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Computation of Euler-Type Integrals Involving Generalized M-Series

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Abstract: : This research presents new evaluations of Euler-type integrals that encompass the generalized M-series, a comprehensive extension of special functions. These results not only deepen our understanding of the structural properties of the generalized M-series but also suggest potential applications in various fields, including mathematical physics, engineering, and applied mathematics. Several specific cases are analysed to illustrate the versatility and utility of the derived integral formulas by selecting particular parameter values of the generalized M-series.

Keywords: : Euler-type integrals, generalized M-series and Fox-wright functions.

1. Introduction

Special functions are mathematical functions that have been assigned specific names and symbols due to their importance in various fields, including functional analysis, geometry, physics, and mathematical analysis. Integral transforms are widely used in many applied mathematics and mathematical physics problems. Several mathematicians have developed integral transforms, such as the Euler integral, Laplace transform, Fourier transform, Mellin transform, and Hankel transform, each incorporating different special functions. In numerous studies, the term exp(t), which appears in the integral representation of the gamma function, is often substituted with more complex functions to facilitate generalizations. These extensions of the beta function are presented while maintaining its symmetry properties [1] . Furthermore, integral and derivative formulas for Gauss hypergeometric and confluent hypergeometric functions, along with descriptions of their generalizations, are derived using the generalized beta function.

Recently, many researchers have been presenting extensions and generalizations for special functions, such as the Pochhammer symbol, gamma function, k-gamma function, p-k-gamma and beta functions, along with the confluent hypergeometric function (see [2-5, 9-10, 15]). Sharma and Jain [12] characterized a generalized M-series, which extends various special functions such as the Mittag-Leffler function, Wright function, Prabhaker function, and Gauss hypergeometric function. In 2018, Suthar et al. [15] assessed integral expressions of the product of M-series and Jacobi polynomials. Subsequently, Sachan et al. [11] presented a new extension of the M-series and discovered its properties, including recurrence relations, integral representation, and formulas for fractional integrals and derivatives. Following this excellent work, we obtained composition formulas for Euler-type integrals featuring a generalized M-series kernel. Additionally, we demonstrated applications of our findings as specific instances by selecting appropriate values for the parameters of generalized M-series

Sachan et al. [12] defined generalized M-series, which is novel an extension of generalized hypergeometric function, Wright function and Mittag-Leffler is defined as:

$$ {}_{p}^{\sigma}M_{q}^{\rho} = {}_{p}^{\sigma}M_{q}^{\rho}(k_1, \dots, k_p, l_1 \dots . l_q : z) = \sum_{m=0}^{\infty} \frac{(k_1)_m \cdots (k_p)_m}{(l_1)_m \cdots (l_q)_m} \frac{z^m}{\Gamma(\rho+\sigma m)}, \quad (1.1) $$

where $\rho, \sigma, z \in C, R(\sigma) > 0, (k_i)_m (i = 1, \dots, p)$ and $(l_j)_m$ $(j = 1, \dots, q)$. Well known Pochhamer symbols $(k)_m$ defined as:

$$ (k)_m = \begin{cases} 1 & m = 0, \quad k \neq 0 \\ k(k+1)\dots(k+m-1), & m \in N, k \in C \end{cases} $$

$$ = \frac{\Gamma(k+m)}{\Gamma(k)}. $$

The series (1.1) is convergent for all z if $p \leq q$. it is convergent for $|z| < \eta = \sigma^{\sigma}$ if $p = q + 1$ and divergent if $p > q + 1$. when $p = \nu + 1$ and $|z| = \eta$, there is convergent under conditions that depend on parameters. The detailed accound of the M-series can be found in paper written by Sharma and Jain [12].

The generalized M-series can be represented as a special case of Wright generalized hypergeometric function, called Fox-Wright function ${}_r\Psi_s[x]$, of the Fox H-function, and of Meijer G-function [6].

$$ {}_{p}^{\sigma}M_{q}^{\rho}\left(k_1, \dots, k_p, l_1 \dots . l_q : z\right) = \frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{j=1}^{p}\Gamma(k_p)} \sum_{k=0}^{\infty} \frac{\Gamma(\lambda_1+\lambda_1 k), \dots, \Gamma(\lambda_r+\lambda_s k) x^k}{\Gamma(l_1+l_1 k), \dots, \Gamma(l_1+l_1 k) k!}, \quad (1.2) $$

$$ = {}_{p+1}\Psi_{q+1}\left[\begin{matrix} (\lambda_1, \lambda_1), \dots, & (\lambda_r, \lambda_s); \\ (l_1, l_1), \dots, & (l_r, l_s); \end{matrix} x\right] \quad (1.3) $$

$$ {}_{p}^{\sigma}M_{q}^{\rho}\left(k_1, \dots, k_p, l_1 \dots . l_q : z\right) = \frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{j=1}^{p}\Gamma(k_p)} H_{p+1,q+2}^{1,p+1}\left[-z \middle| \begin{matrix} (1-k_j)_1^p, & (0,1) \\ (0,1), (1-l_j)_1^q, & (0,1) \end{matrix}\right], \quad (1.4) $$

where $H_{r,s+1}^{1,r}[x]$ is a Fox-H function [1] and the coefficients $\gamma_1', \dots, \gamma'_r, l'_1, \dots, l'_s \in R^+$ such that $1 + \sum_{j=1}^{s} l_j' - \sum_{i=1}^{r} \gamma_j'$ for suitable bounded value of $|x|$.

Taking suitable values of the parameters in (1.1), we conclude special cases and connections which are enumerated as follows:

1. For$\sigma = 1$, the generalized M-series reduces in the M-series defined by Sharma [12].

$$ {}_{p}^{1}M_{q}^{\rho}\left(k_1, \dots, k_p, l_1 \dots . l_q : z\right) = \sum_{m=0}^{\infty} \frac{(k_1)_m \dots (k_p)_m}{(l_1)_m \dots (l_q)_m} \frac{z^m}{\Gamma(\rho+\sigma m)} \quad (1.5) $$

$$= {}_pM_q^{\rho}\left(k_1,\ldots,k_p,l_1\ldots l_q:z\right)$$

2. For $p = q = 0$, the generalized M-series reduces in the Mittag-Leffler function[17]

$${}_0^{\sigma}M_0^{\rho}\left(k_1,\ldots,k_p,l_1\ldots l_q:z\right) = \sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\rho+\sigma m)} = E_{\rho,\sigma}(z) \quad (1.6)$$

3. For $p = q = 1, k = \alpha\epsilon\, C, l = 1$, equation (1.1) reduces in the generalized Mittag-Leffler function [14]

$${}_1^{\sigma}M_1^{\rho}\ (-,-:z) = \sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\rho+\sigma m)} = E_{\rho,\sigma}^{\alpha}(z) \quad (1.7)$$

4. For $p = q = 1, k = \alpha\epsilon\, C, l = 1$, equation (1.1) reduces in the Wright function [17]

$${}_1^{1}M_1^{\rho}\ (-,1:z) = \sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\rho+m)} = W_{\rho,\sigma}(z) \quad (1.8)$$

5. For $\rho = \sigma = 1$, equation (1.1) reduces in term of Gauss hypergeometric function

$${}_p^{1}M_q^{1}\left(k_1,\ldots,k_p,l_1\ldots l_q:z\right) = \sum_{m=0}^{\infty}\frac{(k_1)_m\cdots(k_p)_m}{(l_1)_m\cdots(l_q)_m}\frac{z^m}{m!} == {}_pF_q\left[\begin{matrix}k_1,\ldots & k_p; \\ l_1,\ldots & l_q;\end{matrix}\ z\right]. \quad (1.9)$$

In the present study, we also need to recall the following interesting and useful integral identities in term of gamma function [1, 14-15] as follows:

1. $\int_0^{\infty} e^{-\alpha t}[\sinh(\beta t)]^{\gamma}dt = \beta^{-1}2^{-\gamma-1}\frac{\Gamma\left(\frac{\alpha}{2\beta}-\frac{\gamma}{2}\right)\Gamma(1+\gamma)}{\Gamma\left(\frac{1}{2\beta}+\frac{\gamma}{2}+1\right)}$, (1.10)
   a. $R(\gamma) > -1,\ R(\beta) > 0, R(\frac{\alpha}{\beta}) > R(\gamma)$
2. $\int_0^1 x^{\sigma}(1-x^2)^{-\frac{\mu}{2}}P_{\gamma}^{\mu}(x)dx = 2^{\mu-1}\frac{\Gamma\left(\frac{1}{2}+\frac{\sigma}{2}\right)\Gamma(1+\frac{\sigma}{2})}{\Gamma\left(1+\frac{\sigma}{2}-\frac{\nu}{2}-\frac{\mu}{2}\right)\Gamma(\frac{\sigma}{2}+\frac{\nu}{2}-\frac{\mu}{2}+\frac{3}{2})}$, (1.11)
   a. $Re(\mu) < 1, Re(\sigma) > -1$
3. $.\int_1^{\infty} x^{-\rho}(x^2-1)^{-\frac{\mu}{2}}P_{\nu}^{\mu}(x)dx = 2^{\rho+\mu-2}\frac{\Gamma\left(\frac{\rho+\mu+\nu}{2}\right)\Gamma\left(\frac{\rho+\mu-\nu-1}{2}\right)}{\pi^{\frac{1}{2}}\Gamma(\rho)}$, (1.12)
   $Re(\mu) < 1, Re(\rho+\mu+\nu) > 0, Re(\rho+\mu-\nu) > 1$

Integral containing Tchebichef polynomial

4. $\int_{-1}^{1}(1-x)^{\frac{1}{2}}(1+x)^{\alpha}U_n(x)dx = \pi^{\frac{1}{2}}2^{\alpha+2\mu+\frac{3}{2}}\frac{[n+1]^2\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\alpha+1)}{(2n+2)\Gamma\left(\alpha+n+\frac{1}{2}\right)\Gamma\left(\alpha-n+\frac{1}{2}\right)}$ (1.13)
   a. where, $Re(\alpha) > -1;\ n = 1,2,\ldots\ldots$
5. $\int_0^1 x^{\lambda}P_{2\gamma}(x)dx = \frac{(-1)^{\gamma}\Gamma\Gamma}{2\Gamma\left(\gamma+\frac{\lambda}{2}+\frac{3}{2}\right)\Gamma(-\frac{\lambda}{2})}$, $Re(\lambda) > -1, \gamma$ is non-negative integer. (1.14)
6. $\int_0^1 x^{\lambda}P_{2\nu+1}(x)dx = (-1)^{\nu}\frac{\Gamma\left(\frac{1}{2}-\frac{\lambda}{2}+\nu\right)\Gamma(1+\frac{\lambda}{2})}{\Gamma\left(\frac{1}{2}-\frac{\lambda}{2}\right)\Gamma(2+\nu+\frac{\lambda}{2})}$, $Re(\lambda) > -2$ (1.15).

Integral associated with generalized Laguerre polynomial as:

7. $\int_0^{\infty} x^{\beta-1}e^{-x}L_n^{(\alpha)}(x)dx = \frac{\Gamma(\alpha-\beta+n+1)\Gamma(\beta)}{\Gamma(\alpha-\beta+1)\, n!}$, $Re(\beta) > 0$, n is non negative integer. (1.16)

8. $\int_0^{\infty} u^{\lambda-1}(1+u)^{-\mu}du = \frac{\Gamma(\lambda)\Gamma(\mu-1)}{\Gamma(\mu)}, 0 < Re(\lambda) < Re(\mu).$ (1.17)

9. $\int_0^{\frac{\pi}{2}} \sin^{\mu}\vartheta.\cos^{\nu}\vartheta d\vartheta = \frac{\Gamma\left(\frac{\mu+1}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}{2\Gamma\left(\frac{\mu+\nu+2}{2}\right)}, Re(\mu) > 0, Re(\nu) > 0.$ (1.18)

2. Composition of Euler type Integrals:

Theorem 1. If $Re(\eta) > -1, Re(\xi) > 0, Re(\nu) > 0, Re\left(\frac{\mu}{\nu}\right) > Re(\mu)$, $\sigma, \rho \in C$, $\xi = \vartheta_4$, $Re(\sigma) > 0$, we get following results

$$\int_0^{\infty} e^{-\mu t}[\sinh(\nu t)]^{\eta}\, {}_p^{\sigma}M_q^p\left(k_1, \dots, k_p, l_1 \dots l_q : ze^{-\xi t}\right)dt = \nu^{-1}2^{-\eta-1}\frac{\prod_{i=1}^{q}\Gamma(l_q)\Gamma(1+\eta)}{\prod_{j=1}^{p}\Gamma(k_p)}\, {}_{p+2}\Psi_{q+2}\left[\begin{matrix}(k_1,1),\dots\dots, & (k_p,1),\left(\frac{\mu}{2\nu}-\frac{\eta}{2}+\frac{\xi}{2\nu}\right),(1,1)\\ (l_1,1),\dots\dots, & (l_q,1),(\rho,\sigma),\left(\frac{\mu}{2\nu}+\frac{\eta}{2}+1+\frac{\xi}{2\nu}\right)\end{matrix} Z\right].$$
(2.1)

Proof. To prove above Theorem 1, expressing ${}_p^{\sigma}M_q^p$ is the L.H.S. of (2.1). Changing the order of integral and on evaluating the inner integral with help of (1.10).

$$\int_0^{\infty} e^{-\mu t}[\sinh(\nu t)]^{\eta}\, {}_p^{\sigma}M_q^p\left(k_1, \dots, k_p, l_1 \dots l_q : ze^{-\xi t}\right)dt$$

$$= \int_0^{\infty} e^{-\mu t}[\sinh(\nu t)]^{\eta} \sum_{m=0}^{\infty} \frac{(k_1)_m \dots (k_p)_m}{(l_1)_m \dots (l_q)_m}\frac{z^m e^{-\xi m t}}{\Gamma(\rho+\sigma m)}dt$$

$$= \sum_{m=0}^{\infty} \frac{(k_1)_m \dots (k_p)_m}{(l_1)_m \dots (l_q)_m}\frac{z^m e^{-\xi m t}}{\Gamma(\rho+\sigma m)}\int_0^{\infty} e^{-(\mu+\xi m)t}[\sinh(\nu t)]^{\eta}dt$$

$$= \sum_{m=0}^{\infty} \frac{(k_1)_m \dots (k_p)_m}{(l_1)_m \dots (l_q)_m}\frac{z^m e^{-\xi m t}}{\Gamma(\rho+\sigma m)}\nu^{-1}2^{-\eta-1}\frac{\Gamma\left(\frac{\mu+\xi m}{2\nu}-\frac{\eta}{2}\right)\Gamma(\eta+1)}{\Gamma\left(\frac{\mu+\xi m}{2\nu}+\frac{\eta}{2}+1\right)}$$

$$= \nu^{-1}2^{-\eta-1}\Gamma(\eta+1)\sum_{m=0}^{\infty} \frac{(k_1)_m \dots (k_p)_m}{(l_1)_m \dots (l_q)_m}\frac{z^m e^{-\xi m t}}{\Gamma(\rho+\sigma m)}\frac{\Gamma\left(\frac{\mu}{2\nu}-\frac{\eta}{2}+\frac{\xi m}{2\nu}\right)}{\Gamma\left(\frac{\mu}{2\nu}+\frac{\eta}{2}+1+\frac{\xi m}{2\nu}\right)}.$$

Finally, we use equation (1.3), and we get desired result (2.1).

Theorem 2. If $Re(\xi) > 0, Re(\eta) > -1, Re(\sigma) > 0, Re\left(\frac{\mu}{2}\right) > Re(\eta)$: $\sigma, \rho \in C$, following results holds:

$$\int_0^{\infty} e^{-\mu}[\sinh(\nu t)]^{\eta}\, {}_p^{\sigma}M_z^p\left(k_1, \dots, k_p, l_1 \dots l_q : z(2\sinh(\nu t))^{\xi m}\right)dt = \nu^{-1}2^{-\eta-1}\frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)}$$

$$\times {}_{p+3}\Psi_{q+2}\left[\begin{matrix}(k_1,1),\ldots\ldots, & (k_p,1),\left(\frac{\mu}{2\nu}-\frac{\eta}{2}+\frac{\xi}{2}\right),(1+\eta,\xi),(1,1)\\ (l_1,1),\ldots\ldots, & (l_q,1),(\rho,\sigma),\left(\frac{\mu}{2\nu}+\frac{\eta}{2}+1+\xi\right)\end{matrix};Z\right]. \qquad (2.2)$$

Proof. For solving the above integral formula (2.2), using definition (1.1) in the L.H.S. of (2.2) and then changing order of integration, we have

$$=\sum_{m=0}^{\infty}\frac{(k_1)_m\ldots\ldots(k_p)_m}{(l_1)_m\ldots\ldots(l_q)_m}\frac{z^m 2^{\xi m}(\sinh(\nu t))^{\xi m}}{\Gamma(\rho+\sigma m)}\int_0^{\infty}e^{-\mu}[\sinh(\nu t)]^{\eta}\,dt$$

$$=\frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)}\sum_{m=0}^{\infty}\frac{\Gamma(1+k_1)\ldots\ldots\Gamma(m+k_p)\,(z2\xi)^m}{\Gamma(1+l_1)\ldots\ldots\Gamma(m+l_q)\Gamma(\rho+\sigma m)}\int_0^{\infty}e^{-\mu}[\sinh(\nu t)]^{\eta+\xi m}\,dt$$

$$=\frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)}\sum_{m=0}^{\infty}\frac{\Gamma(1+k_1)\ldots\ldots\Gamma(m+k_p)(z2\xi)^m}{\Gamma(1+l_1)\ldots\ldots\Gamma(m+l_q)\Gamma(\rho+\sigma m)}\frac{\nu^{-1}2^{-(\eta+\xi m)-1}\Gamma\left(\frac{\mu}{2\nu}-\frac{\eta+\xi m}{2}\right)\Gamma(1+\eta+\xi m)}{\Gamma(\frac{\mu}{2\nu}+\frac{\eta+\xi m}{2}+1)}$$

$$=\nu^{-1}2^{-\eta-1}\frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)}\sum_{m=0}^{\infty}\frac{\Gamma(1+k_1)\ldots\ldots\Gamma(m+k_p)}{\Gamma(1+l_1)\ldots\ldots\Gamma(m+l_q)\Gamma(\rho+\sigma m)}\frac{\Gamma\left(\frac{\mu}{2\nu}-\frac{\eta}{2}+\frac{\xi m}{2}\right)\Gamma(1+\eta+\xi m)}{\Gamma\left(\frac{\mu}{2\nu}+\frac{\eta}{2}+1+\xi m\right)}\frac{z^m}{m!}\Gamma(1+m).$$

Hence, we call equation (1.3) and then reached at desired result (2.2).

Theorem 3. If $\mathrm{Re}(\xi)>0, \mathrm{Re}(\nu)<1, \mathrm{Re}(\mu)>0, \sigma,\xi\in C, \mathrm{Re}(\sigma)>0$, following integral formula holds

$$\int_0^1 u^{\mu}(1-u^2)^{-\frac{\nu}{2}}P_{\eta}^{\nu}(u)\,{}_p^{\sigma}M_z^{p}(k_1,\ldots,k_p,l_1\ldots.l_q:zu^{\xi})du = 2^{\nu-1}\frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)}$$

$$\times {}_{p+3}\Psi_{q+3}\left[\begin{matrix}(k_1,1),\ldots\ldots, & (k_p,1),\left(\frac{\mu}{2}+\frac{1}{2},\frac{\xi}{2}\right),\left(1+\frac{\eta}{2},\frac{\xi}{2}\right),(1,1)\\ (l_1,1),\ldots\ldots, & (l_q,1),(\rho,\sigma),\left(\frac{\mu}{2}-\frac{\eta}{2}-\frac{\nu}{2}+1,\frac{\xi}{2}\right)\left(\frac{\mu}{2}+\frac{\eta}{2}-\frac{\nu}{2}+\frac{3}{2},\frac{\xi}{2}\right)\end{matrix};Z\right]. \quad (2.3)$$

Proof. To prove Theorem 3, using definition of generalized M-series in the left hand side of (2.3). After simple simplification, we get

$$\int_0^1 u^{\mu}(1-u^2)^{-\frac{\nu}{2}}P_{\eta}^{\nu}(u)\,{}_p^{\sigma}M_z^{p}(k_1,\ldots,k_p,l_1\ldots.l_q:zu^{\xi})du$$

$$= \sum_{m=0}^{\infty} \frac{(k_1)_m \ldots\ldots (k_p)_m}{(l_1)_m \ldots\ldots (l_q)_m} \frac{z^m}{\Gamma(\rho+\sigma m)} \int_0^1 u^{\mu+\xi m}(1-u^2)^{-\frac{\nu}{2}} P_\eta^\nu(u)du$$

$$= \frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)} \sum_{m=0}^{\infty} \frac{\Gamma(1+k_1)\ldots\ldots\Gamma(m+k_p)}{\Gamma(1+l_1)\ldots\ldots\Gamma(m+l_q)\Gamma(\rho+\sigma m)} \frac{2^{\nu-1}\Gamma\left(\frac{1}{2}+\frac{\mu+\xi m}{2}\right)\Gamma\left(1+\frac{\mu+\xi m}{2}\right)\Gamma(1+m)z^m}{\Gamma\left(1+\frac{\mu+\xi m}{2}-\frac{\eta}{2}-\frac{\nu}{2}\right)\Gamma(\frac{\mu+\xi m}{2}+\frac{\eta}{2}-\frac{\nu}{2}+\frac{3}{2})m!}$$

Hence Proved.

Theorem 4. If $\mathrm{Re}(\xi) > 0, \mathrm{Re}(\nu) < 1, \mathrm{Re}(\mu+\nu+\eta) > 0, \mathrm{Re}(\mu+\nu-\eta) > 1, \mathrm{Re}(\sigma) > 0$, following integral formulas holds:

$$\int_0^\infty u^{-\mu}(u^2-1)^{-\frac{\nu}{2}} P_\eta^\nu(u)\, {}_p^\sigma M_z^p(k_1,\ldots,k_p,l_1\ldots l_q : zu^{-\xi})du$$
$$= \frac{\prod_{j=1}^{q}\Gamma(l_q)}{(\pi)^{\frac{1}{2}}\prod_{i=1}^{p}\Gamma(k_p)}\ {}_{p+3}\Psi_{q+2}\left[\begin{matrix}(k_1,1),\ldots\ldots, & (k_p,1),\left(\frac{\mu+\nu+2}{2},\frac{\xi}{2}\right),\left(\frac{\mu+\nu-\eta-1}{2},\frac{\xi}{2}\right),(1,1) \\ (l_1,1),\ldots\ldots, & (l_q,1),\ (\rho,\sigma),\ (\mu,\xi)\end{matrix} ; Z\right]. \qquad (2.4)$$

Proof. Using equation (1.2) and changing the order of integration in the left hand side of (2.4), we have

$$= \frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)} \sum_{m=0}^{\infty} \frac{\Gamma(1+k_1)\ldots\ldots\Gamma(m+k_p)\, z^m}{\Gamma(1+l_1)\ldots\ldots\Gamma(m+l_q)\Gamma(\rho+\sigma m)} \int_0^\infty u^{-(\mu+\xi m)}(u^2-1)^{-\frac{\nu}{2}} P_\eta^\nu(u)du$$

$$= \frac{\prod_{j=1}^{q}\Gamma(l_q)}{(\mu)^{\frac{1}{2}}\prod_{i=1}^{p}\Gamma(k_p)} \sum_{m=0}^{\infty} \frac{\Gamma(1+k_1)\ldots\ldots\Gamma(m+k_p)\, z^m}{\Gamma(1+l_1)\ldots\ldots\Gamma(m+l_q)\Gamma(\rho+\sigma m)}$$
$$\times \frac{\Gamma\left(\frac{\mu+\xi m+\nu+2}{2}\right)\Gamma\left(\frac{\mu+\xi m+\nu-\eta-1}{2}\right)\Gamma(1+m)}{\Gamma(\mu+\xi m)m!}$$

Thus, from equation (1.3), reached at required result of Theorem 4.

Theorem 5. If $\mathrm{Re}(\mu) > -1, \mathrm{Re}(\lambda) > 0, \mathrm{Re}(\sigma) > 0, n = 1,2.,,$ following integral formula holds:

$$\int_0^1 (1-u)^{\frac{1}{2}}(1+u)^{\mu} U_n(u)\, {}_p^\sigma M_z^p(k_1,\ldots,k_p,l_1\ldots l_q : z(1+x)^{\lambda})du$$
$$=$$
$$\frac{(\pi)^{\frac{1}{2}}2^{2n+\frac{3}{2}}\{(n+1)!\}^2}{(2n+2)} \frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)}\ {}_{p+3}\Psi_{q+2}\left[\begin{matrix}(k_1,1),\ldots\ldots, & (k_p,1),\left(\frac{2\mu+1}{2},\lambda\right),(\mu,\lambda),(1,1) \\ (l_1,1),\ldots\ldots, & (l_q,1),\ (\rho,\sigma),\ \left(\mu+\frac{5}{2},\lambda\right),\left(\mu`-n+\frac{1}{2},\lambda\right)\end{matrix} ; Z\right]$$
$$. \qquad (2.5)$$

Proof. To prove above Theorem, using definition (1.1) and relation (1.3) in the left hand side of Theorem 5, and then Simply, we get required result.

$$= \frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)}\sum_{m=0}^{\infty}\frac{\Gamma(1+k_1)\ldots\ldots\Gamma(m+k_p)\,(\frac{z}{2\lambda})^m}{\Gamma(1+l_1)\ldots\ldots\Gamma(m+l_q)\Gamma(\rho+\sigma m)}\int_0^1(1-u)^{\frac{1}{2}}(1+u)^{\mu+\lambda n}U_n(u)\,du$$

$$= \frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)}\sum_{m=0}^{\infty}\frac{\Gamma(1+k_1)\ldots\ldots\Gamma(m+k_p)\,(\frac{z}{2\lambda})^m}{\Gamma(1+l_1)\ldots\ldots\Gamma(m+l_q)\Gamma(\rho+\sigma m)}\frac{(\pi)^{\frac{1}{2}}2^{\mu+\lambda n+2n+\frac{3}{2}}\{(n+1)!\}^2}{(2n+2)\Gamma\left(\mu+\lambda m+\frac{5}{2}\right)\Gamma(\mu+\lambda m-\frac{n}{2}+\frac{1}{2})}$$

$$= \frac{(\pi)^{\frac{1}{2}}2^{2n+\frac{3}{2}}\{(n+1)!\}^2}{(2n+2)}\frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)}\sum_{m=0}^{\infty}\frac{\Gamma(1+k_1)\ldots\ldots\Gamma(m+k_p)\,\left(\frac{z}{\gamma}\right)^m}{\Gamma(1+l_1)\ldots\ldots\Gamma(m+l_q)\Gamma(\rho+\sigma m)}$$

$$\times\frac{\Gamma\left(\mu+\frac{1}{2}+\lambda m\right)\Gamma(\mu+\lambda m)z^m}{\Gamma\left(\mu+\frac{5}{2}+\lambda m\right)\Gamma(\mu` - n+\frac{1}{2}+\lambda m)}$$

Theorem 6. Let$\mathrm{Re}(\mu) > 0, \mathrm{Re}(\sigma) > 0, \mathrm{Re}(\alpha-\mu+n) > -1, \mathrm{Re}(\alpha-\mu) > -1$. The identity holds:

$$\int_0^{\infty}u^{\mu-1}e^{-u}L_n^{\alpha}(u)\,{}_p^{\sigma}M_z^{p}\left(k_1,\ldots,k_p,l_1\ldots l_q : z(2u)^{\xi}\right)du$$

$$= \frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)}\;{}_{p+3}\Psi_{q+2}\left[\begin{matrix}(k_1,1),\ldots\ldots, & (k_p,1),(\alpha-\mu+n+1,-\xi),(\mu,\xi),(1,1)\\ (l_1,1),\ldots\ldots, & (l_q,1),\;\;(\rho,\sigma),\;\;(\alpha-\mu+1,-\xi)\end{matrix};Z2^{\xi}\right]. \qquad (2.6)$$

Proof. Using (1.1) in the left hand side of (2.6) and simple simplification, we get following steps

$$\int_0^{\infty}u^{\mu-1}e^{-u}L_n^{\alpha}(u)\,{}_p^{\sigma}M_z^{p}\left(k_1,\ldots,k_p,l_1\ldots l_q : z(2u)^{\xi}\right)du$$

$$= \sum_{m=0}^{\infty}\frac{(k_1)_1\ldots\ldots(k_p)_m\,z^m2^{\xi m}}{(l_1)_1\ldots\ldots(l_q)_m\Gamma(\rho+\sigma m)}\int_0^{\infty}u^{\mu+\xi m}e^{-u}L_n^{\alpha}(u)du$$

$$= \frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)}\sum_{m=0}^{\infty}\frac{\Gamma(1+k_1)\ldots\ldots\Gamma(m+k_p)\,(z2^{\xi})^m}{\Gamma(1+l_1)\ldots\ldots\Gamma(m+l_q)\Gamma(\rho+\sigma m)}\frac{\Gamma(\alpha-(\mu+\xi m)+n+1)\Gamma(\mu+\xi m)\Gamma(1+m)}{\Gamma((\alpha-\mu-\xi m+1))n!}$$

Hence proved.

Theorem 7. If $\mathrm{Re}(\sigma) > 0,\; 0 < \mathrm{Re}(\lambda) < \mathrm{Re}(\mu)$, following compostion formula holds

$$\int_0^{\infty}u^{\lambda-1}(1+u)^{-\mu}\,{}_p^{\sigma}M_z^{p}\left(k_1,\ldots,k_p,l_1\ldots l_q : z(1+u)^{-\nu}\right)du$$

$$= \frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)}\;{}_{p+2}\Psi_{q+2}\left[\begin{matrix}(k_1,1),\ldots\ldots, & (k_p,1),(\mu-\lambda,\nu),(1,1)\\ (l_1,1),\ldots\ldots, & (l_q,1),\;\;(\rho,\sigma),\;\;(\mu,\nu)\end{matrix};Z\right]. \qquad (2.7)$$

Proof. For evaluating (2.7), applying definition of generalized M-series in the left hand side of (2.7), we get

$$=\sum_{m=0}^{\infty}\frac{(k_1)_1 \ldots\ldots (k_p)_m\, z^m}{(l_1)_1 \ldots\ldots (l_q)_m \Gamma(\rho+\sigma m)}\int_0^{\infty} u^{\lambda-1}(1+u)^{-\mu-2m}du$$

$$=\sum_{m=0}^{\infty}\frac{(k_1)_1 \ldots\ldots (k_p)_m\, z^m}{(l_1)_1 \ldots\ldots (l_q)_m \Gamma(\rho+\sigma m)}\frac{\Gamma(\lambda)\Gamma(\mu+\nu m-\lambda)}{\Gamma(\mu+\nu m)}$$

$$=\frac{\prod_{j=1}^{q}\Gamma(l_q)\Gamma(\lambda)}{\prod_{i=1}^{p}\Gamma(k_p)}\sum_{m=0}^{\infty}\frac{\Gamma(1+k_1)\ldots\ldots\Gamma(m+k_p)\, z^m}{\Gamma(1+l_1)\ldots\ldots\Gamma(m+l_q)\Gamma(\rho+\sigma m)}\frac{\Gamma(\mu+\nu m-\lambda)\Gamma(1+m)}{\Gamma(\mu+\nu m)m!}$$

Hence Proved.

Theorem 8. Let $Re(\mu)>0, Re(\nu)>0, Re(\sigma)>0, Re(\eta)>0, Re(\xi)>0$. Then

$$\int_0^{\frac{\pi}{2}} \sin^{\mu}\vartheta.\cos^{\nu}\vartheta\, {}_p^{\sigma}M_z^{p}\left(k_1,\ldots.,k_p,l_1\ldots.l_q: z\{\sin^{\eta}\vartheta.\cos^{\xi}\vartheta\}\right)dud\vartheta$$

$$=\frac{\prod_{j=1}^{q}\Gamma(l_q)}{2\prod_{i=1}^{p}\Gamma(k_p)}\ {}_{p+3}\Psi_{q+2}\left[\begin{matrix}(k_1,1),\ldots\ldots, & (k_p,1),\left(\frac{\mu+1}{2},\frac{\eta}{2}\right)\left(\frac{\nu+1}{2},\frac{\xi}{2}\right),(1,1) \\ (l_1,1),\ldots\ldots, & (l_q,1),\ (\rho,\sigma),\ \left(\frac{\mu+\nu}{2},\frac{\eta+\xi}{2}\right)\end{matrix};Z\right]. \qquad (2.8)$$

Proof. Proof of Theorem 8 is same as the roof of Theorem 7.

Theorem 9. If $Re(\xi)>0, Re(\nu)<1, Re(\mu)>0, \sigma,\xi\in C, Re(\sigma)>0$, following integral formula holds

$$\int_0^{\infty} u^{\mu-1}e^{-\frac{au}{2}}W_{\eta,\nu}\,{}_p^{\rho}M_q^{\sigma}\left(k_1,\ldots.,k_p,l_1\ldots.l_q: zu^{\xi}\right)du$$

$$=\frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)}a^{\mu}\ {}_{p+3}\Psi_{q+2}\left[\begin{matrix}(k_1,1),\ldots\ldots, & (k_p,1),\left(\mu+\nu+\frac{1}{2},\xi\right)\left(\mu-\nu+\frac{1}{2},\xi\right),(1,1) \\ (l_1,1),\ldots\ldots, & (l_q,1),\ (\rho,\sigma),\ (\mu-k+1,\xi)\end{matrix};\frac{z}{a^{\xi}}\right]$$

(2.9)

Proof. For evaluating (2.7), applying definition of generalized M-series in the left hand side of (2.7), we get

$$=\frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)}a^{\mu}\sum_{m=0}^{\infty}\frac{\Gamma(1+k_1)\ldots\ldots\Gamma(m+k_p)\, z^m}{\Gamma(1+l_1)\ldots\ldots\Gamma(m+l_q)\Gamma(\rho+\sigma m)}\int_0^{\infty} u^{\mu+\xi}e^{-\frac{au}{2}}W_{\eta,\nu}du$$

$$=\frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)}a^{\mu}\sum_{m=0}^{\infty}\frac{\Gamma(1+k_1)\ldots..\Gamma(m+k_p)\,(\frac{z}{a^{\xi}})^m}{\Gamma(1+l_1)\ldots..\Gamma(m+l_q)\Gamma(\rho+\sigma m)}\frac{\Gamma\left(\mu+\xi m+\nu+\frac{1}{2}\right)\Gamma(\mu+\xi m-\nu+\frac{1}{2})}{\Gamma(1-k+\mu+\xi m)}$$

Hence proved.

3. Special cases:

Corollary 1. If we take $p=q=0$ in Theorem 1 and using formula (1.6) , we get the following result

$$\int_0^{\infty} e^{-\mu t}[\sinh(\nu t)]^{\eta}\, E_{\rho,\sigma}\left(ze^{-\xi t}\right)dt =$$
$$\nu^{-1}2^{-\eta-1}\frac{\Gamma(1+\eta)}{1}\;{}_2\Psi_2\left[\begin{matrix}(\frac{\mu}{2\nu}-\frac{\eta}{2},\frac{\xi}{2\nu}) & ,(1,1)\\ (\rho,\sigma) & ,(\frac{\mu}{2\nu}+\frac{\eta}{2}+1+\frac{\xi}{2\nu})\end{matrix} Z\right].$$

Corollary 2. If we take If $p=0, q=1$ in Theorem 1 and using (1.7), we get following result.

$$\int_0^{\infty} e^{-\mu t}[\sinh(\nu t)]^{\eta}\, W_{\rho,\sigma}\left(ze^{-\xi t}\right)dt =$$
$$\nu^{-1}2^{-\eta-1}\frac{\Gamma(1+\eta)}{1}\;{}_1\Psi_2\left[\begin{matrix}(\frac{\mu}{2\nu}-\frac{\eta}{2},\frac{\xi}{2\nu}) & ,\\ (\rho,\sigma) & ,(\frac{\mu}{2\nu}+\frac{\eta}{2}+1+\frac{\xi}{2\nu})\end{matrix} Z\right].$$

Corollary 3. If we take If $p=q=1, k=a$ and $l=1$, an equation (2.3) and using (1.8), we get

$$\int_0^{1} u^{\mu}(1-u^2)^{-\frac{\nu}{2}}P_{\eta}^{\nu}(u)\, E_{\rho,\sigma}^{a}\left(zu^{\xi}\right)du =$$
$$2^{\nu-1}\frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)}\;{}_3\Psi_3\left[\begin{matrix}(a,1) & ,\left(\frac{\mu}{2}+\frac{1}{2},\frac{\xi}{2}\right),\left(\frac{\mu+1}{2},\frac{\xi}{2}\right)\\ (\rho,\sigma) & ,(\frac{\mu}{2}-\frac{\eta}{2}-\frac{\nu}{2}+1,\frac{\xi}{2})(\frac{\mu}{2}+\frac{\eta}{2}-\frac{\nu}{2}+\frac{3}{2},\frac{\xi}{2})\end{matrix};Z\right].$$

Corollary 4. If we take $\rho=\sigma=1$ in Theorem 3 and using (1.5), we get

$$\int_0^{1} u^{\mu}(1-u^2)^{-\frac{\nu}{2}}P_{\eta}^{\nu}(u)\;{}_pF_q\left[\begin{matrix}k_1,\ldots.. & k_p;\\ l_1,\ldots\ldots & l_q;\end{matrix} z\right]du =$$
$$2^{\nu-1}\frac{\prod_{j=1}^{q}\Gamma(l_q)}{\prod_{i=1}^{p}\Gamma(k_p)}\;{}_{p+2}\Psi_{q+2}\left[\begin{matrix}(k_1,1),\ldots\ldots, & (k_p,1),\left(\frac{\mu}{2}+\frac{1}{2},\frac{\xi}{2}\right),\left(1+\frac{\mu}{2},\frac{\xi}{2}\right)\\ (l_1,1),\ldots\ldots, & (l_q,1),(\frac{\mu}{2}-\frac{\eta}{2}-\frac{\nu}{2}+1,\frac{\xi}{2})(\frac{\mu}{2}+\frac{\eta}{2}-\frac{\nu}{2}+\frac{3}{2},\frac{\xi}{2})\end{matrix};Z\right],$$

Corollary 5. If we take $p=q=0$ in Theorem 4 and using (1.6) , we get following result

$$\int_0^{\infty} u^{-\mu}(u^2-1)^{-\frac{\nu}{2}}P_{\eta}^{\nu}(u)\, E_{\rho,\sigma}\left(zu^{-\xi}\right)du=\frac{1}{(\pi)^{\frac{1}{2}}}\;{}_3\Psi_2\left[\begin{matrix}\left(\frac{\mu+\nu+2}{2},\frac{\xi}{2}\right) & ,\left(\frac{\mu+\nu-\eta-1}{2},\frac{\xi}{2}\right),(1,1)\\ & ,\ (\rho,\sigma),\ \ (\mu,\xi)\end{matrix};Z\right].$$

Corollary 6. If we take $p=0, q=1$ in Theorem 5 and using (1.8), we get

$$\int_0^1 (1-u)^{\frac{1}{2}}(1+u)^{\mu} U_n(u)\, W_{\rho,\sigma}\left(z\left(\frac{1+\mu}{2}\right)^{\lambda}\right) du =$$

$$\frac{(\pi)^{\frac{1}{2}} 2^{2n+\frac{3}{2}}\{(n+1)!\}^2}{(2n+2)}\ {}_2\Psi_3\left[\begin{matrix}\left(\frac{2\mu+1}{2},\lambda\right) & ,(\mu,\lambda) \\ (\rho,\sigma), & \left(\mu+\frac{5}{2},\lambda\right),\left(\mu` - n+\frac{1}{2},\lambda\right)\end{matrix};Z\right].$$

Corollary 7. If we take $p = q = 1, k=1$ in Theorem 8 and (1.7), we get

$$\int_0^{\frac{\pi}{2}} \sin^{\mu}\vartheta.\cos^{\nu}\vartheta E^{a}_{\rho,\sigma}(z\sin^{\mu}\vartheta.\cos^{\nu}\vartheta)d\vartheta = \frac{\Gamma(L)}{2\Gamma(K)}\ {}_3\Psi_2\left[\begin{matrix}(a,1), & \left(\frac{\mu+1}{2},\frac{\eta}{2}\right)\left(\frac{\nu+1}{2},\frac{\xi}{2}\right) \\ (\rho,\sigma) & ,\ \left(\frac{\mu+\nu}{2},\frac{\eta+\xi}{2}\right)\end{matrix};Z\right].$$

Corollary 8. If we take $p = q = 0$. In Theorem 9 and using (1.6) , we get

$$\int_0^{\infty} u^{\mu-1} e^{-\frac{au}{2}} W_{\eta,\nu}(au) E_{\rho,\sigma}(zu^{\xi})\, du = a^{\mu}\ {}_3\Psi_2\left[\begin{matrix}\left(\mu+\nu+\frac{1}{2},\xi\right) & ,\left(\mu-\nu+\frac{1}{2},\xi\right),(1,1) \\ (\rho,\sigma), & (\mu-k+1,\xi)\end{matrix};Z/a^{\xi}\right]$$

4. Conclusion remark

In this investigation, we have managed to obtain new forms for the Eulerian type integrals that are associated with the generalized M-series, which in turn has enriched the theory of the special functions. The integral expressions derived not only extend our knowledge of the structural properties of the generalised M-series, but also reveal the flexibility by analysing special cases involving special parameter values. These results offer opportunities for possible applications in a broad range of disciplines such as mathematical physics, engineering, and applied mathematics. Future research may explore further generalizations and applications of these integrals in solving complex problems across different scientific disciplines. We concluded from the present research work by giving some comments on the results of Theorem 1-Theorem 9and their corollaries. The integrals can be further generalized and applied in solving some long-standing problems in that has been presented in this letter. We ended present research work with comments of Theorem 1-Theorem 9and their corollaries. The proposed generalized M-series is an interesting function which is equivalent to one of the several families of transcendental and special functions namely exponential function, binomial series, cosine function, sine function, Mittag-leffer function, Wright function, gauss hypergeometric function, Fox-H function and Meijer G-function which appears to be new even in the case of special cases. Thus, we can deduce more useful results and their equivalent forms from Theorem 1 to Theorem 9 in terms of Fox H-function and Meijer G-function.

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