Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 7s (2024)

Complementary Tree Domination Number of Corona Product of Complete graph with Some Graphs

***S. Jayalakshmi [1] and P. Vidhya[2]**
[1]*Research Scholar (Part Time), School of Mathematics*
Madurai Kamaraj University, Madurai - 625021, Tamilnadu, India
E-mail: jayark83@gmail.com , jayalakshmi.s1@sdnbvc.edu.in
[2]*Associate Professor , Department of Mathematics,*
EMG Yadava Women's College
Madurai - 625014, Tamilnadu, India
E-mail: vidhyaramman@gmail.com ,
p.vidhya-mat@emgywomenscollege.ac.in

Article History:
Received: 12-08-2024
Revised: 15-09-2024
Accepted: 25-10-2024

Abstract: A set D of a graph G = (V, E) is a dominating set, if every vertex in V − D is adjacent to some vertex in D. The domination number γ(G) of G is the minimum cardinality of a dominating set. A dominating set D is called a complementary tree dominating set if the induced subgraph <V − D> is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G and is denoted by γctd(G). The corona G1 ∘ G2 of two graphs G1 and G2 are defined as the graph G obtained by taking one copy of G1 of order p1 and p1 copies of G2 and then joining the ith vertex of G1 to every vertex in the ith copy of G2. The corona G1 ∘ G2 has p1(1 + p2) vertices and q1 + p1q2 + p1p2 edges. In this paper, we discussed complementary tree domination number of corona product of complete graph with some graphs.

AMS Subject Classification: 05C69.

Keywords: Dominating set, Complementary tree domination number.A subset D of the vertex set V(G) of a graph G is said to be a dominating set if every vertex not in D is adjacent to at least one vertex in D. A dominating set D is said to be an eccentric dominating set if for every , there exists at least one eccentric point of v in D. An eccentric dominating set D of G is a non split eccentric dominating set if the induced sub graph < V- D > is connected. The minimum of the cardinalities of the non split eccentric dominating sets of G is called the non split eccentric domination number of G. This paper evaluates the non split eccentric domination number of Corona product and join of some standard graphs.

Keywords: Domination, Eccentric Domination, Non Split Eccentric Domination, Corona product, Join.

1 Introduction

A Graph $G(V, E)$ discussed in this paper be a simple, finite, undirected, connected graph with p vertices and q edges. Roberto Frucht and Frank Harary [1] introduced the binary product of two graphs named Corona in 1970. The corona $G_1 \circ G_2$ of two graphs G_1 and G_2 are defined as the graph G obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2 and then joining the i$^{\text{th}}$ vertex of G_1 to every vertex in the i^{th} copy of G_2. The Corona $G_1 \circ G_2$ has $p_1(1 + p_2)$ vertices and $q_1 + p_1q_2 + p_1p_2$ edges. The concept of domination in graphs was introduced by Ore [4]. A set $D \subseteq V$ is said to be a dominating set of G, if every vertex in V-D is adjacent to some vertex in D. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. The complementary tree domination number of a graph was introduced by S. Muthammai, M. Bhanumathi and P. Vidhya [3] have established some results on complementary tree domination number of graphs. A set $D \subseteq V(G)$ is said to be complementary tree dominating set (ctd-set) if the induced subgraph $<V(G)\text{-}D>$ is a tree. The minimum cardinality of a ctd-set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$. Sergio canoy Jr and Carmelito E.Go [5] have obtained the domination number of corona graphs. Any undefined term in this paper may be found in Harary[2]

For notation convenience G_2^v be a copy of G_2 corresponding to the vertex $v \in V(G_1)$. Also u_{ij} be the vertex of G which are adjacent to the vertex $v_i \in V(G_1)$.

In this paper we discussed complementary tree domination number of corona product of complete graph and their bounds are determined.

2 Prior Results

Observation 2.1. [3]

(i) For any path P_n with n vertices, $\gamma_{ctd}(P_n) = n - 2$, $n \geq 4$.

(ii) For any cycle C_n with n vertices, $\gamma_{ctd}(C_n) = n - 2$, $n \geq 3$.

(iii) For any complete graph K_n with n vertices, $\gamma_{ctd}(K_n) = n - 2$, $n \geq 3$.

(iv) For any star $K_{1,n}$, $\gamma_{ctd}(K_{1,n}) = n$, $n \geq 2$.

(v)For any complete bipartite graph $K_{m,n}$ with $m, n \geq 2$, $\gamma_{ctd}(K_{m,n}) = \min\{m, n\}$.

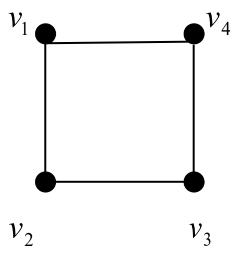
(vi) $\gamma_{ctd}(C_n \circ K_1) = n + 1$, $n \geq 3$, where $C_n \circ K_1$ is the corona of C_n and K_1.

(vii) For any wheel W_n with n vertices, $\gamma_{ctd}(W_n) = 2$, $n \geq 4$.

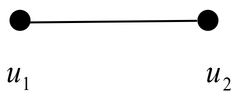
Preposition 2.2 . [3]

If $\gamma_{ctd}(G) \leq p - 2$,then pendant vertices are the members of every ctd-set.

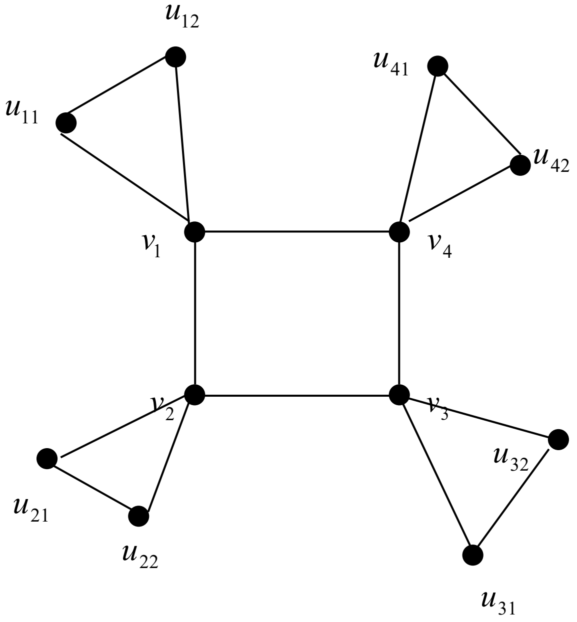
Example 2.3.



G1



G2



$G_1 \circ G_2$
Figure 1:

For the graph $G_1 \circ G_2$ given in Figure 1.

In the following, a necessary and sufficient condition for a ctd-set of a corona product of graphs $G_1 \circ G_2$ is found.

Theorem 2.4

Let G_1 and G_2 be connected graphs then $D \subseteq V(G_1 \circ G_2)$ is a ctd-set in $G_1 \circ G_2$ if and only if one of the following conditions holds.

(i) For each $v \in V(G_1)$, $V(G_2^v) \cap D$ is a dominating in G_2^v and $D \subseteq \bigcup_{u \in V(G)} V(G_2^u)$.

(ii) $V(G_1) \cap D$ is a complementary tree dominating in G_1 and $V(G_2^v) \subseteq D$ whenever $v \in V(G_1) \cap D$ and $V(G_2^v) \cap D$ is dominating in G_2^v whenever $v \in V(G_1) - D$.

Proof. Suppose $V(G_1) \cap D = \phi$.

Let $v \in V(G_1)$ and $x \in V(G_2^v) - D$. Hence $x \in V(G_1 \circ G_2) - D$. Since D is a ctd-set of $G_1 \circ G_2$. There exists $y \in D$ such that $d_{G_1 \circ G_2}(x, y) = 1$. Since $x \in V(G_2^v)$ and $d_{G_1 \circ G_2}(x, y) = 1$ either $y \in V(G_2^v)$ or $y = v$. If $y = v$ then $y \in V(G_1) \cap D$, a contradiction. Therefore $y \in V(G_2^v) \cap D$ is a dominating set of G_2^v. Since $V(G_1) \cap D = \phi$, $D \subseteq \bigcup_{u \in V(G_1)} V(G_2^u)$. Hence (i) holds.

Suppose $V(G_1) \cap D \neq \phi$ and $V(G_1) \cap D \neq V(G_1)$.

Let $x \in V(G_1) - D$ it follows $x \in V(G_1 \circ G_2) - D$. Let D is a dominating set of $G_1 \circ G_2$ there exists $y \in D$ such that $d_{G_1 \circ G_2}(x, y) = 1$. If $y \in V(G_1)$ then $y \in V(G_1) \cap D$ then $d_G(x, y) = 1$. Hence $V(G_1) \cap D$ is a dominating set in G_1. Now we have to prove $V(G_1) \cap D$ is a complementary tree dominating set in G_1 it is enough to prove $\langle V(G_1) - D \rangle$ is a tree. Let $x, y \in V(G_1) - D$. Since $< V(G_1 \circ G_2) - D >$ is a tree T. Therefore $x, y \in V(G_1)$ there exist $u \in V(G_2^x)$ and $v \in V(G_2^y)$. Since T is connected and acyclic. There exists a unique path between u and v .Hence x and y are the vertices transverse from u and v. Hence $V(G_1) \cap D$ is a ctd-set in G1.

Suppose $v \in V(G_1) \cap D$ and $V(G_2^v) - D \neq \phi$.

Let $u \in V(G_2^v) - D$. Since $V(G_1) \cap D \neq V(G_1)$ there exists $w \in V(G_1) - D$. Since $< V(G_1 \circ G_2) - D >$ is a tree. There exists a unique path between u - w with vertices from $V(G_1 \circ G_2) - D$. However any u-w path must contain a vertex v which is impossible. Hence $V(G_2^v) - D = \phi$. Hence $V(G_2^v) \subseteq D$.

Suppose $v \in V(G_1) - D$.

Let $x \in V(G_2^v) - D$. This implies that $x \in < V(G_1 \circ G_2) - D >$. Since D is a ctd-set in $G_1 \circ G_2$ there exists $y \in D$ such that $d_{G_1 \circ G_2}(x, y) = 1$. Consequently $y = v$ or $y \in V(G_2^v)$. Since $y \in D$ and $v \in V(G_1) - D$. Hence $y \neq v$ then it follows $y \in V(G_2^v)$ now $d_{G_1 \circ G_2}(x, y) = 1$ implies $d_{G_2^v}(x, y) = 1$. Hence $V(G_2^v) \cap D$ is a dominating set in G_2^v .Therefore (iii) holds.

Conversely, Suppose (i) holds . Let $x \in V(G_1 \circ G_2) - D$.

Suppose $x \in V(G_1)$ .Since $V(G_2^x) \cap D$ is dominating set in G_2^x and $V(G_2^x) \cap D \neq \phi$. Let $u \in V(G_2^x) \cap D$. Then $d_{G_1 \circ G_2}(x, y) = 1$.

Suppose $x \notin V(G_1)$ then there exist $y \in V(G_1)$ such that $x \in V(G_2^y)$ since $V(G_2^y) \cap D$ is a dominating set in G_2^y and $x \in V(G_2^y) - D$ there exist $t \in V(G_2^y) \cap D$ such that $d_{G_1 \circ G_2}(x, t) = 1$ then D is a dominating set in $G_1 \circ G_2$. Since $D \subseteq \bigcup_{u \in V(G_1)} V(G_2^u)$ *and* $V(G_1) \cap D = \phi$.

Consequently, $V(G_1 \circ G_2) - D = \bigcup_{v \in V(G)} V(G_2^v) - D \cup V(G_1)$ .Let $p, q \in V(G_1 \circ G_2) - D, p \neq q$. If $p, q \in V(G_2^v) - D$ for some $v \in V(G_1)$ then there is a path with vertices p, v, q in $V(G_1 \circ G_2) - D$.

If $p, q \in V(G_1)$ .Then there is a tree which contains a p-q path in $V(G_1 \circ G_2) - D$ .Since G1 is connected.

If $p \in V(G_1)$ and $q \in V(G_2^v) - D$ for some $v \in V(G_1)$ then $V(G_1 \circ G_2) - D$ contains a tree with vertices p and v . Suppose $p \neq v$ since G1 is connected then $V(G_1 \circ G_2) - D$ contains a tree with vertices p, v, q .Suppose $p \in V(G_2^v) - D$ and $q \in V(G_2^w)$ for some

$v,w\in V(G_1)$. Since G_1 is connected then there exist a tree with vertices p,v,w,q in $V(G_1\circ G_2)-D$ .Hence $V(G_1\circ G_2)-D$ is connected and acyclic. Therefore $V(G_1\circ G_2)-D$ is a tree . Hence D is a complementary tree dominating in $G_1\circ G_2$.

Suppose (ii) holds. Let $x\in V(G_1\circ G_2)-D.$ Suppose $x\in V(G_1)$ $(i.e)$ $x\in V(G_1)-D.$ Since $V(G_1)\cap D$ is a dominating in G_1, there exist $y\in V(G_1)\cap D$ such that $d_G(x,t)=1$ for some $t\in V(G_1)$ is follows that $d_{G_1\circ G_2}(x,y)=1.$ Suppose $x\in V(G_2^v)$ for some $v\in V(G_1)$ $(i.e)\, x\in V(G_2^v)-D$ $(i.e)\, d_{G_1\circ G_2}(x,v)=1.$ If $v\in D$ which is a contradiction to $x\in V(G_2^v)-D.$ Hence $v\notin D$ is $v\in V(G_1)-D$. In this case $V(G_2^v)\cap D$ is dominating in G_2^v $(i.e)$ there exist $w\in V(G_2^w)\cap D$ such that $d_{G_2^v}(x,w)=1.$ It follows that $d_{G_1\circ G_2}(x,w)=1.$ Hence D is a dominating set in $G_1\circ G_2$.

Let $x,y\in V(G_1\circ G_2)$ suppose $x,y\in V(G_1)$ $(i.e)$ $x,y\in V(G)-D$. Since $V(G)\cap D$ is a complementary tree dominating set in G_1 and $<V(G_1)-D>$ is a tree. From this $<V(G_1\circ G_2)-D>$ is a tree which contains a path x-y in $V(G_1)-D.$

Suppose $x\in V(G_1)$ and $y\in V(G_2^v)$ for some $v\in V(G_1)$ $(i.e)$ $x\in V(G_1)-D$ and $y\in V(G_2^v)-D.$ If $x=v$ there exists path between x-y. Suppose $x\neq v$ if $v\in V(G_1)\cap D$ then $V(G_2^v)\subseteq D.$ This contradicts the fact that $V(G_2^v)-D\neq 0.$ Then $v\in V(G_1)-D,$ consequently, $V(G_2^v)-D$ is a dominating set in G_2^v $(i.e)$ $V(G_2^v)\cap D\neq\phi.$ Suppose $x\in V(G_2^v)\cap D.$ Since $v\in V(G_1)-D$ and $<V(G_1)-D>$ contains a path with vertices $x-v.$ Thus $<V(G_1\circ G_2)-D>$ contains a tree with vertices $x-v.$

Suppose $x,y\in V(G_2^v)-D,$ $x\neq y$ for some $v\in V(G).$ If $v\in D,$ then $V(G_2^v)\subseteq D$ .This contradicts the fact that $V(G_2^v)-D\neq\phi.$ Thus $v\notin D$ (i.e) $v\in V(G_1)-D.$ Now there exists a path with vertices x,v,y in $V(G_1\circ G_2)-D.$

Suppose $x\in V(G_2^v)$ and $y\in V(G_2^w)$ for some $v,w\in V(G),$ $v\neq w.$ Then $x\in V(G_2^v)-D$ and $y\in V(G_2^w)-D$ if $v\in D$ or $w\in D$ then $V(G_2^v)\subseteq D$ and $V(G_2^w)\subseteq D.$ This contradicts the facts that $V(G_2^v)-D\neq\phi$ and $V(G_2^w)-D\neq\phi$. Thus $v,w\notin D$ that is $v,w\in V(G_1)-D\subseteq V(G_1\circ G_2)-D.$ Since $<V(G_1)-D>$ is a tree. There is a tree with support vertices v and w in $V(G_1\circ G_2)-D$ .Hence $<V(G_1\circ G_2)-D>$ is a tree. Hence D is a complementary tree dominating set in $G_1\circ G_2$.

Corollary 2.5.

Let G_1 and G_2 be any connected graph . Then $\gamma_{ctd}(G_1\circ G_2)\geq 2(|G_1|-1)\gamma(G_2).$
G_1 is not a tree.

Proof. Let $|G_1|=n.$ Suppose there exist D such that D satisfies (ii) of theorem [4.1.2]

$$D \geq (n-2)|G_2| + 2\gamma(G).$$

Since

$$\gamma(G_2) \leq \frac{|G_2|}{2}$$

$$(i.e)\ 2\gamma(G_2) \leq |G_2|$$

$$\begin{aligned} D &\geq (n-2)2\gamma(G_2) + 2\gamma(G_2) \\ &= 2(n-1)\gamma(G_2) \\ &\geq 2(|G_1|-1)\gamma(G_2) \end{aligned}$$

$$\gamma_{ctd}(G_1 \circ G_2) \geq 2(|G_1|-1)\gamma(G_2).$$

Corollary 2.6.

Let G_1 be a tree and G_2 be any connected graph respectively. Then $\gamma_{ctd}(G_1 \circ G_2) = G_1\gamma(G_2).$

Proof. For each $v \in V(G_1)$. Let $G_2^{\,v}$ be a copy of G_2 corresponding to vertex *v*. Further, for each $v \in V(G_1)$. Let D^v be a minimum dominating set in G_2^v. By theorem [4.1.2] $D = \bigcup_{v \in V(G_1)} D^v$ is a complementary tree dominating set in $G_1 \circ G_2$. Thus

$$\begin{aligned} \gamma_{ctd}(G_1 \circ G_2) &\leq D \\ &= \left| \bigcup_{v \in V(G_1)} D^v \right| \\ &= \sum_{v \in V(G_1)} |D^v| \\ &= |G_1|\gamma(G_2) \end{aligned}$$

Therefore, $\gamma_{ctd}(G_1 \circ G_2) \leq |G_1|\gamma(G_2).$

Here , we consider G_1 and G_2 be any connected graph of order n and m respectively. Then the vertex set $\{u_{ij}/1 \leq i \leq n, 1 \leq j \leq m\}$ is the ith copy of G_2 is adjacent to the ith vertex of G_1 and let D is a minimum ctd-set of $G_1 \circ G_2$. Hence $< V(G_1 \circ G_2) - D >$ is a tree.

3 Complementary Tree Domination Number of Corona Product of complete graph with Some Graphs

In this section, for $n \geq 4$ complementary tree domination number of $K_n \circ G_1$, where G_1 is any connected graph with $m \geq 3$ vertices are obtained.

Here, we consider D_1 is a ctd-set of K_n, hence $|D_1| = n-2$ and D_2 be the set whose elements are the vertices of G_1 which are adjacent to each vertex of

D_1.Hence $|D_2| = (n-2)m$.Clearly, $D_1 \cup D_2 \subseteq D$,where D is a minimum ctd-set of $K_n \circ G_1$ and $|D_1| \cup |D_2| = (m+1)(n-2)$.

Proposition 3.1. $\gamma_{ctd}(K_n \circ K_1) = 2n-2$.

Proof. Let $G = K_n \circ K_1$.

Let $V(K_n) = \{v_1, v_2, \ldots\ldots. v_n\}$ and vertex u_i be the i^{th} copy of K_1 attached to the vertex v_i.Then $V(G) = \{v_i, u_i / 1 \leq i \leq n\}$.Here $u_1, u_2 \ldots\ldots u_n$ are the pendant vertices of G. We have pendant vertices are members of ctd-set of G [3]. Hence, $D = \{v_i : 1 \leq i \leq n-2\} \cup \{u_i : 1 \leq i \leq n\} = n-2+n = 2n-2$. is a minimum ctd-set of G. Hence, $|D| = \gamma_{ctd}(G) = 2n-2$.

Proposition 3.2. $\gamma_{ctd}(K_n \circ K_2) = 3n-4$.

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(K_2) = \{u_1, u_2\}$. Then

$V(K_n \circ K_2) = \{v_i / 1 \leq i \leq n\} \cup \{u_{ij} / 1 \leq i \leq n, 1 \leq j \leq 2\}$ .We have, $\gamma_{ctd}(K_n) = n-2$ [3]. By choosing $(n-2)$ vertices of K_n say $v_1, v_2, \ldots, v_{n-2}$ which form a minimum ctd-set in K_n and vertices which are adjacent to $\{v_1, v_2, \ldots, v_{n-2}\}$ are $\{u_{ij} / 1 \leq i \leq n-2, 1 \leq j \leq 2\}$. Therefore,

$D = \{v_i : 1 \leq i \leq n-2\} \cup \{u_{ij} : 1 \leq i \leq n-2, 1 \leq j \leq 2\} \cup \{u_{n-1,1}, u_{n,1}\}$

$= n - 2 + 2n - 4 + 2$

$= 3n - 4$

Proposition 3.3 . *For* $m \geq 2$, $\gamma_{ctd}(K_n \circ \overline{K_m}) = mn + n - 2$.

Proof. Take $G = K_n \circ \overline{K_m}$. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and $\{u_1, u_2, \ldots, u_m\}$ be the vertex set of the i^{th} copy of $\overline{K_m}$ is adjacent to the vertex v_i. Then $V(G) = \{v_i / 1 \leq i \leq n\} \cup \{u_{ij} / 1 \leq i \leq n, 1 \leq j \leq m\}$ where u_{ij}*'s* is the i^{th} copy of $\overline{K_m}$ is adjacent to the vertex v_i in K_n. Let D be a minimum ctd-set of G. Since pendant vertices are members of every ctd-set G. By choosing pendant vertices, it dominates all the vertices of K_n but $\langle V(G) - D \rangle$ forms a cycle. So we are choosing $\{v_i\}_{i=1}^{n-2}$ and all the pendant vertices in a graph G. Therefore,

$D = \{v_i : 1 \leq i \leq n-2\} \cup \{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$

$= n - 2 + nm$

$= mn + n - 2.$

Proposition 3.4. *For* $m \geq 3$, $\gamma_{ctd}(K_n \circ K_{1,m-1}) = n + (m+1)(n-2)$.

Proof. Take $G = K_n \circ K_{1,m-1}$. Let $V(K_n) = \{v_i / 1 \leq i \leq n\}$ and $V(K_{1,m-1}) = \{w, u_1, u_2, \ldots, u_{m-1}\}$. Then $V(G) = \{v_i / 1 \leq i \leq n\} \cup \{w_i u_{ij},\ 1 \leq i \leq n,\ 1 \leq j \leq m-1\}$. By choosing a vertex $\{c_i / 1 \leq i \leq n\}$ which dominates all the vertices of $K_n \circ K_{1,m-1}$. But $\langle V(G) - D \rangle \cong K_n \circ \overline{P_m}$ which contains a cycle. Let $D = D_1 \cup D_2 \cup \{c_1, c_2, \ldots, c_n\}$ is a minimum ctd-set of G.
$|D| = n + (m+1)(n-2)$.

Proposition 3.5. *For,* $m \geq 3$, $\gamma_{ctd}(K_n \circ P_m) = (m+1)(n-2) + 2\left\lfloor \frac{m}{2} \right\rfloor$.

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(P_n) = \{u_1, u_2, \ldots, u_m\}$. Then $V(K_n \circ P_m) = \{v_i / 1 \leq i \leq n\} \cup \{u_{ij} / 1 \leq i \leq n,\ 1 \leq j \leq m\}$ .We have, $\gamma_{ctd}(P_m) = m - 2$, $\gamma_{ctd}(K_n) = n - 2$ [3]. By choosing $(n-2)$ vertices of K_n say $v_1, v_2, \ldots, v_{n-2}$ which form a minimum ctd-set in K_n and vertices which are adjacent to $\{v_1, v_2, \ldots, v_{n-2}\}$ are $\{u_{ij} / 1 \leq i \leq n-2,\ 1 \leq j \leq m\}$. By Proposition 2.3

Case i : m is even

$D = D_1 \cup D_2 \cup \{u_{n-1,1}, u_{n-1,3}, u_{n-1,5}, \ldots . u_{n-1,m}\} \cup \{u_{n,1}, u_{n,3}, \ldots . u_{n,m}\}$ is a minimum ctd-set of $K_n \circ P_m$ and $\langle V(K_n \circ P_m) - D \rangle \cong S_{\frac{m}{2}, \frac{m}{2}}$

$$|D| = |D_1| + |D_2| + \left(\frac{m}{2}\right) + \left(\frac{m}{2}\right)$$

$$= (m+1)(n-2) + 2\left(\frac{m}{2}\right). \tag{1}$$

Case ii : m is odd

$D = D_1 \cup D_2 \cup \{u_{n-1,1}, u_{n-1,3}, u_{n-1,5}, \ldots . u_{n-1,m-1}\} \cup \{u_{n,1}, u_{n,3}, \ldots . u_{n,m-1}\}$ is a minimum ctd-set of $K_n \circ P_m$ and $\langle V(K_n \circ P_m) - D \rangle \cong S_{\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor}$

$$|D| = |D_1| + |D_2| + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \tag{2}$$

From 1 & 2

$$|D| = \gamma_{ctd}(K_n \circ P_m) = (m+1)(n-2) + 2\left\lfloor \frac{m}{2} \right\rfloor.$$

Proposition 3.6. *For* $m \geq 3$, $\gamma_{ctd}(K_n \circ C_m) = (m+1)(n-2) + 2\left\lfloor \frac{m}{2} \right\rfloor$.

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and the set $\{u_1, u_2, \ldots, u_m\}$ be the *vertices* of C_m. Then $V(K_n \circ C_m) = \{v_i/1 \leq i \leq n\} \cup \{u_{ij}/1 \leq i \leq n, 1 \leq j \leq m\}$ By preposition 2.4

Case i : m is even

$D = D_1 \cup D_2 \cup \{u_{n-1,1}, u_{n-1,3}, u_{n-1,5}, \ldots . u_{n-1,m-1}\} \cup \{u_{n,1}, u_{n,3}, \ldots . u_{n,m-1}\}$ is a minimum ctd-set of $K_n \circ C_m$ and $\langle V(K_n \circ C_m) - D \rangle \cong S_{\frac{m}{2},\frac{m}{2}}$

$$\begin{aligned} |D| &= |D_1| + |D_2| + \left(\frac{m}{2}\right) + \left(\frac{m}{2}\right) \\ &= (m+1)(n-2) + 2\left(\frac{m}{2}\right). \end{aligned} \tag{3}$$

Case ii : m is odd

$D = D_1 \cup D_2 \cup \{u_{n-1,1}, u_{n-1,3}, u_{n-1,5}, \ldots . u_{n-1,m}\} \cup \{u_{n,1}, u_{n,3}, \ldots . u_{n,m}\}$ is a minimum ctd-set of $K_n \circ C_m$ and $\langle V(K_n \circ C_m) - D \rangle \cong S_{\frac{m+1}{2} \frac{m+1}{2}}$

$$|D| = |D_1| + |D_2| + 2\left(\frac{m}{2}\right) \tag{4}$$

From 3 & 4

$$|D| = \gamma_{ctd}(K_n \circ C_m) = (m+1)(n-2) + 2\left\lceil \frac{m}{2} \right\rceil.$$

Proposition 3.7. *For* $m \geq 4$, $\gamma_{ctd}(K_n \circ W_m) = (m+1)(n-2) + 2\left\lceil \frac{m-1}{2} \right\rceil$

Proof. Let $V(K_n) = \{v_i / 1 \leq i \leq n\}$ and $V(W_m) = \{c, u_j / 1 \leq j \leq m-1\}$ and $V(K_n \circ W_m) = \{v_1, v_2, \ldots, v_n\} \cup \{c_i\, u_{ij} / 1 \leq i \leq n, 1 \leq j \leq m-1\}$ .Let

$D = D_1 \cup D_2 \cup \{C_{n-1}, C_n\} \cup \{$ctd-set of $K_n \circ C_{m-1}\}$and $\langle V(K_n \circ W_m) - D \rangle \cong S_{\left\lfloor \frac{m}{2} \right\rfloor}$

$|D| = (n-2) + m(n-2) + 2\left\lceil \frac{m-1}{2} \right\rceil$

Therefore,

$|D| = (m+1)(n-2) + 2\left\lceil \frac{m-1}{2} \right\rceil$.

Proposition 3.8. *For* $m \geq 4$, $\gamma_{ctd}(K_n \circ K_m) = n(m+1) - 4$.

Proof. Take $G = K_n \circ K_m$. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(K_m) = \{u_1, u_2, \ldots, u_m\}$. Then $V(G) = \{v_i / 1 \leq i \leq n\} \cup \{u_{ij} / 1 \leq i \leq n, 1 \leq j \leq m\}$ .We have, $\gamma_{ctd}(K_n) = n - 2$ [3].

Suppose by choosing $(n-2)$ vertices of K_n which dominates all the vertices of K_m^i $1 \leq i \leq n-2$ but it does not dominates n and $n-1$ copy of K_m which contradict the ctd-set.

Suppose by choosing $(m-2)$ vertices of K_m which dominates all the vertices of K_n but $\langle V(G) - D \rangle$ contains a cycle which is contradict to ctd-set. Let $D = D_1 \cup D_2 \cup \{u_{ij} / i = n-1, n, \quad 1 \leq j \leq m-1\}$ is a minimum ctd-set of G.

$= n - 2 + m(n-2) + m - 1 + m - 1$

Hence, $|D| = n(m+1) - 4$.

Proposition 3.9. *For* $m_1, m_2 \geq 2, \gamma_{ctd}(K_n \circ K_{m1,m2}) = (n-2)(m_1 + m_2 + 1) + 2\min(m_1, m_2)$.

Proof. Take $G = K_n \circ K_{m1,m2}$. Let $V(K_n) = \{v_i / 1 \leq i \leq n\}, V(K_{m1,m2}) = \{u_j / 1 \leq j \leq m_1\} \cup \{w_j / 1 \leq j \leq m_2\}$. Then $V(G) = \{v_i / 1 \leq i \leq n\} \cup \{u_{ij} / 1 \leq i \leq n; 1 \leq j \leq m_1\} \cup \{w_{ij} / 1 \leq i \leq n; 1 \leq j \leq m_2\}$. Let

D be the minimum ctd-set of $K_n \circ K_{m1,m2}$.

Case i. $m_1 < m_2$

By choosing $\{u_{ij}/1 \leq i \leq n,\ 1 \leq j \leq m_1\}$ which dominates all the vertices of G and $\langle V(G) - D\rangle \cong$ $K_n \circ \overline{K_{m2}}$ forms a cycle. Let $D = D_1 \cup D_2 \cup \{u_{n-1,1}, u_{n-1,2}, \ldots, u_{n-1,m1}\} \cup \{u_{n,1}, u_{n,2}, \ldots, u_{n,m1}\}$ is a minimum ctd-set of G. Hence,

$|D| = n - 2 + m_1(n - 2) + m_2(n - 2) + 2m_1$

$= (n - 2)(m_1 + m_2 + 1) + 2m_1$ (5)

Case ii. $m_2 < m_1$

By choosing a vertex $D = D_1 \cup D_2 \cup \{w_{n-1,1}, w_{n-1,2}, \ldots, w_{n-1,m2}\} \cup \{w_{n,1}, w_{n,2}, \ldots, w_{n,m2}\}$ which dominates all the vertices of $V(G)$ and $\langle V(G) - D\rangle$ is a tree. Here D is a ctd-set of G. Hence,

$|D| = n - 2 + m_1(n - 2) + m_2(n - 2) + 2m_2$

$= (n - 2)(n_1 + n_2 + 1) + 2m_2$ (6)

From (5) and (6)

$|D| = (n - 2)(m_1 + m_2 + 1) + 2\min(m_1, m_2)$.

Proposition 3.10. *For,* $m \geq 2$, $\gamma_{ctd}(K_n \circ K_m) \geq n(m + 1) - 4$.

Proof. Take $G = K_n \circ K_m$. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and $\{u_1, u_2, \ldots, u_m\}$be the vertex set of the i^{th} copy of K_m is adjacent to the vertex v_i in K_n.Then $V(G) = \{v_i/1 \leq i \leq n\} \cup \{u_{ij}/1 \leq i \leq n,\ 1 \leq j \leq m\}$. Let D be a minimum ctd-set of G. By choosing $\{u_{ij}/1 \leq i \leq n,\ 1 \leq j \leq \left\lfloor \frac{m+1}{2} \right\rfloor\}$ dominates all the vertices in a graph G but $\langle V(G) - D\rangle$ forms a cycle so we are choosing $(n - 2)$ vertices from $\{vi\}$ then their corresponding $\{u_{ij}\}$ vertices and for the remaining $n - 1$ and n vertices we choose $\left\lfloor \frac{m+1}{2} \right\rfloor$ vertices from $\{u_{ij}\}$.Therefore, $|D| \geq n(m + 1) - 4$.

4 Bounds for Complementary Tree Domination Number of Corona Product of Complete Graph with Some Graphs

Theorem 4.1.

For any connected graph G, with $m \geq 2$ vertices then

$3n - 4 \leq \gamma_{ctd}(K_n \circ G) \leq (m + 1)(n - 2) + 2(m - \beta 0)$.

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Let T be any induced subgraph of K_n having maximum number of edges such that $T \cong K_2$ is a tree. Then $|T| = 2$. Let S be a maximum independent set of G such that $|S| = \beta_0$.Let D_1 be the set of vertices of S in copies of G which are adjacent to the

vertices of T. Then $|D_1| = 2\beta_0$. Let $D = V(K_n \circ G) - (V(K_2) \cup D_1)$. Then $V(K_n \circ G) - D = V(K_2) \cup D_1$ and each vertex in $V(K_2)$ is adjacent to $m-\beta_0$ vertices in a copy of G. Also, each vertex in D_1 is adjacent to atleast $m-\beta_0$ vertex in a copy of G. Therefore D is a dominating set of $K_n \circ G$ and $\langle V(K_n \circ G) - D\rangle$ is the tree obtained from T by attaching $m - \beta_0$ pendant edges at each vertex of K_2. Therefore D is a ctd-set of $K_n \circ G$,

$\gamma_{ctd}(K_n \circ G) \le |D| = |V(K_n \circ G) - (V(K_2) \cup D_1)|$

$= mn + n - (2 + 2\beta_0)$

$= m(n - 2) + (n - 2) + 2(m - \beta_0)$

$= (n - 2)(m + 1) + 2(m - \beta_0)$

The lower bound equality holds if $G \cong K_2$.

References

[1] R.Frucht and F.Harary, “On the corona of two graphs”, *Aequationes Math.*, Vol. 4,pp. 322–325, 1970.

[2] F.Harary,“*Graph Theory*”, Addison Wesley, Reading Mass,1972.

[3] S.Muthammai, M.Bhanumathi and P.Vidhya, “Complementary tree domination number of a graph”, *International Mathematical Forum*, Vol. 6, No. 26,pp. 1273–1282, 2011.

[4] O.Ore, “Theory of Graphs”, *Amer. Math. Soc. Colloq. Publ.*, Vol. 38,1962.

[5] Sergio CanoyJr and Carmelito E.Go,“Domination in the corona and join of graphs”, *International Mathematical Forum*,Vol. 6, No. 13,2011.

[6] P. Vidhya and S. Jayalakshmi, “Complementary Tree Domination of Corona Product of Cycle C_n with Some Standard Graphs”, *Design Engineering*, Vol. 9,2021.

[7] S. Muthammai and P. Vidhya, “More Results on Complementary Tree Domination Number of Graphs”, *International Journal of Mathematics And its Applications*, Vol. 4, No. 1-D, pp. 17-20, 2016.