Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 10s (2025)

On Additive Complementary Dual Codes over $\mathbb{Z}_2RS$ and their MacWilliams identities

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Article History:

Received: 12-01-2025

Revised: 15-02-2025

Accepted: 01-03-2025

Abstract:

In this paper, we study Additive Complementary Dual (ACD) codes over a mixed alphabet $\mathbb{Z}_2RS$, where $R = \mathbb{Z}_2 + u\mathbb{Z}_2$ and $S = \mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2 + uv\mathbb{Z}_2$, under the conditions $u^2 = 0$, $v^2 = 0$, and $uv = vu$. An additive code will prove to be an ACD code under certain conditions. In addition, it is a necessary and sufficient condition for a separable additive code to be an ACD code. Certain additive codes improve into binary linear complementary dual codes under a gray map that we investigate. Furthermore, a few types of weight enumerators are also calculated, and the associated MacWilliams identities are discussed with supportive examples.

Keywords: Linear code, Additive code, Gray map, Weight Enumerator, Mac Williams Identities.

1. Introduction

A linear code with a complementary dual property was defined in [11]. If a linear code C satisfies $C \cap C^{\perp} = \{0\}$, it is called an LCD code. In the same paper, Massey showed that asymptotically good LCD codes exist and highlighted applications such as providing an optimal linear coding solution for the two-user binary adder channel. In [4], LCD codes were utilized in the context of digital communication security. Recently, many authors have studied codes over rings due to their emerging role in algebraic coding theory and successful application in combined coding and modulation. Li et al. established families of reversible codes and demonstrated that some are optimal [7]. In [8], LCD codes were explored over finite chain rings. The concept of LCD codes was generalized to additive complementary dual (ACD) codes, and these were studied over $\mathbb{Z}_2 \times \mathbb{Z}_4$ in [2].

MacWilliams established a relationship between a code's weight distribution and that of its dual in [10], leading to the development of MacWilliams identities and consideration of weight enumerators for codes over finite Frobenius rings. Various MacWilliams identities over $\mathbb{Z}_4$ were studied in [6]. Weight enumerators over $\mathbb{Z}_l$ and conditions ensuring the presence of such identities were discussed in [12] and [14]. In 2014, the investigation of linear codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ and their weight enumerators was presented in [16]. $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes and their standard generator matrices, along with the link between their weight enumerators and their duals, were introduced in [1]. Further research into MacWilliams identities for various weight enumerators was conducted in [13], examining linear codes over the new ring $S = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$, with identities for the Lee weight enumerator derived using a Gray map from S^n to $(\mathbb{F}_2 + u\mathbb{F}_2)^n$. Additionally, self-dual and cyclic codes over S were studied.

Motivated by the aforementioned research, we explore ACD codes over the finite commutative Frobenius non-chain ring $\mathbb{Z}_2$ RS, where $R = \mathbb{Z}_2 + u\mathbb{Z}2$ and $S = \mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2 + uv\mathbb{Z}_2$. Throughout this discussion, we let Q = RS. We discuss a necessary condition for a $\mathbb{Z}_2$Q-additive code to be a $\mathbb{Z}_2$Q-ACD code. We examine a Gray map wherein specific additive codes over $\mathbb{Z}_2$Q correspond to binary LCD codes. We explore different weight enumerators of additive codes over $\mathbb{Z}_2$Q and the corresponding MacWilliams identities, providing examples to demonstrate our findings. All computations in this paper are performed using the Magma Computational Algebra System [3].

This paper is structured as follows: In Section 3, we examine $\mathbb{Z}_2$Q-ACD codes and determine a condition for an additive code to be an ACD code. In Section 4, we explore a Gray map and demonstrate how some additive codes are mapped to binary LCD codes. Weight enumerators of $\mathbb{Z}_2$Q-additive codes are studied in Section 5, and the corresponding MacWilliams identities are discussed.

2. **Preliminary**

Let $\mathbb{Z}_2$ be the binary finite field. Then $Q = (\mathbb{Z}_2 + u\mathbb{Z}_2) \times (\mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2 + uv\mathbb{Z}_2)$, where $u^2 = 0$, $v^2 = 0$ and uv = vu, is a commutative ring with characteristic 2. Observe that Q is a local ring with its group of units $U(Q) = \{(1 + ub, 1 + ub + vc + uvd) \mid b, c, d \in \mathbb{Z}_2\}$ and maximal ideal $M = \{(ub, ub + vc + uvd) \mid b, c, d \in \mathbb{Z}_2\}$ such that $M = Q \setminus U(Q)$. Multiplication in $\mathbb{Z}_2$Q is defined as follows:

$$x \star (z, r, s) = (\eta(x)z, \delta(x)r, xs), \qquad (2.1)$$

where $\eta : S \to \mathbb{Z}_2$ is defined by $\eta(a + ub + vc + uvd) = a$ and $\delta : S \to R$ is defined by $\delta(a + ub + vc + uvd) = a + ub$. Throughout the manuscript, we will use (x, y) to represent (z, r, s), where x = z and y = (r, s). Multiplication will be represented by

$$b \star (x, y) = (\eta(b)x, by). \qquad (2.2)$$

Let us take $\mathbb{Z}_2$Q as a cross product of the rings $\mathbb{Z}_2$ and Q. Observe that the ring $\mathbb{Z}_2$Q is an S-module with respect to usual addition and $\star$-multiplication.

Definition 2.1. A subset C of $\mathbb{Z}_2^p Q^q$ is said to be an additive code with block length (p, q) if it is a S-submodule of $\mathbb{Z}_2^p Q^q$.

Note that if p = q, then $\mathbb{Z}_2^p Q^p = (\mathbb{Z}_2 Q)^p$. It can be treated as a $\mathbb{Z}_2$Q-additive code with block length (p, p), equivalent to a $\mathbb{Z}_2$Q-additive code of length p.

Definition 2.2. The inner product in $\mathbb{Z}_2^p Q^q$ is defined as follows:

- Let w = (α, β) and w' = (α', β') be elements of $Q^p = (R^p \times S^p)$. Then ⟨·|·⟩ : $Q^p \times Q^p \rightarrow S$ is defined as follows:

$$\langle w, w' \rangle = \langle (\alpha, \beta), (\alpha', \beta') \rangle = \alpha\alpha' + \beta\beta'. \quad (2.3)$$

- By using Equation 2.3, we define an inner product over $\mathbb{Z}_2^p Q^p$ as ⟨·|·⟩ : $\mathbb{Z}_2^p Q^p \times \mathbb{Z}_2^p Q^p \rightarrow S$. Let x = (v, w) and x' = (v', w') be elements of $\mathbb{Z}_2^p Q^p$. Then

$$\langle x, x' \rangle = \langle (v, w), (v', w') \rangle = uv\langle v, v' \rangle + \langle w, w' \rangle. \quad (2.4)$$

Definition 2.3. Let $C \subseteq \mathbb{Z}_2^p Q^q$ be an additive code. The dual code $C^{\perp}$ is defined as:

$$C^{\perp} = \{x \in \mathbb{Z}_2^p Q^q \mid \langle x, y \rangle = 0 \text{ for all } y \in C\}$$

where ⟨·, ·⟩ denotes an appropriate inner product over the module $\mathbb{Z}_2^p Q^q$.

Definition 2.4. The subset A = $\{v_1, v_2, \ldots, v_t\}$ of Q^q is called a Q-linearly independent set if the equation

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_t v_t = 0, \text{ where } \lambda_1, \ldots, \lambda_t \in Q,$$

has only the trivial solution $\lambda_i = 0$ for all i ∈ {1, 2, . . . , t}.

Definition 2.5. An additive code C over $\mathbb{Z}_2Q$ is said to be an Additive Complementary Dual (ACD) code if $C \cap C^{\perp} = \{0\}$. If p = 0, then C is an ACD code of length q, and if q = 0, then C is a binary LCD code of length p.

An additive code C over a finite ring $\mathbb{Z}_2Q$ is called self-orthogonal (self-dual) if $C \subseteq C^{\perp}$ ($C=C^{\perp}$). If C has a basis over $\mathbb{Z}_2Q$, then it is called a free additive code and the cardinality of the basis is called the rank of C.

3. Additive Complementary Dual Code over $\mathbb{Z}_2Q$

Let C_p be a code over $\mathbb{Z}_2$ generated by an m × p matrix G_p, and let C_q be a code over Q generated by an m × q matrix G_q. Then the m × (p + q) matrix $G = [G_p \mid G_q]$ over $\mathbb{Z}_2Q$ generates an additive code C. Now, we have using the inner product equation 2.4. We define the matrix product as follows:

$$G \circ G^t = uvG_pG_p^t + G_qG_q^t, \quad (3.1)$$

where G^t is the transpose matrix of G.

Theorem 3.1. Let C be a $\mathbb{Z}_2Q$- additive code with the generator matrix G of size m × (p + q). Let nonzero vector v_i be the i^{th} row of G such that $\langle v_i, v_j \rangle \in \{0, uv\}$ and $\langle v_i, v_i \rangle \in U(Q)$ for all i, j ∈ {1, 2, . . . , m} such that i ≠ j, then C is a $\mathbb{Z}_2Q$ – ACD code.

Proof. Let u be any nonzero codeword of C. If $u \neq C^{\perp}$, then C is a $\mathbb{Z}_2Q$-ACD codes. Since u ∈ C, then $u = \sum_{i \in J} \lambda_i v_i$, where J = {1, 2, . . . , m} and $\lambda_i \in Q$. Firstly, we assume that there exist j ∈ J such that $\lambda_j \in U(Q)$. Then,

$$\langle u, vj\rangle = \sum_{i\in J} \lambda_i \langle v_i, v_j\rangle$$
$$=\sum_{i\in J\{j} \lambda_i \langle v_i, v_j\rangle + \lambda_j\langle v_j, v_j\rangle.$$

For $i \neq j$, since $\langle v_i , v_j \rangle \in \{0, uv\}$, then $\lambda_i \langle v_i , v_j \rangle \in \{0, uv\}$. Since $\langle v_j , v_j \rangle \in U(Q)$, then $\lambda_j \langle v_j , v_j \rangle \in U(Q)$. Thus $\langle u, v_j \rangle \neq 0$ and $u \neq C^{\perp}$. Further, if $\lambda_i \in m$, the maximal ideal. Let $j \in J$ such that $\lambda_j \in m \setminus \{0\}$. Since $\langle v_i , v_j \rangle \in \{0, uv\}$, then $\lambda_i \langle v_i , v_j \rangle = 0$. Thus,

$$\langle u, v_j\rangle = \sum_{i\in J\{j} \lambda_i \langle v_i, v_j\rangle + \lambda_j\langle v_j, v_j\rangle.$$
$$= \lambda_j\langle v_j, v_j\rangle.$$

Since $\langle v_j , v_j \rangle \in U(Q)$, then $\lambda_j \langle v_j , v_j \rangle \neq 0$. Thus $\langle u, v_j \rangle \neq 0$ and $u \neq C^{\perp}$. Thus C is a $\mathbb{Z}_2$Q- ACD codes. Hence the proved.

Theorem 3.1 directly leads to the deduction of the following corollaries.

Corollary 3.2. Let C be a $\mathbb{Z}_2$Q- additive code with the generator matrix G of size $m \times (p + q)$ and $G \circ G^t = [v_{ij}]_{m\times m}$. If $v_{ii} \in U(Q)$ and $v_{ij} \in \{0, uv\}$ for $i \neq j$, $1 \leq i, j \leq m$, then C is a $\mathbb{Z}_2$Q-ACD code.

It is important to note that the conditions presented in Theorem 3.1 and Corollaries 3.2 are solely sufficient to establish that C is a $\mathbb{Z}_2$Q-ACD code. In general, the converse statements do not hold true. Following example justifies this.

Example 3.3. Consider an additive code C of block length (2, 4) over $\mathbb{Z}_2$Q generated by

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+u & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+u & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+v & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+u+v \end{pmatrix} = [G_p \mid (G_q = R \mid S)].$$

From this C is an ACD code and it is $C \cap C^{\perp} = \{0\}$. But if u_1 is a first row of G, then the (1, 1) entry of $G \circ G^t$ is $v_{11} = \langle v_1 , v_1 \rangle = uv \notin U(Q)$.

Through the use of corollary 3.2, the following result is obtained.

Proposition 3.4. Let $G = (G_p \mid \lambda I_m)$ where $\lambda \in U(Q)$, G_p is an $m \times p$ matrix over $\mathbb{Z}_2$ and I_m is the identity matrix of size $m \times m$. Then G generates an ACD code over $\mathbb{Z}_2$Q.

Proof. Consider

$$G \circ G^t = uvG_p G^t_p + \lambda I_m I^t_m$$
$$= uvG_p G^t_p + \lambda I_m$$
$$G \circ G^t = [v_{ij}]_{m\times m}.$$

So, $v_{ij} \in \{0, uv\}$ for $i \neq j$ and $v_{ii} \in \{\lambda, \lambda + uv\} \subset U(Q)$ for $1 \leq i \leq m$. Therefore, using Corollary 3.2 G generates an ACD code over $\mathbb{Z}_2$Q.

Example 3.5. Let us take

$$G_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \text{over } \mathbb{Z}_2$$

and λ = 1 + u. Then the $\mathbb{Z}_2$Q-additive code generated by G

$$G = (G_2|\lambda I_4) = \begin{pmatrix} 0 & 1 & 1+u & 0 & 0 & 0 \\ 1 & 0 & 0 & 1+u & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+u & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+u \end{pmatrix}$$

is an ACD code.

In [2], the authors defined separable codes over a ring. Now, we generalize it. An additive code C over $\mathbb{Z}_2$Q of block length (p, q) is said to be a separable code if C is the direct product of C_p and C_q . In this case, $C^{\perp}$ is the direct product of $C_p^{\perp}$ and $C_q^{\perp}$. Thus, $C^{\perp}$ is separable.

The following theorem gives a necessary and sufficient condition for an additive separable code to be an ACD code.

Theorem 3.6. Let C be an additive separable code over $\mathbb{Z}_2$Q of block length (p, q). Then C_p and C_q are LCD codes over $\mathbb{Z}_2$ and Q, respectively, if and only if C is an ACD code.

Proof. Given that C = $C_p \times C_q$ is an additive separable code over $\mathbb{Z}_2$Q of block length (p, q). Then $C_p \subseteq \mathbb{Z}_2^p$ and $C_q \subseteq Q^q$, and hence $C^{\perp} = C_p^{\perp} \cap C_q^{\perp}$. Suppose C_p and C_q are LCD codes over $\mathbb{Z}_2$ and Q, respectively. Let (u, u')$\in C \cap C^{\perp}$,then we have u$\in C_p \cap C_p^{\perp} = \{0\}$and u'$\in C_q \cap C_q^{\perp} = \{0\}$, since C is separable. Therefore, the intersection of C and $C^{\perp}$ is trivial, and hence C is an ACD code.

Conversely, assume that C is an ACD code. Let u $\in C_p \cap C_p^{\perp}$ and u' $\in C_q \cap C_q^{\perp}$, then (u, u') $\in C \cap C^{\perp} = \{0\}$, and hence u = 0 and u′ = 0. This implies that $C_p \cap C_p^{\perp} = \{0\}$and $C_q \cap C_q^{\perp} = \{0\}$. Hence, C_p and C_q are LCD codes over $\mathbb{Z}_2$ and Q.

Example 3.7. Consider the additive code C over $\mathbb{Z}_2$Q generated by

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1+u & 0 & 0 & 1+uv & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1+u & 0 & 0 & 0 & 1+uv & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1+u & 1+u & 0 & 1+uv & 0 & 1+uv & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Clearly, C is a separable code. Then $C \cap C^{\perp} = \{0\}$. Thus C is ACD code over $\mathbb{Z}_2$Q. Moreover, we have $C_p \cap C_p^{\perp} = \{0\}$and $C_q \cap C_q^{\perp} = \{0\}$. Thus C_p and C_q is an LCD code over $\mathbb{Z}_2$ and Q.

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 10s (2025)

4. Gray map

In this section, we have introduced a new Gray map from $\mathbb{Z}_2$Q to $\mathbb{Z}_2^7$ and have discussed the properties in it.

Define the gray map ϕ: Q $\rightarrow \mathbb{Z}_2^6$ by

$$\phi(b + ud, e + uf + vg + uvh) = (d + b, d, h + g + f + e, h + g, h + f, h)$$

for all element in Q.

Let (u, u′), (u_1, u'_1)$\in \mathbb{Z}_2^p Q^q$, then the Hamming weight of (u, u′) is the number of non-zero coordinates of (u, u′) and is denoted by W_H (u, u′). We define the Lee weight of (u, u′) $\in \mathbb{Z}_2^p Q^q$ by W_L (u, u′) = W_H (ϕ(u, u′)) = W_H (u) + W_H (ϕ(u′)) and the Lee distance between (u, u′) and (u_1 , u'_1) by d_L ((u, u′), (u_1 , u'_1)) = W_L (u − u_1 , u′ − u'_1) = W_H (u − u_1) + W_H (ϕ(u′ − u'_1)). The Hamming distance between (u, u′) and (u_1 , u'_1) is defined by d_H ((u, u′), (u_1 , u'_1)) = W_H (u − u_1 , u′ − u'_1).

Then the Lee weight of elements of Q :

Element in the ring Q	its Lee weight
(0,0)	0
(0,1), (0,1+u), (0,1+v), (0,1+u+v+uv), (1,0), (1+u,0)	1
(0,u), (0,v), (0,u+v), (0,u+uv), (0,v+uv), (0,u+v+uv), (1,1), (1,1+u),(1,1+v), (1,1+u+v+uv), (u,0), (1+u,1), (1+u,1+u), (1+u,1+v),(1+u,1+u+v+uv)	2
(0,1+uv), (0,1+u+v), (0,1+u+uv), (0,1+v+uv), (1,u), (1,v), (1,u+v),(1,u+uv), (1,v+uv), (1,u+v+uv), (u,1), (u,1+u), (u,1+v), (u,1+u+v+uv),(1+u,u), (1+u,v), (1+u,u+v), (1+u,u+uv), (1+u,v+uv), (1+u,u+v+uv)	3
(0,uv), (1,1+uv), (1,1+u+v), (1,1+u+uv), (1,1+v+uv), (u,u), (u,v), (u,u+v),(u,u+uv), (u,v+uv), (u,u+v+uv), (1+u,1+uv), (1+u,1+u+v), (1+u,1+u+uv),(1+u,1+v+uv)	4
(1,uv), (u,1+uv), (u,1+u+v), (u,1+u+uv), (u,1+v+uv), (1+u,uv)	5
(u,uv)	6

Lemma 4.1. The Gray map ϕ is $\mathbb{Z}_2$ -linear.

Proof. The Lee weight of x ∈ Q is the Hamming weight of ϕ(x). Then the Lee distance between two elements is given as d_L (x, y) is a Lee weight of x − y for all x, y ∈ Q. See that table 1 is the Lee weight of element in Q.

For p, q ∈ ℤ, p, q ≥ 0,the gray mapping ϕ extended to

$$\phi : \mathbb{Z}_2^p \times Q^q \rightarrow \mathbb{Z}_2^{p+6q}$$

where $\phi(u_0, u_1, \ldots, u_{p-1}, u'_0, u'_1, \ldots, u'_{q-1}) = \big((u_0, u_1, \ldots, u_{p-1}), \phi(u'_0), \phi(u'_1) \ldots, \phi(u'_{q-1})\big)$. The Lee weight of $\big(u_0, u_1, \ldots, u_{p-1}, \phi(u'_0), \phi(u'_1), \ldots, \phi(u'_{q-1})\big) \in \mathbb{Z}_2^p Q^q$ is $\Sigma_{i=0}^{p-1} wt_H(u_i) + \Sigma_{j=0}^{q-1} wt_L(u'_j)$ and the Lee distance between x, y $\in \mathbb{Z}_2^p Q^q$ is $d_L(x, y) = wt_L(x - y)$.

In this section, we study certain conditions that make the gray image of an additive code over $\mathbb{Z}_2Q$ the binary LCD code.

Theorem 4.2. Let C be an additive code over $\mathbb{Z}_2Q$ generated by the matrix $G = (G_p | G_q)_{m\times(p+q)}$ where G_p generates a self-orthogonal code C_p over $\mathbb{Z}_2$ and G_q generates a additive code C_q over Q. If the rows of the matrics G_q is Q-linealy independent and ϕ (C_q) is an binary LCD codes, then the binary Gray image ϕ (C) is an LCD code.

Proof. Let $\phi(x) = (u, \phi(u')) \in \phi(C) \cap \phi(C)^{\perp}$ where x = (u, u′) ∈ C. Since for all y = (v, v′) ∈ C, $\phi(y) = (v, \phi(v')) \in \phi(C)$and $\phi(x) \in \phi(C)^{\perp}$,

$$(u, \phi(u')).(v, \phi(v')) = 0 \text{ for all } y = (v, v') \in C$$

$$(u.v) + (\phi(u').\phi(v')) = 0$$

Since u, v ∈ C_p and C_P is a self-orthogonal code, (u, v)=0. Therefore,$(\phi(u'), \phi(v')) = 0$. Since $\phi(u'), \phi(v') \in \phi(C_q)$and C_q is a binary LCD code, implies $(\phi(u'), \phi(v')) = 0$ and hence $\phi(u') = 0$, thus u′ = 0 because ϕ is linear.

We will prove that for any c = (u, u′) ∈ C, if u′ = 0, then the c = 0. For i ∈ {0, 1, · · · , m − 1}, let $g_i = (v_i, v_i')$ be the i^{th} row of G where v_i and v_i' denotes the i^{th} row of G_p and G_q , respectively. Since c ∈ C, there exist scalars $\lambda_0, \lambda_1, \cdots, \lambda_{m-1} \in Q$ such that

$$c = (u, u') = \sum_{i=0}^{m-1} \lambda_i g_i = (\sum_{i=0}^{m-1} \eta(\lambda_i) v_i, \sum_{i=0}^{m-1} \lambda_i v_i')$$

Thus $u = \sum_{i=0}^{m-1} \eta(\lambda_i) v_i$ and $u' = \sum_{i=0}^{m-1} \lambda_i v_i = 0$. Since the set of all rows of G_q is Q-linealy independent, $\lambda_i = 0$ for all i ∈ {0, 1, · · · , m − 1}. Therefore (u, u′) = 0 and hence $\phi(u, u') = 0$.

Here is an example to illustrate Theorem 4.2

Example 4.3. Let C be an additive code over $\mathbb{Z}_2Q$ generated by the matrix

$$G=\begin{pmatrix}1&1&0&0&u+1&u&uv+v+u+1&v+1\\0&0&1&1&0&u+1&uv+v&v+u\\0&0&0&0&0&0&v+1&v\end{pmatrix}$$= [G_p | (G_q = R | S)].

Then clearly, G_p generates a self-orthogonal code C_p and the set of all row vectors of G_q are Q-linearly independent. The Gray image $\phi(C_q)$ of C_q is a binary [64, 12, 25] linear LCD code.

5. **Weight Enumerators and MacWilliams Identities**

In this section, we present the weight enumerators of some $\mathbb{Z}_2$Q-additive codes and discuss related results.

5. 1. Complete Weight Enumerators and MacWilliams Identity

We arrange the elements of $\mathbb{Z}_2$Q in a fixed order as follows and denote them as $\{f_1, f_2, f_3, \ldots, f_{128}\}$:

$\mathbb{Z}_2$Q = {(0, 0, 0), (0, 0, 1), (0, 0, u), (0, 0, v), (0, 0, uv), (0, 0, 1 + u), (0, 0, 1 + v), (0, 0, 1 + uv), (0, 0, u + v), (0, 0, u + uv), (0, 0, v + uv), (0, 0, u + v + uv), (0, 0, 1 + u + v), (0, 0, 1 + u + uv), (0, 0, 1 + v + uv), (0, 0, 1 + u + v + uv), (0, 1, 0), (0, 1, 1), (0, 1, u), (0, 1, v), (0, 1, uv), (0, 1, 1 + u), (0, 1, 1 + v), (0, 1, 1 + uv), (0, 1, u + v), (0, 1, u + uv), (0, 1, v + uv), (0, 1, u + v + uv), (0, 1, 1 + u + v), (0, 1, 1 + u + uv), (0, 1, 1 + v + uv), (0, 1, 1 + u + v + uv), (0, u, 0), (0, u, 1), (0, u, u), (0, u, v), (0, u, uv), (0, u, 1 + u), (0, u, 1 + v), (0, u, 1 + uv), (0, u, u + v), (0, u, u + uv), (0, u, v + uv), (0, u, u + v + uv), (0, u, 1 + u + v), (0, u, 1 + u + uv), (0, u, 1 + v + uv), (0, u, 1 + u + v + uv), (0, 1 + u, 0), (0, 1 + u, 1), (0, 1 + u, u), (0, 1 + u, v), (0, 1 + u, uv), (0, 1 + u, 1 + u), (0, 1 + u, 1 + v), (0, 1 + u, 1 + uv), (0, 1 + u, u + v), (0, 1 + u, u + uv), (0, 1 + u, v + uv), (0, 1 + u, u + v + uv), (0, 1 + u, 1 + u + v), (0, 1 + u, 1 + u + uv), (0, 1 + u, 1 + v + uv), (0, 1 + u, 1 + u + v + uv),
(1, 0, 0), (1, 0, 1), (1, 0, u), (1, 0, v), (1, 0, uv), (1, 0, 1 + u), (1, 0, 1 + v), (1, 0, 1 + uv), (1, 0, u + v), (1, 0, u + uv), (1, 0, v + uv), (1, 0, u + v + uv), (1, 0, 1 + u + v), (1, 0, 1 + u + uv), (1, 0, 1 + v + uv), (1, 0, 1 + u + v + uv), (1, 1, 0), (1, 1, 1), (1, 1, u), (1, 1, v), (1, 1, uv), (1, 1, 1 + u), (1, 1, 1 + v), (1, 1, 1 + uv), (1, 1, u + v), (1, 1, u + uv), (1, 1, v + uv), (1, 1, u + v + uv), (1, 1, 1 + u + v), (1, 1, 1 + u + uv), (1, 1, 1 + v + uv), (1, 1, 1 + u + v + uv), (1, u, 0), (1, u, 1), (1, u, u), (1, u, v), (1, u, uv), (1, u, 1 + u), (1, u, 1 + v), (1, u, 1 + uv), (1, u, u + v), (1, u, u + uv), (1, u, v + uv), (1, u, u + v + uv), (1, u, 1 + u + v), (1, u, 1 + u + uv), (1, u, 1 + v + uv), (1, u, 1 + u + v + uv), (1, 1 + u, 0), (1, 1 + u, 1), (1, 1 + u, u), (1, 1 + u, v), (1, 1 + u, uv), (1, 1 + u, 1 + u), (1, 1 + u, 1 + v), (1, 1 + u, 1 + uv), (1, 1 + u, u + v), (1, 1 + u, u + uv), (1, 1 + u, v + uv), (1, 1 + u, u + v + uv), (1, 1 + u, 1 + u + v), (1, 1 + u, 1 + u + uv), (1, 1 + u, 1 + v + uv), (1, 1 + u, 1 + u + v + uv)}.

The complete weight enumerator of an additive code C over $\mathbb{Z}_2$Q is defined by

$$\mathrm{CWE}_C(x_1, x_2, \ldots, x_{128}) = \sum_{c\in C} x_1^{w_{f_1}(c)}\, x_2^{w_{f_2}(c)}\, x_3^{w_{f_3}(c)} \cdots x_{128}^{w_{f_{128}}(c)} \quad (5.1)$$

where $w_{f_i}(c)=|\{j \mid c_j = f_i,\ 1 \le j \le n\}|$ for $1 \le i \le 128$ and $c = (c_1, c_2, c_3, \ldots, c_n) \in C$.

Remark 5.1. Note that $CWE_C(x_1, x_2, \ldots, x_{128})$ is a homogeneous polynomial. The total degree of each monomial in $CWE_C(x_1, x_2, \ldots, x_{128})$ is n. The term x^n_1 always appears in $CWE_C(x_1, x_2, \ldots, x_{128})$ as $(0, 0, 0, \ldots, 0)$ is a codeword in C. Furthermore, note that $CWE_C(1, 1, 1, \ldots, 1) = |C|$ and $CWE_C(a, 0, 0, \ldots, 0) = a^n$.

Let R be a finite commutative ring with unity. A character χ of R is a group homomorphism from R to C^*. In [5], the authors defined that if R is a Frobenius ring and $\hat{R}$ is the character module of R such that $\hat{R}$ is R-module isomorphic to R, and if $\gamma : R \to \hat{R}$ is an R-module isomorphism, then $\chi := \gamma(1)$ is said to be a generating character of R.

Lemma 5.2. [5] Let χ be a character of a finite commutative ring R. Then χ is a generating character if and only if $\ker(\chi)$ contains no nonzero ideals of R.

Definition 5.3. The generating character $: \mathbb{Z}_2Q \to C^*$ is defined as

$$\chi(a, b + d, e + f + g + h) = (-1)^{a+b+d+e+f+g+h}. \quad (5.2)$$

Observe that the restriction of χ to every nonzero ideal of $\mathbb{Z}_2Q$ is nontrivial. Therefore, by Lemma 5.2, it is a generating character of the ring $\mathbb{Z}_2Q$.

Lemma 5.4. For a nonzero ideal I of $\mathbb{Z}_2Q$, we have $\sum_{x\in I} \chi(x) = 0$ where χ is defined in Equation 5.2.
Proof. Suppose that I is an ideal generated by (0, u, uv). Then

I = {(0, 0, 0), (0, 0, u), (0, 0, v), (0, 0, uv), (0, 0, u + v), (0, 0, u + uv),
(0, 0, v + uv), (0, 0, u + v + uv), (0, u, 0), (0, u, u), (0, u, v), (0, u, uv),
(0, u, u + v), (0, u, u + uv), (0, u, v + uv), (0, u, u + v + uv)}

is a maximal ideal of $\mathbb{Z}_2Q$. Observe that,

$$\sum_{x\in I} \chi(x) = 1 + (-1) + (-1) + (-1) + 1 + 1 + 1 + (-1) + (-1) + 1 + 1 + 1 + (-1) + (-1) + (-1) + 1 = 0.$$

Similarly, we can verify the equation if I is replaced by any other nonzero ideal of $\mathbb{Z}_2Q$.
Suppose that $T = [t_{i,j}]_{128\times128}$ is a matrix such that $t_{i,j} = \chi(f_{i,j})$, where $f_i, f_j \in \mathbb{Z}_2Q$ for $1 \le i, j \le 128$, and χ is the generating character defined in Equation 5.2. Then

$$T = \begin{bmatrix} A & A & A & A & A & A & A & A \\ A & A & A & A & A & A & A & A \\ A & A & A & A & A & A & A & A \\ A & A & A & A & A & A & A & A \\ A & A & A & A & A & A & A & A \\ A & A & A & A & A & A & A & A \\ A & A & A & A & A & A & A & A \\ A & A & A & A & A & A & A & A \end{bmatrix} \tag{5.3}$$

$where, A$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

For further discussion, we need the following result from [15].

Theorem 5.5. If C is an additive code of length n over $\mathbb{Z}_2Q$, then $CWE_{C\perp}(x_1, x_2, x_3, \ldots, x_{128}) = \frac{1}{|c|} CWE_C(T \cdot (x_1, x_2, x_3, \ldots, x_{128})^t)$, where t is denoted by the transpose and T is an in Equation 5.3.

Example 5.6. Let C be the R-submodule of $(\mathbb{Z}_2Q)^2$ spanned by {(1, 0; 0, 0; 0, 0), (0, 0; 0, u; 0, 0), (0, 0; 0, 0; 0, uv), (0, 0; 0, u; 0, uv)}. Clearly, C is a vector space over $\mathbb{Z}_2$ of dimension 7. Then the dual $C^\perp$ of the additive code C is the R-submodule of $(\mathbb{Z}_2Q)^2$ spanned by {(0, 1; 0, 0; 0, 0), (0, 0; 0, u; 0, 0), (0, 0; 0, 0; 0, uv), (0, 0; 0, u; 0, uv)}, which is also a 7-dimensional vector space over $\mathbb{Z}_2$. The complete weight enumerator of C and $C^\perp$ are:

$CWE_C(x_1, x_2, x_3, \ldots, x_{128}) = x_1^2 + x_5^2 + x_{33}^2 + x_{37}^2 + 2x_1 x_5 + 2x_1 x_{33} + 2x_1 x_{37} + x_1 x_3 + x_1 x_4 + x_1 x_9 + x_1 x_{10} + x_1 x_{11} + x_1 x_{12} + x_1 x_{35} + x_1 x_{36} + x_1 x_{41} + x_1 x_{42} + x_1 x_{43} + x_1 x_{44} + x_1 x_{65} + x_1 x_{67} + x_1 x_{68} + x_1 x_{69} + x_1 x_{73} + x_1 x_{74} + x_1 x_{75} + x_1 x_{76} + x_1 x_{97} + x_1 x_{99} + x_1 x_{100} + x_1 x_{101} + x_1 x_{105} + x_1 x_{106} + x_1 x_{107} + x_1 x_{108} + 2x_5 x_{33} + 2x_5 x_{37} + x_5 x_3 + x_5 x_4 + x_5 x_9 + x_5 x_{10} + x_5 x_{11} + x_5 x_{12} + x_5 x_{35} + x_5 x_{36} + x_5 x_{41} + x_5 x_{42} + x_5 x_{43} + x_5 x_{44} + x_5 x_{65} + x_5 x_{67} + x_5 x_{68} + x_5 x_{69} + x_5 x_{73} + x_5 x_{74} + x_5 x_{75} + x_5 x_{76} + x_5 x_{97} + x_5 x_{99} + x_5 x_{100} + x_5 x_{101}$

$$+ x_5 x_{105} + x_5 x_{106} + x_5 x_{107} + x_5 x_{108} + 2x_{33} x_{37} + x_{33} x_3 + x_{33} x_4 + x_{33} x_9 + x_{33} x_{10} + x_{33} x_{11} + x_{33} x_{12} + x_{33} x_{35} + x_{33} x_{36} + x_{33} x_{41} + x_{33} x_{42} + x_{33} x_{43} + x_{33} x_{44} + x_{33} x_{65} + x_{33} x_{67} + x_{33} x_{68} + x_{33} x_{69} + x_{33} x_{73} + x_{33} x_{74} + x_{33} x_{75} + x_{33} x_{76} + x_{33} x_{97} + x_{33} x_{99} + x_{33} x_{100} + x_{33} x_{101} + x_{33} x_{105} + x_{33} x_{106} + x_{33} x_{107} + x_{33} x_{108} + x_{37} x_3 + x_{37} x_4 + x_{37} x_9 + x_{37} x_{10} + x_{37} x_{11} + x_{37} x_{12} + x_{37} x_{35} + x_{37} x_{36} + x_{37} x_{41} + x_{37} x_{42} + x_{37} x_{43} + x_{37} x_{44} + x_{37} x_{65} + x_{37} x_{67} + x_{37} x_{68} + x_{37} x_{69} + x_{37} x_{73} + x_{37} x_{74} + x_{37} x_{75} + x_{37} x_{76} + x_{37} x_{97} + x_{37} x_{99} + x_{37} x_{100} + x_{37} x_{101} + x_{37} x_{105} + x_{37} x_{106} + x_{37} x_{107} + x_{37} x_{108} ,$$

$$CWE_{C^\perp} (x_1 , x_2 , x_3 , \dots , x_{128}) = x^2_1 + x^2_5 + x^2_{33} + x^2_{37} + 2x_1 x_5 + 2x_1 x_{33} + 2x_1 x_{37} + x_1 x_3 + x_1 x_4 + x_1 x_9 + x_1 x_{10} + x_1 x_{11} + x_1 x_{12} + x_1 x_{35} + x_1 x_{36} + x_1 x_{41} + x_1 x_{42} + x_1 x_{43} + x_1 x_{44} + x_1 x_{65} + x_1 x_{67} + x_1 x_{68} + x_1 x_{69} + x_1 x_{73} + x_1 x_{74} + x_1 x_{75} + x_1 x_{76} + x_1 x_{97} + x_1 x_{99} + x_1 x_{100} + x_1 x_{101} + x_1 x_{105} + x_1 x_{106} + x_1 x_{107} + x_1 x_{108} + 2x_5 x_{33} + 2x_5 x_{37} + x_5 x_3 + x_5 x_4 + x_5 x_9 + x_5 x_{10} + x_5 x_{11} + x_5 x_{12} + x_5 x_{35} + x_5 x_{36} + x_5 x_{41} + x_5 x_{42} + x_5 x_{43} + x_5 x_{44} + x_5 x_{65} + x_5 x_{67} + x_5 x_{68} + x_5 x_{69} + x_5 x_{73} + x_5 x_{74} + x_5 x_{75} + x_5 x_{76} + x_5 x_{97} + x_5 x_{99} + x_5 x_{100} + x_5 x_{101} + x_5 x_{105} + x_5 x_{106} + x_5 x_{107} + x_5 x_{108} + 2x_{33} x_{37} + x_{33} x_3 + x_{33} x_4 + x_{33} x_9 + x_{33} x_{10} + x_{33} x_{11} + x_{33} x_{12} + x_{33} x_{35} + x_{33} x_{36} + x_{33} x_{41} + x_{33} x_{42} + x_{33} x_{43} + x_{33} x_{44} + x_{33} x_{65} + x_{33} x_{67} + x_{33} x_{68} + x_{33} x_{69} + x_{33} x_{73} + x_{33} x_{74} + x_{33} x_{75} + x_{33} x_{76} + x_{33} x_{97} + x_{33} x_{99} + x_{33} x_{100} + x_{33} x_{101} + x_{33} x_{105} + x_{33} x_{106} + x_{33} x_{107} + x_{33} x_{108} + x_{37} x_3 + x_{37} x_4 + x_{37} x_9 + x_{37} x_{10} + x_{37} x_{11} + x_{37} x_{12} + x_{37} x_{35} + x_{37} x_{36} + x_{37} x_{41} + x_{37} x_{42} + x_{37} x_{43} + x_{37} x_{44} + x_{37} x_{65} + x_{37} x_{67} + x_{37} x_{68} + x_{37} x_{69} + x_{37} x_{73} + x_{37} x_{74} + x_{37} x_{75} + x_{37} x_{76} + x_{37} x_{97} + x_{37} x_{99} + x_{37} x_{100} + x_{37} x_{101} + x_{37} x_{105} + x_{37} x_{106} + x_{37} x_{107} + x_{37} x_{108}.$$

Definition 5.7. Let C be an additive code of length n over $\mathbb{Z}_2Q$. Then the Hamming weight enumerator of C over $\mathbb{Z}_2Q$ is defined as

$$W_C(x,y) = \sum_{c\in C} x^{n-wt_H(c)}\, y^{wt_H(c)} = CWE_C(x,y,y,\dots,y). \qquad (5.4)$$

Note that the polynomial W_C (x, y) is homogeneous of degree n.

Theorem 5.8. Let C be an additive code of length n over $\mathbb{Z}_2Q$. Then

$$W_{C^\perp}(x, y) = \frac{1}{|c|} W_C(x + 127y, x - y)$$

Proof. Consider

$$\begin{aligned} W_{C^\perp}(x, y) &= CWE_{C^\perp}(x, y, y, y, \dots, y) \\ &= \frac{1}{|c|} CWE_C(T \cdot (x, y, y, y, \dots, y)^t) \\ &= \frac{1}{|c|} CWE_C(x + 127y, x - y, x - y, \dots, x - y) \\ W_{C^\perp}(x, y) &= \frac{1}{|c|} W_C(x + 127y, x - y). \end{aligned}$$

Example 5.9. Let C be the code in Example 5.6. Then

$$W_C(x, y) = x^2 + 34xy + 93y^2$$

and

$$W_{C^\perp}(x, y) = \frac{1}{128}\{(x + 127y)^2 + 93(x - y)^2 + 34(x + 127y)(x - y)\}$$
$$= \frac{1}{128}\{128x^2 + 4,352xy + 11,904y^2\}$$
$$W_{C^\perp}(x, y) = x^2 + 34xy + 93y^2 .$$

5. 2. Symmetrized Weight Enumerator and Lee Weight Enumerator

The Lee weight of each element of $\mathbb{Z}_2Q$ is listed below:

Element in $\mathbb{Z}_2$Q	Lee weight
(0,0,0)	0
(0,0,1), (0,0,1+u), (0,0,1+v), (0,0,1+u+v+uv), (0,1,0), (0,1+u,0), (1,0,0)	1
(0,0,u), (0,0,v), (0,0,u+v), (0,0,u+uv), (0,0,v+uv), (0,0,u+v+uv), (0,1,1), (0,1,1+u), (0,1,1+v), (0,1,1+u+v+uv), (0,u,0), (0,1+u,1), (0,1+u,1+u), (0,1+u,1+v), (0,1+u,1+u+v+uv), (1,0,1), (1,0,1+u), (1,0,1+v), (1,0,1+u+v+uv), (1,1,0), (1,1+u,0)	2
(0,0,1+uv), (0,0,1+u+v), (0,0,1+u+uv), (0,0,1+v+uv), (0,1u), (0,1,v), (0,1,u+v), (0,1,u+uv), (0,1,v+uv), (0,1,u+v+uv), (0,u,1), (0,u,1+u), (0,u,1+v), (0,u,1+u+v+uv), (0,1+u,u), (0,1+u,v), (0,1+u,u+v), (0,1+u,u+uv), (0,1+u,v+uv), (0,1+u,u+v+uv), (1,0,u), (1,0,v), (1,0,u+v), (1,0,u+uv), (1,0,v+uv), (1,0,u+v+uv), (1,1,1), (1,1,1+u), (1,1,1+v), (1,1,1+u+v+uv), (1,u,0), (1,1+u,1), (1,1+u,1+u), (1,1+u,1+v), (1,1+u,1+u+v+uv)	3
(0,0,uv), (0,1,1+uv), (0,1,1+u+v), (0,1,1+u+uv), (0,1,1+v+uv), (0,u,u), (0,u,v), (0,u,u+v), (0,u,u+uv), (0,u,v+uv), (0,u,u+v+uv), (0,1+u,1+uv), (0,1+u,1+u+v), (0,1+u,1+u+uv), (0,1+u,1+v+uv), (1,0,1+uv), (1,0,1+u+v), (1,0,1+u+uv), (1,0,1+v+uv), (1,1,u), (1,1,v), (1,1,u+v), (1,1,u+uv), (1,1,v+uv), (1,1,u+v+uv), (1,u,1), (1,u,1+u), (1,u,1+v), (1,u,1+u+v+uv), (1,1+u,u), (1,1+u,v), (1,1+u,u+v), (1,1+u,u+uv), (1,1+u,v+uv), (1,1+u,u+v+uv)	4
(0,1,uv), (0,u,1+uv), (0,u,1+u+v), (0,u,1+u+uv), (0,u,1+v+uv), (0,1+u,uv), (1,0,uv), (1,1,1+uv), (1,1,1+u+v), (1,1,1+u+uv), (1,1,1+v+uv), (1,u,u), (1,u,v), (1,u,u+v), (1,u,u+uv), (1,u,v+uv), (1,u,u+v+uv), (1,1+u,1+uv), (1,1+u,1+u+v), (1,1+u,1+u+uv), (1,1+u,1+v+uv)	5
(0,u,uv), (1,1,uv), (1,u,1+uv), (1,u,1+u+v), (1,u,1+u+uv), (1,u,1+v+uv), (1,1+u,uv)	6

(1,u,uv)	7

Definition 5.10. Let C be an additive code of length n over $\mathbb{Z}_2$Q. Its symmetrized weight enumerator is defined by

SWE$_C$ (X, Y, Z, W, T, P, Q, N) =CWE$_C$ (X, Y, Z, Z, T, Y, Y, W, Z, Z, Z, Z, W, W, W, Y, Y, Z, W, W,P, Z, Z, T, W, W, W, W, T, T, T, Z, Z, W, T, T, Q, W, W, P, T, T, T, T, P, P, P, W, Y, Z, W, W, P, Z, Z, T, W, W, W, W, T, T, T, Z, Y, Z, W, W, P, Z, Z, T, W, W, W, W, T, T, T, Z, Z, W, T, T, Q, W, W, P, T, T, T, T, P, P, P, W, W, T, P, P, N, T, T, Q, P, P, P, P, Q, Q, Q, T, Z, W, T, T, Q, W, W, P, T, T, T, T, P, P, P, W).

where the Lee weight of the elements 0, 1, 2, 3, 4, 5, 6, and 7 is represented by the variables X, Y, Z, W, T, P, Q, and N , respectively.

By definition, we have

$$SWE_C(X,Y,Z,W,T,P,Q,N) = \sum_{c\in C} X^{n_0(c)}\, Y^{n_1(c)} Z^{n_2(c)} W^{n_3(c)} T^{n_4(c)} P^{n_5(c)} Q^{n_6(c)} N^{n_7(c)}$$

(5.5)

where $n_0(c) = w_{f_1}(c)$,

$n_1(c) = w_{f_2}(c) + w_{f_6}(c) + w_{f_7}(c) + w_{f_{16}}(c) + w_{f_{17}}(c) + w_{f_{49}}(c) + w_{f_{65}}(c)$,

$n_2(c) = w_{f_3}(c) + w_{f_4}(c) + w_{f_9}(c) + w_{f_{10}}(c) + w_{f_{11}}(c) + w_{f_{12}}(c) + w_{f_{18}}(c) + w_{f_{22}}(c) + w_{f_{23}}(c) + w_{f_{32}}(c) + w_{f_{33}}(c)$
$+w_{f_{50}}(c) + w_{f_{54}}(c) + w_{f_{55}}(c) + w_{f_{64}}(c) + w_{f_{66}}(c) + w_{f_{70}}(c) + w_{f_{71}}(c) + w_{f_{80}}(c) + w_{f_{81}}(c) + w_{f_{113}}(c)$,

$n_3(c) = w_{f_8}(c) + w_{f_{13}}(c) + w_{f_{14}}(c) + w_{f_{15}}(c) + w_{f_{19}}(c) + w_{f_{20}}(c) + w_{f_{25}}(c) + w_{f_{26}}(c) + w_{f_{27}}(c) + w_{f_{28}}(c) + w_{f_{34}}(c)$
$+w_{f_{38}}(c) + w_{f_{39}}(c) + w_{f_{48}}(c) + w_{f_{51}}(c) + w_{f_{52}}(c) + w_{f_{57}}(c) + w_{f_{58}}(c) + w_{f_{59}}(c) + w_{f_{60}}(c) + w_{f_{67}}(c) + w_{f_{68}}(c) + w_{f_{73}}(c)$
$+w_{f_{74}}(c) + w_{f_{75}}(c) + w_{f_{76}}(c) + w_{f_{82}}(c) + w_{f_{86}}(c) + w_{f_{87}}(c) + w_{f_{90}}(c) + w_{f_{97}}(c) + w_{f_{114}}(c) + w_{f_{118}}(c) + w_{f_{119}}(c) + w_{f_{128}}(c)$,

$n_4(c) = w_{f_5}(c) + w_{f_{24}}(c) + w_{f_{29}}(c) + w_{f_{30}}(c) + w_{f_{31}}(c) + w_{f_{35}}(c) + w_{f_{36}}(c) + w_{f_{41}}(c) + w_{f_{42}}(c) + w_{f_{43}}(c) + w_{f_{44}}(c)$
$+w_{f_{56}}(c) + w_{f_{61}}(c) + w_{f_{62}}(c) + w_{f_{63}}(c) + w_{f_{72}}(c) + w_{f_{77}}(c) + w_{f_{78}}(c) + w_{f_{79}}(c) + w_{f_{83}}(c) + w_{f_{84}}(c) + w_{f_{89}}(c)$
$+w_{f_{90}}(c) + w_{f_{91}}(c) + w_{f_{92}}(c) + w_{f_{98}}(c) + w_{f_{102}}(c) + w_{f_{103}}(c) + w_{f_{112}}(c) + w_{f_{115}}(c) + w_{f_{116}}(c) + w_{f_{121}}(c) + w_{f_{122}}(c)$
$+w_{f_{123}}(c) + w_{f_{124}}(c)$,

$n_5(c) = w_{f_{21}}(c) + w_{f_{40}}(c) + w_{f_{45}}(c) + w_{f_{46}}(c) + w_{f_{47}}(c) + w_{f_{53}}(c) + w_{f_{69}}(c) + w_{f_{88}}(c) + w_{f_{93}}(c) + w_{f_{94}}(c) + w_{f_{95}}(c)$

$+w_{f_{99}}(c) + w_{f_{100}}(c) + w_{f_{105}}(c) + w_{f_{106}}(c) + w_{f_{107}}(c) + w_{f_{108}}(c) + w_{f_{120}}(c) + w_{f_{125}}(c) + w_{f_{126}}(c) + w_{f_{127}}(c),$

$n_6(c) = w_{f_{37}}(c) + w_{f_{85}}(c) + w_{f_{104}}(c) + w_{f_{109}}(c) + w_{f_{110}}(c) + w_{f_{111}}(c) + w_{f_{117}}(c),$

$n_7(c) = w_{f_{101}}(c).$

Combining Theorem 5.5 and Definition 5.10, we have the following:

Theorem 5.11. Suppose C is an additive code over $\mathbb{Z}_2$Q of length n. Then

$$SWE_{C^\perp}(X, Y, Z, W, T, P, Q, N) = \frac{1}{|c|} SWE_C(S \cdot (X, Y, Z, W, T, P, Q, N)^t),$$

where
$$S = \begin{pmatrix} 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\ 1 & 5 & 9 & 5 & -5 & -9 & -5 & -1 \\ 1 & 3 & 1 & -5 & -5 & 1 & 3 & 1 \\ 1 & 1 & -3 & -3 & 3 & 3 & -1 & -1 \\ 1 & -1 & -3 & 3 & 3 & -3 & -1 & 1 \\ 1 & -3 & 1 & 5 & -5 & -1 & 3 & -1 \\ 1 & -5 & 9 & -5 & -5 & 9 & -5 & 1 \\ 1 & -7 & 21 & -35 & 35 & -21 & 7 & -1 \end{pmatrix}$$

Proof. Since C is an additive code over $\mathbb{Z}_2$Q, by Definition 5.10 and Theorem 5.5, we obtain

$SWE_{C^\perp}$ (X, Y, Z, W, T, P, Q, N) = $CWE_{C^\perp}$ (X, Y, Z, Z, T, Y, Y, W, Z, Z, Z, Z, W, W, W, Y, Y, Z, W, W, P, Z, Z, T, W, W, W, W, T, T, T, Z, Z, W, T, T, Q, W, W, P, T, T, T, T, P, P, P, W, Y, Z, W, W, P, Z, Z, T, W, W, W, W, T, T, T, Z, Y, Z, W, W, P, Z, Z, T, W, W, W, W, T, T, T, Z, Z, W, T, T, Q, W, W, P, T, T, T, T, P, P, P, W, W, T, P, P, N, T, T, Q, P, P, P, P, Q, Q, Q, T, Z, W, T, T, Q, W, W, P, T, T, T, T, P, P, P, W)

$=\frac{1}{|c|}CWE_C${T · (X, Y, Z, Z, T, Y, Y, W, Z, Z, Z, Z, W, W, W, Y, Y, Z, W, W, P, Z, Z, T, W, W, W, W, T, T, T, Z, Z, W, T, T, Q, W, W, P, T, T, T, T, P, P, P, W, Y, Z, W, W, P, Z, Z, T, W, W, W, W, T, T, T, Z, Y, Z, W, W, P, Z, Z, T, W, W, W, W, T, T, T, Z, Z, W, T, T, Q, W, W, P, T, T, T, T, P, P, P, W, W, T, P, P, N, T, T, Q, P, P, P, P, Q, Q, Q, T, Z, W, T, T, Q, W, W, P, T, T, T, T, P, P, P,W $)^t$}

$=\frac{1}{|c|}SWE_C$ {X + 7Y + 21Z + 35W + 35T + 21P + 7Q + N, X + 4Y + 3Z + 2W − 2T − 3P − 4Q − N, X + 2Y − Z − 2W − 2T − P + 2Q + N, X − Y − Z − W + T + P + Q − N, X − Y − Z + W − T + P − Q + N, X − 2Y − Z − 2W+ 2T + P + 2Q − N, X − 4Y + 3Z

$+ 2W - 2T + 3P - 4Q + N, X - 7Y + 21Z - 35W + 35T - 21P + 7Q - N \}$.

Hence,

$$SWE_{C^{\perp}} (X, Y, Z, W, T, P, Q, N) = \frac{1}{|c|} SWE_C (S \cdot (X, Y, Z, W, T, P, Q, N)^t).$$

Example 5.12. Let C be the code in Example 5.6. Then

$$SWE_C (X, Y, Z, W, T, P, Q, N) = X^2 + 7Z^2 + 7T^2 + Q^2 + 8XZ + 8XT + 2XQ + XN + XY + 7XP + 7XW + 14ZT + 8ZQ + ZY + 7ZP + 7ZW + ZN + 8TQ + TY + 7TW + 7TP + TN + QY + 7QW + 7QP + QW.$$

And

$$SWE_{C^{\perp}} (X, Y, Z, W, T, P, Q, N) = X^2 + 7Z^2 + 7T^2 + Q^2 + 8XT + 7ZP + XN + 8XZ + 14ZT + 7XW + 7XP + 8ZQ + ZY + 2XQ + 7ZW + ZN + 8TQ + 7TW + TY + 7TP + TN + QY + 7QW + 7QP + QW + XY.$$

Let $c \in (Z_2Q)^n$. The Lee weight of c is $wt_L (c) = n_1(c) + 2n_2(c) + 3n_3(c) + 4n_4(c) + 5n_5(c) + 6n_6(c) + 7n_7(c)$, where $n_i (c)$ is defined in Equation 5.5 for $1 \leq i \leq 7$.

Definition 5.13. The Lee weight enumerator of an additive code C over $\mathbb{Z}_2Q$ is given by

$$L_C(x, y) = \sum_{c \in C} x^{7n - wt_L(c)} y^{wt_L(c)}.$$

Theorem 5.14. Let C be a additive code of length n over $\mathbb{Z}_2Q$. Then

$$L_C (x, y) = SWE_C (x^7 , x^6 y, x^5 y^2 , x^4 y^3 , x^3 y^4 , x^2 y^5 , xy^6 , y^7).$$

Proof. Let $c \in C$. Then $wt_L (c) = \sum_{i=0}^{7} i\, n_i(c)$ and $n = \sum_{i=0}^{7} n_i(c)$ **.** By Definition 5.13, we have

$$L_C (x, y) = \sum_{c \in C} x^{7n - wtL (c)} y^{wtL (c)}$$

$$= \sum_{c \in C} x^{7n_0(c)+6n_1(c)+5n_2(c)+4n_3(c)+3n_4(c)+2n_5(c)+6n_6(c)} y^{n_1(c)+2n_2(c)+3n_3(c)+4n_4(c)+5n_5(c)+6n_6(c)+7n_7(c)}$$

$$= \sum_{c \in C} (x^7)^{n_0(c)} (x^6y)^{n_1}(c) (x^5y^2)^{n_2}(c) (x^4y^3)^{n_3}(c) (x^3y^4)^{n_4}(c) (x^2y^5)^{n_5}(c) (xy^6)^{n_6}(c).$$

By Equation 5.5,

$$L_C (x, y) = SWE_C (x^7 , x^6 y, x^5 y^2 , x^4 y^3 , x^3 y^4 , x^2 y^5 , xy^6 , y^7).$$

Theorem 5.15. Let C be an additive code of length n over $\mathbb{Z}_2Q$. Then

$$L_{C^{\perp}} (x, y) = \frac{1}{|c|} L_C (x + y, x - y).$$

Proof. By Definition 5.13,

$$L_{C^{\perp}}(x, y) = SWE_{C^{\perp}}(x^7, x^6y, x^5y^2, x^4y^3, x^3y^4, x^2y^5, xy^6, y^7)$$

$$= \frac{1}{|C|} SWE_C(S \cdot (x^7, x^6y, x^5y^2, x^4y^3, x^3y^4, x^2y^5, xy^6, y^7)^t) \text{ (By Theorem 5.11)}$$

$$= \frac{1}{|C|} SWE_C((x+y)^7, (x+y)^6(x-y), (x+y)^5(x-y)^2, (x+y)^4(x-y)^3, (x+y)^3(x-y)^4, (x+y)^2(x-y)^5, (x+y)(x-y)^6, (x-y)^7)$$

$$= \frac{1}{|C|} L_C(x+y, x-y) \quad \text{(By Theorem 5.14).}$$

Example 5.16. Suppose C is as in Example 5.6. Then by using Theorem 5.14,

$$L_{C^{\perp}}(x, y) = SWE_{C^{\perp}}(x^7, x^6y, x^5y^2, x^4y^3, x^3y^4, x^2y^5, xy^6, y^7)$$

$$= x^{14} + x^{13}y + 8x^{12}y^2 + 8x^{11}y^3 + 15x^{10}y^4 + 15x^9y^5 + 16x^8y^6 + 16x^7y^7 + 15x^6y^8 + 15x^5y^9 + 8x^4y^{10} + 8x^3y^{11} + x^2y^{12} + xy^{13}$$

and

$$L_C(x+y, x-y) = SWE_C((x+y)^7, (x+y)^6(x-y), (x+y)^5(x-y)^2, (x+y)^4(x-y)^3, (x+y)^3(x-y)^4, (x+y)^2(x-y)^5, (x+y)(x-y)^6, (x-y)^7)$$

$$= 128[x^{14} + x^{13}y + 8x^{12}y^2 + 8x^{11}y^3 + 15x^{10}y^4 + 15x^9y^5 + 16x^8y^6 + 16x^7y^7 + 15x^6y^8 + 15x^5y^9 + 8x^4y^{10} + 8x^3y^{11} + x^2y^{12} + xy^{13}].$$

6. Conclusion

In this paper, we studied the ACD codes over the finite commutative Frobenius ring $\mathbb{Z}_2RS$, where $R = \mathbb{Z}_2 + u\mathbb{Z}_2$ and $S = \mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2 + uv\mathbb{Z}_2$. We established a condition to ensure that an additive code is an ACD code. Furthermore, we derived a necessary and sufficient condition for a separable additive code to be an ACD code. We also examined a Gray map that transforms certain additive codes into binary linear complementary dual (LCD) codes. Additionally, the MacWilliams identities corresponding to different weight enumerators were discussed.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

Acknowledgment

The authors declare that, there is no financial support from any of the institutions or personal relationship to affect the quality of the paper.

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