2} & x, y \in [0,1] \\ 0 & otherwise \end{cases} \qquad (2.18)$$

We show that contractive condition of Theorem 2.3 is satisfied. Without loss of generality we assume that $x \geq y$. Then for $x, y \in [0, 1]$ we get

$$\begin{aligned} \alpha(x, y)\rho_c(T\,x, T\,y) = \alpha(x, y)\rho_c\left(\frac{x}{3}, \frac{y}{3}\right) &= \frac{3}{2}\left(\frac{x}{3}, \frac{kx}{3}\right) \\ &= \frac{1}{4}(x, kx) \\ &\leq \frac{1}{2}(x, kx) \\ &= \frac{1}{2}\,max\{(x, kx), (x, kx), (y, ky)\} \\ &= \frac{1}{2}\,max\{\rho_c(x, y), \rho_c(x, T\,x), \rho_c(y, T\,y)\} \\ &= \psi(max\{\rho_c(x, y), \rho_c(x, T\,x), \rho_c(y, T\,y)\}) \quad (2.19) \end{aligned}$$

Theorem 2.5 Let (X, ρ_c) be a complete partial cone metric space P is a normal cone with constant K. Suppose the mapping $T : X \to X$ satisfies the contractive condition

$$\rho_c(T\,x,T\,y)) \le a_1\,\rho_c(x,y) + a_2\,\frac{\rho_c(x,T\,x)\rho_c(y,T\,y)}{\rho_c(x,y)+\rho_c(y,T\,x)+\rho_c(x,T\,y)} + a_3\,\frac{\rho_c(y,T\,y)\,\rho_c(x,T\,x)}{\rho_c(x,y)} + a_4\,\rho_c(x,T\,y) + a_5\,\rho_c(y,T\,x) \qquad (2.20)$$

where $a_1, a_2, a_3, a_4, a_5 \ge 0$ are constants such that $a_1 + a_2 + a_3 + 2a_4 + a_5 < 1$.

Then T has a unique fixed point in X.

Proof Choose $x_0 \in X$ such that $T\,x_0 = x_1, T\,x_1 = T^2x_0 = x_2 \ldots x_n = T\,x_{n-1} = T^n x_0$. Then

$$\rho_c(x_n, x_{n+1}) = \rho_c(T\,x_{n-1}, T\,x_n)$$

$$\le a_1\,\rho_c(x_{n-1},x_n) + a_2\,\frac{\rho_c(x_{n-1},T\,x_{n-1})\rho c(x_n,T\,x_n)}{\rho_c(x_{n-1},x_n)+\rho_c(x_n,T\,x_{n-1})+\rho_c(x_{n-1},T\,x_n)} + a_3\,\frac{\rho_c(x_n,T\,x_n)\rho_c(x_{n-1},T\,x_{n-1})}{\rho_c(x_{n-1},x_n)} + a_4\rho_c(x_{n-1},T\,x_n) + a_5\rho_c(x_n,T\,x_{n-1})$$

$$\le a_1\,\rho_c(x_{n-1},x_n) + a_2\,\frac{\rho_c(x_{n-1},x_n)\rho_c(x_n,x_{n+1})}{\rho_c(x_{n-1},x_n)+\rho_c(x_n,x_n)+\rho_c(x_{n-1},x_{n+1})} + a_3\,\frac{\rho_c(x_n,x_{n+1})\rho_c(x_{n-1},x_n)}{\rho_c(x_{n-1},x_n)} + a_4\rho_c(x_{n-1},x_{n+1}) + a_5\rho_c(x_n,x_n)$$

$$\le a_1\rho_c(x_{n-1},x_n) + a_2\rho_c(x_{n-1},x_n) + a_3\rho_c(x_n,x_{n+1}) + a_4\rho_c(x_{n-1},x_{n+1}) + a_5\rho_c(x_n,x_n)$$

$$\le a_1\rho_c(x_{n-1},x_n) + a_2\rho_c(x_{n-1},x_n) + a_3\rho_c(x_n,x_{n+1}) + a_4\rho_c(x_{n-1},x_n) + a_4\rho_c(x_n,x_{n+1}) - a_4\rho_c(x_n,x_n) + a_5\rho_c(x_n,x_n)$$

$$= (a_1 + a_2 + a_4)\rho_c(x_{n-1},x_n) + (a_3 + a_4)\rho_c(x_n,x_{n+1}) + (a_5 - a_4)\rho_c(x_n,x_n)$$

$$= (a_1 + a_2 + a_4)\rho_c(x_{n-1},x_n) + (a_3 + a_4 + a_5)\rho_c(x_n,x_{n+1})$$

$$\le \frac{(a_1 + a_2 + a_4)}{1 - (a_3 + a_4 + a_5)}\rho_c(x_{n-1},x_n) \qquad (2.21)$$

$$Let\,\lambda = \frac{(a_1 + a_2 + a_4)}{1 - (a_3 + a_4 + a_5)}$$

Since $a_1 + a_2 + a_3 + 2a_4 + a_5 < 1$ and $a_3 + a_4 + a_5 < 1$ implies that $\lambda < 1$. Hence

$$\rho_c(x_n,x_{n+1}) \le \lambda\rho_c(x_{n-1},x_n) \qquad (2.22)$$

for all $n \in \mathbb{N}$. For any $m > n$ where $m, n \in \mathbb{N}$ we have

$$\rho_c(x_m,x_n) \le \rho_c(x_m,x_{m-1}) + \rho_c(x_{m-1},x_{m-2})\ldots\ldots\rho_c(x_{n+1},x_n) - \sum_{k}^{m-n-1} \rho_c(x_{m-k},x_{m-k})$$

$$\le \rho_c(x_m,x_{m-1}) + \rho_c(x_{m-1},x_{m-2})\ldots\ldots\rho_c\,(x_{n+1},x_n)$$

$$\le (\lambda^{m-1} + \lambda^{m-1} + \ldots\ldots\lambda^n)\rho_c(x_0,x_1)$$

$$= \frac{\lambda^n}{1-\lambda}\rho_c(x_0,x_1) \qquad (2.23)$$

Since P is normal cone with normal constant K, we have

$$||\rho_c(x_m, x_n))|| \le\ K||\frac{\lambda^n}{1-\lambda}\rho_c(x_0, x_1)|| \tag{2.24}$$

Now since $\lambda\ <\ 1, ||\rho_c(x_m, x_n))|| \le\ K||\frac{\lambda^n}{1-\lambda}\ \rho_c(x_0, x_1)|| \to\ 0$ as $n\ \to\ \infty$

Hence $\{x_n\}$ is a Cauchy sequence in a partial cone metric space which is complete hence it must be convergent in X, let $\lim_{n\to\infty} x_n\ =\ z$ therefore

$$\rho_c(z, z)\ =\ \lim_{n\to\infty}\rho_c(x_n, z)\ \ = \lim_{n\to\infty}\rho_c(x_n, x_n)\ =\ 0$$

Now we show that $||\rho_c(T\,z, z)|| \ge\ 0$, On contrary, assume $||\rho_c(T\,z, z)|| >\ 0$ we have

$\rho_c(T\,z, z)\ \le\ \rho_c(T\,z, T\,x_n)\ +\ \rho_c(T\,x_n, z) - \rho_c(T\,x_n, T\,x_n)$

$\le a_1\rho_c(z, x_n) +\ a_2\frac{\rho_c(z,T\,z)\rho_c(x_n,T\,x_n)}{\rho_c(z,x_n)+\ \rho_c(x_n,T\,z)+\ \rho_c(z,T\,x_n)}\ +\ a_3\frac{\rho_c(x_n,T\,x_n)\rho_c(z,T\,z)}{\rho_c(z,x_n)} + a_4\rho_c(z, T\,x_n) +$

$$a_5\rho_c(x_n, T\,z)\ +\ \rho_c(T\,x_n, z) - \rho_c(T\,x_n, T\,x_n)$$

$\le a_1\rho_c(z, x_n) +\ a_2\frac{\rho_c(z,T\,z)\rho_c(x_n,x_{n+1})}{\rho_c(z,x_n)+\ \rho_c(x_n,T\,z)+\ \rho_c(z,x_{n+1})}\ +\ a_3\frac{\rho_c(x_n,x_{n+1})\rho_c(z,T\,z)}{\rho_c(z,x_n)} + a_4\rho_c(z, T\,x_n) +$

$$a_5\rho_c(x_n, T\,z)\ +\ \rho_c(\ x_{n+1}, z)$$

$\le a_1\rho_c(z, x_n)\ +\ + a_4\rho_c(z, x_{n+1})\ +\ a_5\rho_c(x_n, T\,z)\ +\ \rho_c(x_{n+1}, z)$

$\le a_1\rho_c(z, x_n) +\ + a_4\ \rho_c(z, x_{n+1})\ +\ a_5\rho_c(T\,z, z)\ +\ a_5\rho_c(z, x_n) - a_5\rho_c(z, z)\ +\ \rho_c(x_{n+1}, z)$

$$\le\ (a_1\ +\ a_5)\rho_c(z, x_n)\ +\ +[a_4\rho_c(x_{n+1}, z)\ +\ \rho_c(x_{n+1}, z)]\ +\ a_5\rho_c(T\,z, z) \tag{2.25}$$

So using (2.25) we have

$$\rho_c(T\,z, z)\ \le\ \frac{(a_1 + a_5)}{1 - a_5}\rho_c(z, x_n)\ +\ \frac{a_4}{1 - a_5}\rho_c(x_{n+1}, z)\ +\ \frac{1}{1 - a_5}\rho_c(x_{n+1}, z) \tag{2.26}$$

Hence

$$||\rho_c(T\,z, z)||\ \le\ \frac{(a_1 + a_5)}{1 - a_5}K||\rho_c(z, x_n)|| + \frac{a_4}{1 - a_5}K||\rho_c(x_{n+1}, z)|| + \frac{1}{1 - a_5}K||\rho_c(x_{n+1}, z)|| \to\ 0 \tag{2.27}$$

So we have $||\rho_c(T\,z, z)|| =\ 0$ therefore $\rho_c(T\,z, z)\ =\ 0$ or T z = z.

Uniqueness If z_1 is s another Fixed Point of T , Then $T\,z_1\ =\ z_1$ replacing x by z

and y by z_1 in (2.1) we get

$$\begin{aligned}\rho_c(z, z_1) &= \rho_c(T\,z, T\,z_1)\\ &\le a_1\rho_c(z, z_1) +\ a_2\frac{\rho_c(z, T\,z)\rho_c(z_1, T\,z_1)}{\rho_c(z, z_1)\ +\ \rho_c(z_1, T\,z)\ +\ \rho_c(z, T\,z_1)}\\ &\quad +\ a_3\frac{\rho_c(z_1, T\,z_1)\rho_c(z, T\,z)}{\rho_c(z, z_1)}\ +\ a_4\rho_c(z, T\,z_1)\ +\ a_5\rho_c(z_1, T\,z)\end{aligned}$$

$$\leq a_1\rho_c(z,z_1) + a_2\frac{\rho_c(z,z)\rho_c(z_1,z_1)}{\rho_c(z,z_1)+\rho_c(z_1,z)+\rho_c(z,z_1)} + a_3\frac{\rho_c(z_1,z_1)\rho_c(z,z)}{\rho_c(z,z_1)} + a_4\rho_c(z,\ z_1) + a_5\rho_c(z_1,\ z) \tag{2.28}$$

Therefore $\rho_c(z,z_1) = 0$ or $z = z_1$.

3. Application

This section is influenced by the findings presented in the papers [19, 20] which aims to offer an application of Theorem 2.3 to the solution of second order differential equation of the form

$$x''(t) = -f\,(t,x(t)),t \in I$$
$$x(0) = x(1) = 0 \tag{3.1}$$

$where\ I = [0,1], f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function.Consider the space X = C(I) of continuous function defined on I. It is well-known that the Problem (3.1) is equivalent to the integral equation

$$x(t) = \int_0^1 g(t,s)f\,(s,x(s))\,ds \tag{3.2}$$

for all $t \in [0,1]$ where g is the Green function defined by

$$g(t,s) = \begin{cases}(1-s)t & 0 \leq \mathrm{t} \leq \mathrm{s} \leq 1\\(1-t)s & 0 \leq \mathrm{s} \leq \mathrm{t} \leq 1\end{cases} \tag{3.3}$$

Then solving problem (3.1) is equivalent to finding fixed point of T in C(I).

Theorem 3.1 Let $X = C(I)$ and $T : X \to X$ be an operator given by

$$T\,x(t) = \int_0^1 g(t,s)f\,(s,x(s))\,ds \tag{3.4}$$

for all $x \in X$ and $t \in I = [0,1]$. Suppose the following conditions hold:

(i) For all $t \in I$, for all $a,b \in \mathbb{R}\ with\ ||a||,||b|| \leq 1$, we have
$$|f\,(t,a) - f\,(t,b)| \leq 8\mu(|a-b|) \tag{3.5}$$

(ii) there exists $x_0 \in C(I)$ such that $\left||x_0|\right|_\infty \leq 1$,

(iii) for all $x \in C(I)$
$$|x||_\infty \leq 1 \to ||\int_0^1 g(t,s)f\left(s,x(s)\right)ds||_\infty \leq 1 \tag{3.6}$$

Then the second order differential equation (3.1) has a solution.

Proof Consider C(I) endowed with the partial metric given by

$$\rho_c(x,y) = \begin{cases}\left||x-y|\right|_\infty & ||x|| \leq 1, ||y|| \leq 1\\ \left||x-y|\right|_\infty + \tau & otherwise\end{cases} \tag{3.7}$$

where $\tau > 0$. Then $(C(I),\rho_c)$ is a partial metric space. Now we define partial cone metric as

$$\rho_c(x,y) \; = \; (\rho_c(x,y), \beta\rho_c(x,y)) \qquad (3.8)$$

where $\beta \; \geq \; 0$. Now, let $x, y \; \in \; C(I)$ such that $||x|| \leq \; 1$, $||y|| \leq \; 1$, then we have

$$\rho_c(T\,x, T\,y) \; = \; (\left||T\,x \; - \; T\,y|\right|_\infty, \beta\left||T\,x \; - \; T\,y|\right|_\infty)$$

$$= \left(\sup_{t\in[0,1]} \int_0^1 g(t,s)\left|f\left(s, x(s)\right) - f\left(s, y(s)\right)\right| ds \, , \beta \sup_{t\in[0,1]} \int_0^1 g(t,s)\left|f\left(s, x(s)\right) - f\left(s, y(s)\right)\right| ds\right)$$

$$= \left(\sup_{t\in[0,1]} \int_0^1 g(t,s) 8\,\mu|x(s) - y(s)| ds \, , \beta \sup_{t\in[0,1]} \int_0^1 g(t,s) 8\,\mu|x(s) - y(s)| ds\right)$$

$$= \left(\sup_{t\in[0,1]} \int_0^1 g(t,s) \times (8\,\mu\left||x-y|\right|_\infty) ds \, , \beta \sup_{t\in[0,1]} \int_0^1 g(t,s) \times (8\,\mu\left||x-y|\right|_\infty) ds\right) \quad (3.9)$$

Now, as we know $\sup_{t\in[0,1]} \int_0^1 g(t,s) ds = \frac{1}{8}$ and taking $\psi(t) \; = \; \mu t$

$$\rho_c(T\,x, T\,y) \; \leq \; \mu(\rho_c(x,y)), \beta(\rho_c(x,y)))$$

$$\leq \; \mu\rho_c(x,y)$$

$$\leq \; \psi(\rho_c(x,y)) \; \leq \; \psi(max\{\rho_c(x,y), \rho_c(x, T\,x), \rho_c(y, T\,y)\}) \qquad (3.10)$$

Define the function $\alpha : \; C(I) \; \times \; C(I) \; \rightarrow \; [0, \infty)$ as

$$\alpha(x,y) = \begin{cases} 1 & ||x|| \leq \; 1, ||y|| \leq \; 1 \\ 0 & otherwise \end{cases} \qquad (3.11)$$

For all $x, y \; \in \; C(I)$

$$\alpha(x,y)\rho_c(T\,x, T\,y) \; \leq \; \psi(max\{\rho_c(x,y), \rho_c(x, T\,x), \rho_c(y, T\,y)\}) \qquad (3.12)$$

Clearly, all the conditions of Theorem 2.3 are satisfied and so Γ has a fixed point. Thus the system of integral equations (3.2) has a solution.

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