Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 10s (2025)

An Analytical and Numerical simulation of Fuzzy Delay Differential Equation Model of HIV infection of CD4+T-Cells

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Article History:

Received: 12-01-2025

Revised: 15-02-2025

Accepted: 01-03-2025

Abstract:

A new fuzzy mathematical model of HIV infection of CD4+ T-Cells consisting of fuzzy Delay Differential Equations (FDDEs) has been suggested with the appropriate theory and analysis. In this fuzzy dynamical system, we consider three equations having parameters uninfected, viral and infected cells. Therefore, utilizing a fuzzy dynamical system and $\alpha-$ cuts, we prepare to suppress the HIV epidemic in this approach. We transform this system into a fuzzy linear system of differential equations, and then we endeavor to solve these FDDEs using the Runge-Kutta method at the order of five. In addition, we presented the stability of fuzzy differential equations before the numerical analysis

Keywords: HIV infection; Fuzzy Delay Differential equation; Stability; Numerical solution; Runge-Kutta Method of order five.

Mathematics Subject Classification: 74H15, 34A07

1. Introduction

HIV infection of CD4+ cells has been a subject of discussion for many studies. Perelson [14] provided a simple explanation of how to form a model for interaction between HIV (Human immune-deficiency virus). Even though the interactions between the human immune system and HIV are extremely complicated, we still don't understand the pathogenicity. They developed the HIV model and evaluated the model's behavior. Uninfected CD4+ T-cells, latently infected CD4+ T-cells, infected CD4+ T-cells, and the free virus was listed as the four components. The fluctuations of a density of uninfected target cells (Uc), infected target cells (Ic), and free virus (Fv) were explored in the basic differential equation model developed by [14] to analyze viral dynamics. Numerous researchers, including [3], [5], [6], [10], [18], [19], [22], detailed the model utilising ODE techniques (ODE). That model does not include an intracellular time delay between a cell becoming affected and a virus-producing a new infection. Herz et al., [8], assumed that virus generation lags behind cell infection by a delay of τ . Finally, there are two techniques to solve the system of equations. This reveals that the density of cells that were freshly infected at time $t - \tau$

and are still alive at time t determines the recruitment of virus-producing cells at time t. The first

one is analytical when it is possible, and the second is numerical methods since they enable the prediction of virus concentrations and target cell behaviour.

We will discuss a fuzzy differential equation model of HIV infection of CD4+ T-cells with delay using the fifth order Runge-Kutta method in this study. Rebecca et al. [4] have described the most basic model of HIV infection of CD4+ T-cells. We must turn the nonlinear fuzzy delay model into a fuzzy linear model in this study. Then it will provide a strategy for determining the analytical solution of the fuzzy linearized system.

Variables and parameters for viral spread[4]		
Terms	***Description of variables and constants***	***Values***
	Dependent variables	
Uc	*Uninfected CD4⁺ T-cell population size*	$1000mm^{-3}$
Ic	*Infected CD4⁺ T-cell density*	0
Fv	*Initial density of HIV RNA*	10^{-3}
	Parameters and Constants	
μ_u	*Rate of natural death for CD4⁺T cells*	$0.02day^{-1}$
μ_{Ic}	*Infectious people die in droves for CD4⁺ T-cells*	$0.26day^{-1}$
μ_b	*Rate of lytic death in infected cells*	$0.24day^{-1}$
μ_{Fv}	*Rate of free virus death*	$2.4day^{-1}$
k_1	*Rate CD4⁺ T-cells become infected with virus*	$2.4 \times 10^{-5}mm^3day^{-1}$
k	*The rate at which infected cells activate*	$2 \times 10^{-5}mm^3day^{-1}$
r	*Growth rate of CD⁺ T-cell population*	$0.03day^{-1}$
N	*The number of virions that infected CD4⁺ T cells produce*	*Varies*
U_{cmax}	*Maximum number of CD4⁺ T-cells in a population*	$1500mm^{-3}$
s	*Uninfected CD4⁺ T-cells are the source term*	$10(day)^{-1}(mm^{-3})$
	derived quantities	
T_0	*The population of CD4⁺ T cells for HIV-negative patients*	$1000mm^{-3}$

2. HIV Infection of CD4+cells with a Fuzzy Delay Model

Consider the FDDEs model of CD4+T-cell HIV infection,

$$\begin{cases} \frac{d}{dt}[U_c(t)]^\alpha = [s]^\alpha - [\mu_t U_c(t)]^\alpha + [rU_c(t)]^\alpha \left(1 - \frac{[U_c(t)]^\alpha + [I_c(t)]^\alpha}{U_c max}\right) - k[F_v(t)]^\alpha [U_c(t)]^\alpha, \\ \frac{d}{dt}[I_c(t)]^\alpha = k'_1[F_v(t-\tau)]^\alpha [U_c(t-\tau)]^\alpha - \mu_I [F_v(t)]^\alpha, \\ \frac{d}{dt}[F_v(t)]^\alpha = [N\mu_b I_c(t)]^\alpha - k_1[F_v(t)]^\alpha [U_c(t)]^\alpha - \mu F_v [F_c(t)]^\alpha, \end{cases} \quad (2.1)$$

with the fuzzy initial function is given by

$$[U_c(t)]^\alpha = [1001 - e^t]F_1(\alpha),$$
$$[I_c(t)]^\alpha = [0]F_2(\alpha),$$
$$[F_v(t)]^\alpha = [1.001 - e^t]F_3(\alpha), t \in [-\tau, 0],$$

where ,

$$F_1(\alpha) = [1000\alpha + 850(1-\alpha), 1000\alpha + 1150(1-\alpha)]$$
$$F_2(\alpha) = [5\alpha + 4(1-\alpha), 5\alpha + 6(1-\alpha)],$$
$$F_3(\alpha) = [7000\alpha + 6750(1-\alpha), 7000\alpha + 7250(1-\alpha)], \alpha \in [0,1].$$

Where $[U_c(t)]^{\alpha}$ symbolizes the number of healthy CD4+T-cells at time t. $[I_c(t)]^{\alpha}$ reflects the accumulation of carrying the virus CD4+T-cells at time t, $[F_v(t)]$ corresponds to the number of free HIV at time t, and the constant of proportionality τ, which reflects the time frame of the delay, is also present. The parameters are as follows: s represents the precursors source of CD4+T-cells, μ_T indicates the natural death rate of CD4+T-cells, r represents their rate of growth (therefore, $r > \mu_T$ generally), and $U_c max$ defines their carrying capacity. Given as a loss term for both healthy cells and virus since they are both lost by binding to one another and as the source term for infected cells, the parameter k_1 describes the rate of infection of T-cells with free virus. Since both healthy cells and viruses are lost via attaching to one another, issued as a loss term for both, while also serving as the parameter for infected cells. The ratio $\frac{k_1'}{k_1}$ represents the proportion of T-cells that ever become actively infected. k_1' indicates the rate at which infected cells become actively infected. To reflect the presumption that we initially do-not know whether the cells die normally or via bursting, μ_I is a general word for the death of infected cells. To describe the dynamics of healthy cells, a time delay is added to the system.

Only healthy cells that the virus infected τ time units ago(i.e.,at time(t−τ) become infectious at time t. Thus, the incidence term of healthy cells is changed from $k_1' U_c(t) F_v(t)$ to $k_1' U_c(t-\tau) F_v(t-\tau)$.

Additionally, the catalytic mortality rate for virus particles is μ_b.Each lysing proved to possess N viral particles, hence this term is multiplied by N to indicate the source of free virus (assuming a one-time initial infection). Finally, μ_V is the viral loss rate.

3. Linearization Technique

(i)To find the equilibrium points

Determining the values of equilibrium is required in order to adequately comprehend the dynamics of the three-component model. A system's equilibrium point is a stable solution, (2.1) indicating that if the system starts out at that value, it will stay there forever. In other words, the inhabitants are static, and as a result, each population's rate of change is zero. Two steady states exist for the system:

(i) The uninfected steady state $E_0 = \left(\left[U_{c_0}(t)\right]^{\alpha}, 0,0\right)$, where $\left[U_{c_0}(t)\right]^{\alpha}$ is given by

$$\left[U_{c_0}(t)\right]^{\alpha} = \left[\frac{r-\mu_T\left[(r-\mu_T)^2 + 4rsU_c^{-1}{}_{max}\right]^{1/2}}{2rU_{cmax}^{-1}}\right][F_1(\alpha)].$$

(ii) The positively infected steady state $[\bar{E}(t)]^{\alpha} = ([\overline{U_c}(t)]^{\alpha}, [\overline{I_c}(t)]^{\alpha}, [\overline{F_v}(t)]^{\alpha})$, where $[\overline{U_c}(t)]^{\alpha}, [\overline{I_c}(t)]^{\alpha}, [\overline{F_v}(t)]^{\alpha}$ are given by

$$[\overline{U_c}(t)]^{\alpha} = \left(\frac{\mu F_v \mu I_c}{k_1' N\mu_b - k_1 \mu I_c}\right)[F_1(\alpha)],\ [\overline{I_c}(t)]^{\alpha} = \left(\frac{k_1'\overline{U_c}\overline{F_v}}{\mu I_c}\right)[F_2(\alpha)],$$

$$[\overline{F_v}(t)]^{\alpha} = \left(\frac{\mu I_c\left[(s+(r-\mu U_c)\overline{U_c})U_{cmax} - r\overline{U_c}^2\right]}{\overline{U_c}[k_1' r\overline{U_c} + k_1\mu_1 U_{cmax}]}\right)[F_3(\alpha)].$$

Using the parameter values in Table, we find the three fixed(or) equilibrium points of the system (2.1) are,

(i) Initially: $[E_0]^\alpha = ([U_{c0}]^\alpha, 0, 0) = (1000, 0, 0)$,

(ii) For N=500;

$$[\bar{E}(t)]^\alpha = ([\overline{U_c}(t)]^\alpha, [\overline{I_c}(t)]^\alpha, [\overline{F_v}(t)]^\alpha) = (260.7[F_1(\alpha)], 35.5[F_2(\alpha)], 1768.2[F_3(\alpha)])$$

(iii) For N=1000;

$$[\bar{E}(t)]^\alpha = ([\overline{U_c}(t)]^\alpha, [\overline{I_c}(t)]^\alpha, [\overline{F_v}(t)]^\alpha) = (130.2[F_1(\alpha)], 34.9[F_2(\alpha)], 3480.1[F_3(\alpha)])$$

iii) To find the Jacobian matrix at the equilibrium points

The nonlinear fuzzy system (2.1) can be written as

$$\begin{cases} \frac{d}{dt}([U_c(t)]^\alpha - [\overline{U_c}(t)]^\alpha) = M([U_c(t)]^\alpha - [\overline{U_c}(t)]^\alpha) - \frac{r\overline{U_c}^\alpha}{U_{cmax}}([I_c(t)]^\alpha - [\overline{I_c}(t)]^\alpha) \\ \qquad -k_1[\overline{U_c}(t)]^\alpha([F_v(t)]^\alpha - [\overline{F_v}(t)]^\alpha), \\ \frac{d}{dt}([I_c(t)]^\alpha - [\overline{I_c}(t)]^\alpha) = k_1'[\overline{F_v}(t)]^\alpha([U_c(t-\tau)]^\alpha - [\overline{U_c}(t)]^\alpha) - \mu_1([I_c(t)]^\alpha - [\overline{I_c}(t)]^\alpha) \\ \qquad +k_1'[\overline{U_c}(t)]^\alpha([F_v(t-\tau)]^\alpha - [\overline{F_v}(t)]^\alpha), \\ \frac{d}{dt}([F_v(t)]^\alpha - [\overline{F_v}(t)]^\alpha) = -k_1[\overline{F_v}(t)]^\alpha([U_c(t)]^\alpha - [\overline{U_c}(t)]^\alpha) + N\mu_b([I_c(t)]^\alpha - [\overline{I_c}(t)]^\alpha) \\ \qquad -(k_1[\overline{U_c}(t)]^\alpha + \mu F_v)([F_v(t)]^\alpha - [\overline{F_v}(t)]^\alpha). \end{cases} \tag{3.1}$$

Where

$$M = \mu_T^\alpha + \left(\frac{r\left(2\overline{U_c}(t) + \overline{I_c}(t)\right)}{U_{cmax}}\right)^\alpha + (k_1F_v(t) - r)^\alpha$$

System(3.1)is a linearized fuzzy system. Then, using a fuzzy matrix, we express the system (3.1) as follows.:

$$[X'(t)]^\alpha = A_1[X(t)]^\alpha + A_2[X(t-\tau)]^\alpha \tag{3.2}$$

where $[X(t)]^\alpha = \begin{bmatrix} [w_1(t)]^\alpha \\ [w_2(t)]^\alpha \\ [w_2(t)]^\alpha \end{bmatrix}$ and A_1, A_2 are 3x3fuzzy matrices are given by,

$$A_1 = \begin{bmatrix} -M & \frac{-r\overline{U_c}}{U_{cmax}} & -k_1\overline{U_c} \\ 0 & -\mu_1 & 0 \\ -k_1\overline{F_v} & N\mu_b & -(k_1\overline{U_c + \mu F_v}) \end{bmatrix} \begin{bmatrix} F_1(\alpha) \\ F_2(\alpha) \\ F_3(\alpha) \end{bmatrix} \text{ and}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ k_1\overline{F_c} & 0 & k_1'\overline{U_c} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} F_1(\alpha) \\ F_2(\alpha) \\ F_3(\alpha) \end{bmatrix} \tag{3.3}$$

Using the parameter values from the table and the equilibrium point for N=500,

$$[\bar{E}]^{\alpha} = (260.7[F_1(\alpha)], 35.5[F_2(\alpha)], 1768.2[F_3(\alpha)])$$

And $\tau = 1$, (3.1) can be written as,

$$\begin{cases} [w'_1(t)]^{\alpha} = -0.0436[w_1(t)]^{\alpha} - 0.0052[w_2(t)]^{\alpha} - 0.00626[w_3(t)]^{\alpha}, \\ [w'_2(t)]^{\alpha} = 0.0354[w_1(t-1)]^{\alpha} - 0.26[w_2(t)]^{\alpha} + 0.0052[w_3(t-1)]^{\alpha}, \\ [w'_3(t)]^{\alpha} = -0.0424[w_1(t)]^{\alpha} + 120[w_2(t)]^{\alpha} - 2.24063[w_3(t)]^{\alpha}. \end{cases} \quad (3.4)$$

At the equilibrium point for N=1000,

$$[\bar{E}]^{\alpha} = (130.2[F_1(\alpha)], 34.9[F_2(\alpha)], 3480.1[F_3(\alpha)]).$$

And $\tau = 1$, (3.1) can be written as,

$$\begin{cases} [w'_1(t)]^{\alpha} = -0.07942[w_1(t)]^{\alpha} - 0.0026[w_2(t)]^{\alpha} - 0.003124[w_3(t)]^{\alpha}, \\ [w'_2(t)]^{\alpha} = 0.0696[w_1(t-1)]^{\alpha} - 0.26[w_2(t)]^{\alpha} + 0.0026[w_3(t-1)]^{\alpha}, \\ [w'_3(t)]^{\alpha} = -0.0835[w_1(t)]^{\alpha} + 240[w_2(t)]^{\alpha} - 2.20312[w_3(t)]^{\alpha}. \end{cases} \quad (3.5)$$

$$[w_1(t)]^{\alpha} = [1001 - e^t][F_1(\alpha)],$$

$$[w_2(t)]^{\alpha} = [0][F_2(\alpha)],$$

$$[w_3(t)]^{\alpha} = [1.001 - e^t][F_3(\alpha)], \quad for\ t \in [-1,0].$$

With initial functions.

The analytical solution is

$$[X(t)]^{\alpha} = [\Phi(t)]^{\alpha} C + [\Phi(t)]^{\alpha} \int_{t_0}^{t} [\Phi^{-1}(s)]^{\alpha} B(s,\alpha) ds, t \in I. \quad (3.6)$$

The fundamental matrix of the system(3.5) is given by,

$$[\Phi(t)]^{\alpha} = \begin{bmatrix} a_1 e^{\propto t} & b_1 e^{\propto t} & c_1 e^{\propto t} \\ a_2 e^{\propto t} & b_2 e^{\propto t} & c_2 e^{\propto t} \\ a_3 e^{\propto t} & b_3 e^{\propto t} & c_3 e^{\propto t} \end{bmatrix} \begin{bmatrix} F_1(\alpha) \\ F_2(\alpha) \\ F_3(\alpha) \end{bmatrix} \quad (3.7)$$

where $\alpha = -0.0434876, \beta = -0.25, \gamma = -2.40641$, and

$$\begin{cases} a_1 = 0.999839, & a_2 = 0, & a_3 = -0.0179587, \\ b_1 = 0.0293406, & b_2 = 0.0178856, & b_3 = 0.9994090, \\ d_1 = 0.000264938, & d_2 = 0, & d_3 = 0.9999960 \end{cases}$$

Let, $b = \begin{bmatrix} 0 \\ 35.4406 - (0.0406e^{t-1}) \\ 0 \end{bmatrix}$ and $\Phi^{-1}(t)B = \frac{1}{M} \begin{bmatrix} m_1 e^{-\alpha t} \\ m_2 e^{-\beta t} \\ m_3 e^{-\gamma t} \end{bmatrix}$ where,

$$M = a_1 h_2 - b_1 h_1 + d_1 h_3,$$

$$m_1 = n_1 h_1 - n_2 h_4 + n_3 h_7,$$

$$m_2 = -n_1 h_2 + n_2 h_5 - n_3 h_8,$$

$$m_3 = n_1 h_3 - n_2 h_6 + n_3 h_9,$$

$$n_1 = 0, \qquad n_2 = 35.4406 - (0.0406e^{t-1})), \qquad n_3 = 0$$

and

$$\begin{cases} h_1 = \gamma_3 a_2 - \gamma_2 a_3,\ h_2 = \gamma_3 b_2 - \gamma_2 b_3,\ h_3 = a_2 b_3 - b_2 a_3, \\ h_4 = \gamma_1 b_3 - \gamma_3 b_1,\ h_5 = \gamma_3 a_1 - \gamma_1 a_3,\ h_6 = a_1 b_3 - a_3 b_1, \\ h_7 = \gamma_1 b_2 - \gamma_2 b_1,\ h_8 = \gamma_2 a_1 - \gamma_1 a_2,\ h_9 = a_1 b_2 - b_1 a_2. \end{cases}$$

The linear system (3.5)'s analytical solution is,

$$x(t) = C_1 a_1 e^{\alpha t} + C_2 b_1 e^{\beta t} + C_3 d_1 e^{\gamma t} + 1/M \left[\frac{a_1 m_1}{\alpha}(e^{\alpha} - 1) + \frac{b_1 m_1}{\beta}\left(e^{\beta} - 1\right) + \frac{d_1 m_1}{\gamma}(e^{\gamma} - 1) \right]$$

$$y(t) = C_1 a_2 e^{\alpha t} + C_2 b_2 e^{\beta t} + C_3 d_2 e^{\gamma t} + 1/M \left[\frac{a_2 m_1}{\alpha}(e^{\alpha} - 1) + \frac{b_2 m_1}{\beta}\left(e^{\beta} - 1\right) + \frac{d_2 m_1}{\gamma}(e^{\gamma} - 1) \right],$$

$$z(t) = C_1 a_3 e^{\alpha t} + C_2 b_3 e^{\beta t} + C_3 d_3 e^{\gamma t} + 1/M \left[\frac{a_3 m_1}{\alpha}(e^{\alpha} - 1) + \frac{b_3 m_1}{\beta}\left(e^{\beta} - 1\right) + \frac{d_3 m_1}{\gamma}(e^{\gamma} - 1) \right],$$

where $C_1 = 10001.4,\ C_2 = 0, C_3 = 179.793.$

4. Stability Analysis

The linearized system's characteristic equation, (3.2), is generated by,

$$\Delta(\lambda) = |\lambda I + A_1 - e^{-\lambda\tau} A_2| = 0$$

That is,

$$\lambda_3 + c_1\lambda_2 + c_2\lambda + c_3 e^{-\lambda\tau} + c_4\lambda e^{-\lambda\tau} + c_5 = 0, \quad (4.1)$$

Compared to the ODE model, it is difficult to find the eigenvalues of (4.1), since it is a transcen dental equation and it has infinitely many eigenvalues. Let $\lambda = \eta(\tau) + i\omega(\tau)(\omega > 0)$, be the eigenvalue off the characteristic equation (4.1), where $\eta(\tau)$ and $\omega(\tau)$ depend only on the delay τ. Suppose $i\omega(\omega > 0)$ is a root of (4.1) if and only if $h(z) = z_3 + \mu z_2 + vZ + \rho = 0$, where $z = \omega_2, \mu = c_1^2 - 2c_2, v = c_2^2 - 2c_1c_5 - c_4^2, \rho = c_5^2 - c_3^2$. The following proposition demonstrates that the steady state of the delay model (2.1) is asymptotically stable for all delay values if the parameter values in Table 1 satisfy the specified conditions.

4.1. Proposition

If (i)$c_1 > 0, c_3 + c_5 > 0, c_1(c_2 + c_4) - (c_3 + c_5) > 0$, (ii) $\rho \geq 0$, and $v > 0$, then the fuzzy delay model's infected steady state (2.1) is completely stable. that is, "E" is stable asymptotically across all $\tau \geq 0$.

Case (i): For N = 500, then$c_1 = 3.07, c_2 = 1.70173, c_3 = -0.2256, c_4 = -0.6255, c_5 = 0.2525, c_3 + c_5 = 0.027 > 0, c_1(c_2 + c_4) - (c_3 + c_5) = 3.27703 > 0$, and ρ = 0.01286 ≥ 0,v = 0.95428 > 0. The infected steady state $E = (260.7[F_1(\alpha)], 35.5[F_2(\alpha)], 1768.2[F_3(\alpha)])$ is asymptotically stable, as demonstrated by the numerical simulations and all of the parameter values in Table I.

Case (ii): There will be a significant increase in the number of infected cells and a decrease in the number of uninfected CD4+ T-cells as the N value is raised, but the steady state will remain stable. If N equals 1000, then $c_1 = 2.7422, c_2 = 0.8360, c_3 = 0.00299, c_4 = -0.62477, c_5 = 0.04955, c_3 + c_5 = 0.0525535 > 0, c_1(c_2 + c_4) - (c_3 + c_5) = 0.526945 > 0,$ and $\rho =$

$0.00244 \geq 0, v = 0.036806 > 0$. As a result, the infected steady state $E = (130.2[F_1(\alpha)], 34.9[F_2(\alpha)], 3480.1[F_3(\alpha)])$is asymptotically stable based on all of the parameter values in Table 1. Therefore, for any delay $\tau \geq 0$, the infected steady state E is statistically stable. If proposition 4.1's conditions (i) and (ii) are not met, the steady state stability is dependent on the delay value, and the delay may even cause oscillations. When τ passes through a critical value τ_j, a Hopf-bifurcation [9] occurs.

$$\tau_j = \frac{1}{\omega_0} arc\ cos\left(\frac{c_4\omega_0^4 + (c_1c_3 - c_2c_4)\omega_0^2 - c_3c_5}{c_3^2 + c_4^2\omega_0^2}\right) + \frac{2j\pi}{\omega_0}, j = 0,1,2,\ldots$$

Such that $\eta(t_0)$=0, $\omega(\tau_0) = \omega_0$. The transversality condition is:

$$\frac{d}{d\tau} Re\big(\lambda(\tau)\big)|_{\tau=\tau_0} = \frac{d}{d\tau} Re\big(\eta(\tau)\big)|_{\tau=\tau_0} > 0.$$

4.2. Proposition

If (i)$c_1 > 0, c_3 + c_5 > 0, c_1(c_2 + c_4) - (c_3 + c_5) > 0$ (ii) $\rho < 0$, (iii) $\rho \geq 0$, and v_0, are satisfied, then the infected steady state of the fuzzy delay model (2.1) is asymptotically stable when τ, τ_0, and unstable when $\tau > \tau_0$, where

$$\tau_0 = \frac{1}{\omega_0} arc\ cos\left(\frac{c_4\omega_0^4 + (c_1c_3 - c_2c_4)\omega_0^2 - c_3c_5}{c_3^2 + c_4^2\omega_0^2}\right).$$

When $\tau = \tau_0$, a Hopf bifurcation occurs; that is, a family of periodic solutions bifurcates from E as τ passes through the critical value τ_0.

4.3. Proposition

The characteristic equation of the linearized system is given by $|A = \lambda I| = \lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0$, where for $N = 500, c_2 = 2.7099, c_1 = 0.741621, c_0 = 0.0272083$, for $N = 1000, c_2 = 2.74225, c_1 = 0.836078, c_0 = 0.049557$. The eigenvalues of the matrix A are for $N = 500, \alpha = -0.0434876, \beta = -0.26, \gamma = -2.40641$, for $N = 1000, \alpha = -0.0793157, \beta = -0.26, \gamma = -2.40324$. If and only if all of the eigenvalues have negative real parts, the system is said to be stable. If any one of the eigenvalues has a positive real part, the system is said to be unstable. We have three eigenvalues that are linearly independent. Negative real parts are present in all eigenvalues. Our system is therefore stable.

5. Numerical Simulation of SIR model

In this section, we are using the RKM-5. We finding value of $U_c(t), I_c(t)$ and $F_v(t)$ at $h = 0.1$ for the best approximation. For $0 \leq \alpha \leq 1$. To evaluate $U_c(t), I_c(t)$ and $F_v(t)$, consider,

$$[U_c(t+1)]^\alpha = [U_c(t)]^\alpha + \frac{1}{90}(7[K_1]^\alpha + 32[K_3]^\alpha + 12[K_4]^\alpha + 32[K_5]^\alpha + 7[K_6]^\alpha),$$

$$[I_c(t+1)]^\alpha = [I_c(t)]^\alpha + \tfrac{1}{90}(7[K_1]^\alpha + 32[K_3]^\alpha + 12[K_4]^\alpha + 32[K_5]^\alpha + 7[K_6]^\alpha), \quad (5.1)$$

$$[F_V(t+1)]^\alpha = [F_v(t)]^\alpha + \frac{1}{90}(7[K_1]^\alpha + 32[K_3]^\alpha + 12[K_4]^\alpha + 32[K_5]^\alpha + 7[K_6]^\alpha),$$

In estimate (5.1), for $1 \leq p \leq 6$ and $0 \leq \alpha \leq 1$, we use

$$\big[K_p\big]^\alpha = \big[\underline{K_p}(t;\alpha), \overline{K}_p(t;\alpha)\big], \big[L_p\big]^\alpha = \big[\underline{L_p}(t;\alpha), \overline{L}_p(t;\alpha)\big]\ \big[M_p\big]^\alpha = [\underline{M_p}(t;\alpha), \overline{M_p}(t;\alpha)].$$

For $0 \leq t \leq n, n = 1,2,3,\ldots$, and for $q = t+1, t = 0,1,2,3,\ldots$

$[K(q)]^{\alpha} = \left[\underline{K_q}(t;\alpha), \overline{K}_q(t;\alpha)\right]$, $[L(q)]^{\alpha} = \left[\underline{L_p}(t;\alpha), \overline{L}_p(t;\alpha)\right]$ $[M(q)]^{\alpha} = [\underline{M_p}(t;\alpha), \overline{M_p}(t;\alpha)]$.

Where,

$[K_1]^{\alpha} = h(s - \mu_t[U_c(t)]^{\alpha} + r[U_c(t)]^{\alpha}(1 - ([U_c(t)]^{\alpha} + [I_c(t)]^{\alpha})/U_{cmax}) - k_1[F_v(t)]^{\alpha}[U_c(t)]^{\alpha})$,
$[L_1]^{\alpha} = h(k_1'[F_v(t-\tau)]^{\alpha}[U_c(t-\tau)]^{\alpha} - \mu_I[I_c(t)]^{\alpha})$,

$[M_1]^{\alpha} = h(N\mu_b[I_c(t)]^{\alpha} - k_1[F_v(t)]^{\alpha}[U_c(t)]^{\alpha} - \mu_v[F_v(t)]^{\alpha})$.

$[K_2]^{\alpha} = h(s - \mu_t([U_c(t)]^{\alpha} + 1/2[K_1]^{\alpha}) + r([U_c(t)]^{\alpha} + 1/2[K_1]^{\alpha})$
$\left(1 - \left(([U_c(t)]^{\alpha} + 1/2[K_1]^{\alpha}) + ([I_c(t)]^{\alpha} + 1/2[L_1]^{\alpha})\right)/U_{cmax}\right) - k_1([F_v(t)]^{\alpha}$
$+ 1/2[M_1]^{\alpha})([U_c(t)]^{\alpha} + 1/2[K_1]^{\alpha}))$,

$[L_2]^{\alpha} = h(k_1'([F_v(t-\tau)]^{\alpha} + 1/2[M_1]^{\alpha})([U_c(t-\tau)]^{\alpha} + 1/2[K_1]^{\alpha}) - \mu_I([I_c(t)]^{\alpha} + 1/2[L_1]^{\alpha}))$,

$[M_2]^{\alpha} = h(N\mu_b([I_c(t)]^{\alpha} + 1/2[L_1]^{\alpha}) - k_1([F_v(t)]^{\alpha} + 1/2[M_1]^{\alpha})([U_c(t)]^{\alpha} + 1/2[K_1]^{\alpha})$
$- \mu_v([F_v(t)]^{\alpha} + 1/2[M_1]^{\alpha}))$.

$[K_3]^{\alpha} = h(s - \mu_t([U_c(t)]^{\alpha} + 3/16[K_1]^{\alpha} + 1/16[K_2]^{\alpha}) + r([U_c(t)]^{\alpha} + 3/16[K_1]^{\alpha} +$
$1/16[K_2]^{\alpha})\left(1 - \left(([U_c(t)]^{\alpha} + 3/16[K_1]^{\alpha} + 1/16[K_2]^{\alpha}) + ([I_c(t)]^{\alpha} + 3/16[L_1]^{\alpha} +\right.\right.$
$\left.\left.1/16[L_2]^{\alpha})\right)/U_{cmax}\right) - k_1([F_v(t)]^{\alpha} + 3/16[M_1]^{\alpha} + 1/16[M_2]^{\alpha})([U_c(t)]^{\alpha} +$
$3/16[K_1]^{\alpha} + 1/16[K_2]^{\alpha}))$,

$[L_3]^{\alpha} = h(k_1'([F_v(t-\tau)]^{\alpha} + 3/16[M_1]^{\alpha} + 1/16[M_2]^{\alpha})([U_c(t-\tau)]^{\alpha} + 3/16[K_1]^{\alpha} +$
$1/16[K_2]^{\alpha}) - \mu I_c([I_c(t)]^{\alpha} + 3/16[L_1]^{\alpha} + 1/16[L_2]^{\alpha}))$,

$[M3]\alpha = h(N\mu b([Ic(t)]\alpha + 3/16[L1]\alpha + 1/16[L2]\alpha) - k1([Fv(t)]\alpha + 3/16[M1]\alpha$
$+ 1/16[M2]\alpha)([Uc(t)]\alpha + 3/16[K1]\alpha + 1/16[K2]\alpha) - \mu v([Fv(t)]\alpha$
$+ 3/16[M1]\alpha + 1/16[M2]\alpha))$.

$[K_4]^{\alpha} = h(s - \mu_t([U_c(t)]^{\alpha} + 1/2[K_3]^{\alpha}) + r([U_c(t)]^{\alpha} + 1/2[K_3]^{\alpha})\left(1 - \left(([U_c(t)]^{\alpha} +\right.\right.$
$\left.\left.1/2[K_3]^{\alpha}) + ([I_c(t)]^{\alpha} + 1/2[L_3]^{\alpha})\right)/U_{cmax}\right) - k_1([F_v(t)]^{\alpha} + 1/2[M_3]^{\alpha})([U_c(t)]^{\alpha} +$
$1/2[K_3]^{\alpha}))$,

$[L_4]^{\alpha} = h(k_1'([F_v(t-\tau)]^{\alpha} + 1/2[M_3]^{\alpha})([U_c(t-\tau)]^{\alpha} + 1/2[K_3]^{\alpha}) - \mu I_c([I(t)]^{\alpha} + 1/2[L_3]^{\alpha}))$,

$[M_4]^{\alpha} = h(N\mu_b([I_c(t)]^{\alpha} + 1/2[L_3]^{\alpha}) - k_1([F_v(t)]^{\alpha} + 1/2[M_3]^{\alpha})([U_c(t)]^{\alpha} + 1/2[K_3]^{\alpha}) -$
$\mu_v([F_v(t)]^{\alpha} + 1/2[M_3]^{\alpha}))$.

$[K_5]^{\alpha} = h(s - \mu_t([U_c(t)]^{\alpha} - 3/16[K_2]^{\alpha} + 6/16[K_3]^{\alpha} + 9/16[K_4]^{\alpha})$
$+ r([U_c(t)]^{\alpha} - 3/16[K_2]^{\alpha} + 6/16[K_3]^{\alpha} + 9/16[K_4]^{\alpha})\left(1\right.$
$\left. - \left(([U_c(t)]^{\alpha} - 3/16[K_2]^{\alpha} + 6/16[K_3]^{\alpha} + 9/16[K_4]^{\alpha})\right.\right.$
$\left.\left. + ([I_c(t)]^{\alpha} - 3/16[L_2]^{\alpha} + 6/16[L_3]^{\alpha} + 9/16[L_4]^{\alpha})\right)/U_{cmax}\right) - k_1([F_v(t)]^{\alpha}$
$- 3/16[M_2]^{\alpha} + 6/16[M_3]^{\alpha} + 9/16[M_4]^{\alpha})([U_c(t)]^{\alpha} - 3/16[K_2]^{\alpha} + 6/16[K_3]^{\alpha}$
$+ 9/16[K_4]^{\alpha}))$,

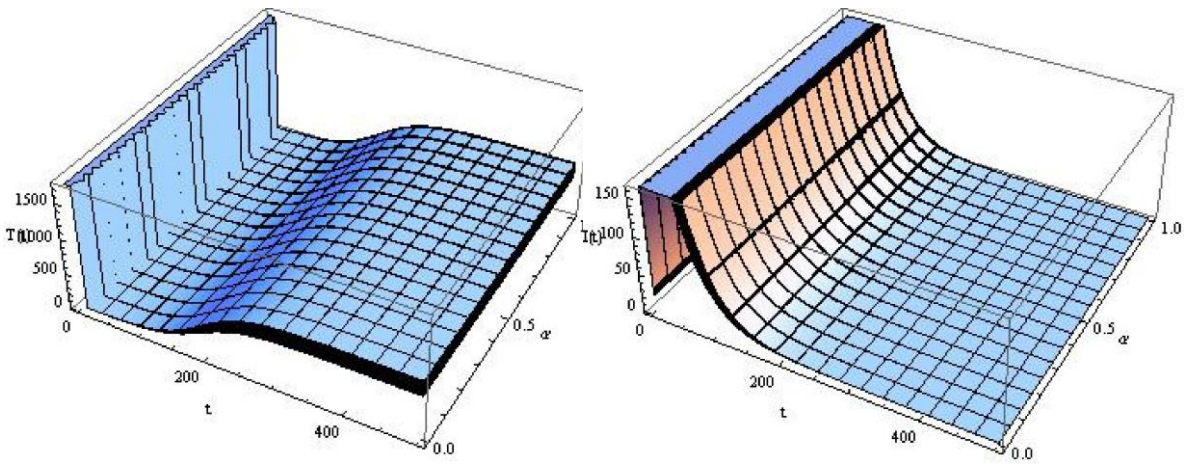
$$[L_5]^\alpha = h\big(k_1'([F_v(t-\tau)]^\alpha - 3/16[M_2]^\alpha + 6/16[M_3]^\alpha + 9/16[M_4]^\alpha)([U_c(t-\tau)]^\alpha - 3/16[K_2]^\alpha + 6/16[K_3]^\alpha + 9/16[K_4]^\alpha) - \mu I_c([I_c(t)]^\alpha - 3/16[L_2]^\alpha + 6/16[L_3]^\alpha + 9/16[L_4]^\alpha)\big),$$

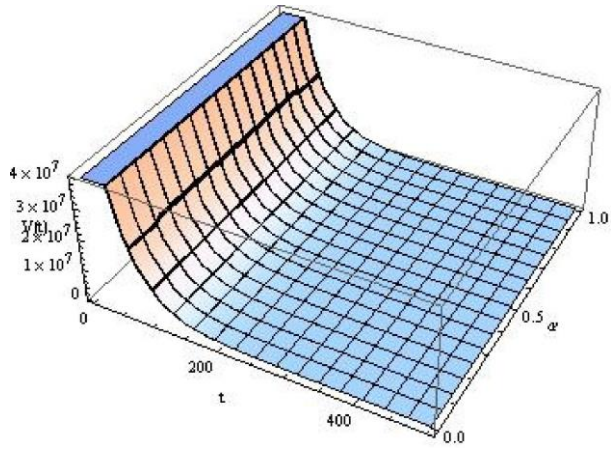
$$[M_5]^\alpha = h(N\mu_b([I_c(t)]^\alpha - 3/16[L_2]^\alpha + 6/16[L_3]^\alpha + 9/16[L_4]^\alpha) - k_1([F_v(t)]^\alpha - 3/16[M_2]^\alpha + 6/16[M_3]^\alpha + 9/16[M_4]^\alpha)([U_c(t)]^\alpha - 3/16[K_2]^\alpha + 6/16[K_3]^\alpha + 9/16[K_4]^\alpha) - \mu_v([F_v(t)]^\alpha - 3/16[M_2]^\alpha + 6/16[M^3]^\alpha + 9/16[M_4]^\alpha)).$$

$$[K_6]^\alpha = h(s - \mu_t([U_c(t)]^\alpha + 1/7[K_1]^\alpha + 4/7[K_2]^\alpha + 6/7[K_3]^\alpha - 12/7[K_4]^\alpha + 8/7[K_5]^\alpha) + r([U_c(t)]^\alpha + 1/7[K_1]^\alpha + 4/7[K_2]^\alpha + 6/7[K_3]^\alpha - 12/7[K_4]^\alpha + 8/7[K_5]^\alpha)(1 - (([U_c(t)]^\alpha + 1/7[K_1]^\alpha + 4/7[K_2]^\alpha + 6/7[K_3]^\alpha - 12/7[K_4]^\alpha + 8/7[K_5]^\alpha) + ([I_c(t)]^\alpha + 1/7[L_1]^\alpha + 4/7[L_2]^\alpha + 6/7[L_3]^\alpha - 12/7[L_4]^\alpha + 8/7[L_5]^\alpha))/U_{cmax}) - k_1([F_v(t)]^\alpha + 1/7[M_1]^\alpha + 4/7[M_2]^\alpha + 6/7[M_3]^\alpha - 12/7[M_4]^\alpha + 8/7[M_5]^\alpha)([U_c(t)]^\alpha + 1/7[K_1]^\alpha + 4/7[K_2]^\alpha + 6/7[K_3]^\alpha - 12/7[K_4]^\alpha + 8/7[K_5]^\alpha)),$$

$$[L_6]^\alpha = h(k_1'([F_v(t-\tau)]^\alpha + 1/7[M_1]^\alpha + 4/7[M_2]^\alpha + 6/7[M_3]^\alpha - 12/7[M_4]^\alpha + 8/7[M_5]^\alpha)([U_c(t-\tau)]^\alpha + 1/7[K_1]^\alpha + 4/7[K_2]^\alpha + 6/7[K_3]^\alpha - 12/7[K^4]^\alpha + 8/7[K_5]^\alpha) - \mu I_c([I_c(t)]^\alpha + 1/7[L_1]^\alpha + 4/7[L_2]^\alpha + 6/7[L_3]^\alpha - 12/7[L_4]^\alpha + 8/7[L_5]^\alpha)),$$

$$[M_6]^\alpha = h(N\mu_b([I_c(t)]^\alpha - 3/16[L_2]^\alpha + 6/16[L_3]^\alpha + 9/16[L_4]^\alpha) - k_1([F_v(t)]^\alpha + 1/7[M_1]^\alpha + 4/7[M_2]^\alpha + 6/7[M_3]^\alpha - 12/7[M_4]^\alpha + 8/7[M_5]^\alpha)([U_c(t)]^\alpha + 1/7[K_1]^\alpha + 4/7[K_2]^\alpha + 6/7[K_3]^\alpha - 12/7[K_4]^\alpha + 8/7[K_5]\alpha) - \mu v([Fv(t)]\alpha + 1/7[M1]\alpha + 4/7[M_2]^\alpha + 6/7[M_3]^\alpha - 12/7[M_4]^\alpha + 8/7[M_5]^\alpha)).$$





6. Result and Discussion

First, we reconstructed Rebecca et al.'s[4] DDE model of HIV infection in CD4 + T cells, into a three system of linear equations. In order to extend the infection, we discover a limit on the number of viral particles released by each infectious cell. Under this constraint, the system is in an infected steady state with a positive equilibrium. We established sufficient parameters for the steady state stability of the infected virus by employing stability analysis. Numerical results demonstrated that the analysis was correct, and all of our stability conditions are met. Despite the fact that our estimation of the number of viral particles released from each infectious cell is lower than Perelson et al.'s [14], It has no effect on the existence or stability of the infected steady state. There is a solution to every nonlinear differential equation. Analytical solutions to the majority of nonlinear differential equations are extremely challenging to find. We must therefore use numerical approaches. It is simple to solve all nonlinear differential equations using numerical techniques. To solve the delay differential equation numerically, Runge-Kutta's fifth-order method is the most effective approach.

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