

## Bounds for Product Eccentricity Energy of Graphs

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### Abstract:

In this article some lower bounds for the product eccentricity energy  $[E_{PE}]$  of  $G$  as well as a bound for the eigenvalues of the product eccentricity matrix  $[PE(G)]$  are obtained.

**Mathematical Subject Classification :** 05C07, 05C50.

**Conclusion:** This article gives an idea on few lower bounds for the product eccentricity energy based on its eigen values and various inequalities are also proved.

**Keywords:** Energy, bounds, Product eccentricity energy.

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## 1. Introduction

Graph theory is a branch of discrete arithmetic that involves considering structures together with their attributes, goals, and relationships. Originally useful for resolving a wide range of mathematical problems, it periodically branched out into new areas of mathematical analysis when applied in complicated science, computer science, chemistry, and other disciplines.

Simple, loop-less, and connected graphs are the types of graphs examined in this article. An essential concept in this theory is a vertex's eccentricity, which assesses the greatest distance between two vertices. The distance between two vertices  $a$  and  $b$  in  $V(G)$  is the shortest a-b path length in  $G$ . The maximum distance between a particular vertex and any other vertex in the graph is determined by the vertex's eccentricity. Formally, it can be expressed as:

$$\xi(b) = \max\{d(b, a) | \forall a \in V(G)\}$$

A graph  $G$ 's eccentricity matrix  $\xi(G)$  is derived from its distance matrix by keeping the largest distances in each row and column and leaving zeros in the others. Summing the absolute values of the eigenvalues of  $\xi(G)$  yields the eccentricity energy of  $G$ .

Let  $G$  be a graph with  $n$  vertices and  $m$  edges. Denote the absolute eigen values of  $G$  as  $\lambda_i, i = 1, 2, \dots, n$  arranged in order that is not increasing as  $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$ . In 1978 Ivan Gutman [5] computed the energy of a graph  $G$  as  $E(G) = \sum_{i=1}^n |\lambda_i|$ . Li, X, Y. Shi and I. Gutman [6] introduced the energy of graph in 2012 in which the adjacency matrix of a graph  $G$  is defined as

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 0 & \text{otherwise} \end{cases}$$

Spectrum of the graph is denoted by

$$Sp(G) = \begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ m_1 & m_2 & \dots & m_n \end{bmatrix}$$

Where  $m_i$ 's denote the multiplicities of the corresponding eigen value. The total of the absolute values of the adjacency matrix's eigenvalues equals the graph's energy.

Later, in 2009, C. Adiga et al. [1] defined the graph's maximum degree energy, which is dependent on the related graph's maximum degree matrix. The maximum degree matrix is defined as

$$d_{ij} = \begin{cases} \max \{d(v_i), d(v_j)\}, & \text{if } v_i v_j \in E \\ 0, & \text{otherwise} \end{cases}$$

In 2016, Ahmed M. Naji et.al [2] defined the concept of maximum eccentricity matrix. Later, Mohammad Issa Sowaity and B.Sharada [7] in 2017 introduced the concept of sum-eccentricity energy of a graph in 2017. Motivated by this we have introduced the product eccentricity energy of a graph  $G$ .

In 2025 Priya Karen S and Arokia Lancy A [10] defined the idea of product eccentricity energy as

$$P_{ij} = \begin{cases} e(v_i) \cdot e(v_j) & \text{if } v_i \sim v_j \\ 0 & \text{otherwise} \end{cases}$$

$P_e(G)$  denotes the product eccentricity energy of the graph.

The characteristic polynomial of the product eccentricity matrix is defined by  $|\eta I - P_e(G)|$  and the corresponding characteristic equation is  $\eta I - P_e(G) = 0$ . Here,  $I$  denotes the identity matrix of order  $n$ .  $P_e(G)$  is a real symmetric matrix with its trace zero. Since  $G$  is a simple loopless graph all  $a_{ii} = 0$  and its eigen values with real sum equals zero ( $tr(P_e(G)) = 0$ ). Eigen values of the product eccentricity matrix are the roots of the corresponding characteristic polynomial.

$E_{PE}(G)$  is defined as the sum of the absolute eigen values,

$$E_{PE}(G) = \sum_{i=1}^n |\eta_i|$$

$\eta_1, \eta_2, \dots, \eta_n$  are the eigen values of the given product eccentricity matrix.

## 2. Preliminaries

A few important theorems that are utilized throughout the work are listed below in order to show the complete results.

**Theorem 2.1:[9]** Suppose  $a_i$  and  $b_i$ ,  $1 \leq i \leq n$  are non-negative real numbers, then

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \leq \frac{1}{4} \left( \sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2 \left( \sum_{i=1}^n a_i b_i \right)^2$$

Where  $M_1 = \max_{1 \leq i \leq n} (a_i)$ ;  $M_2 = \max_{1 \leq i \leq n} (b_i)$ ;  $m_1 = \min_{1 \leq i \leq n} (a_i)$ ;  $m_2 = \min_{1 \leq i \leq n} (b_i)$

**Theorem 2.2:[8]** Let  $a_i$  and  $b_i$ ,  $1 \leq i \leq n$  are non-negative real numbers, then

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left( \sum_{i=1}^n a_i b_i \right)^2 \leq \frac{n^2}{4} (M_1 M_2 - m_1 m_2)^2$$

Where  $M_1 M_2$  and  $m_1 m_2$  are defined similarly to theorem 2.1

**Theorem 2.3:[3]** Suppose  $a_i$  and  $b_i$ ,  $1 \leq i \leq n$  are positive real numbers, then

$$\left| n \sum_{i=1}^n a_i b_i - \sum_{i=1}^n a_i \sum_{i=1}^n b_i \right| \leq \mu(n)(A - a)(B - b)$$

Here  $a, b, A$  and  $B$  are the real constants,  $1 \leq i \leq n, a \leq a_i \leq A$  and  $b \leq b_i \leq B$ .

Further we have,  $\mu(n) = n \left\lfloor \frac{n}{2} \right\rfloor \left( 1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right)$

**Theorem 2.4:[4]** Let  $a_i$  and  $b_i$ ,  $1 \leq i \leq n$  are non-negative real numbers, then

$$\sum_{i=1}^n b_i^2 + rR \sum_{i=1}^n a_i^2 \leq (r + R) \left( \sum_{i=1}^n a_i b_i \right)$$

Here  $r$  and  $R$  are real constants,  $1 \leq i \leq n$  holds  $ra_i \leq b_i \leq Ra_i$

### 3.Bounds for the eigen values of Product Eccentricity Matrix of a graph

The following lemma is required to support the subsequent findings.

**Lemma 3.1.** If the trace of  $PE(G) = 0$ , then the eigen values obtained from  $PE(G)$  matrix satisfies the following

3.  $\sum_{i=1}^n \eta_i = 0$
4.  $\sum_{i=1}^n \eta_i^2 = \text{trace} (PE(G))^2$

$$\begin{aligned} \sum_{i=1}^n \eta_i^2 &= \sum_{i=1}^n (\eta_i)^2 = \left( \text{trace}(P_e(G)) \right)^2 \\ &= \sum_{i=1}^n \sum_{k=1}^n p_{ik} p_{ki} \\ &= 2 \sum_{i=1}^n \sum_{i < k}^n (p_{ik})^2 \\ &= 2 \sum_{i=1, i < k}^n \left( e(v_i) \cdot e(v_j) \right)^2 \\ &= \sum_{i=1}^n (\eta_i)^2 = 2 H \end{aligned}$$

Where  $H = \sum_{i=1, i < k}^n (e(v_i) \cdot e(v_j))^2$

**Theorem 3.2** Let  $G$  be a graph with  $n -$  vertices, then

$$\eta_i \leq \sqrt{\frac{2H(n-1)}{n}}$$

**Proof.** Consider a graph  $G$  with  $n -$ vertices. Let  $PE(G)$  be the product eccentricity matrix of graph  $G$  and  $\eta_1, \eta_2, \dots, \eta_n$  are the eigen values obtained from the  $PE(G)$  matrix, where  $\eta_1$  is the largest eigen value among all the eigen values computed and using the Cauchy-Schwarz inequality is used to obtain the bound for  $\eta_1$ .

$$\left( \sum_{i=1}^n a_i b_i \right)^2 \leq \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right)$$

Let  $a_i = 1$  and  $b_i = \eta_i \forall i = 1, 2, 3, \dots, n$  then the inequality becomes,

$$\left( \sum_{i=1}^n 1 \cdot \eta_i \right)^2 \leq \left( \sum_{i=1}^n 1^2 \right) \left( \sum_{i=1}^n \eta_i^2 \right)$$

Using the idea of lemma 3.1 (i)

$$\begin{aligned} \sum_{i=1}^n \eta_i &= 0 \\ \eta_1 + \sum_{i=2}^n \eta_i &= 0 \\ \sum_{i=2}^n \eta_i &= -\eta_1 \end{aligned}$$

On squaring we obtain

$$\left( \sum_{i=2}^n \eta_i \right)^2 = (-\eta_1)^2 = \eta_1^2$$

By (ii) of Lemma 3.1

$$\begin{aligned} \left( \sum_{i=2}^n \eta_i \right)^2 &= 2H \\ (\eta_1)^2 + \left( \sum_{i=2}^n \eta_i \right)^2 &= 2H \end{aligned}$$

$$\left(\sum_{i=2}^n \eta_i\right)^2 = 2H - (\eta_1)^2$$

Equation one becomes

$$\begin{aligned} (-\eta_1)^2 &\leq (n-1)(2H - \eta_1^2) \\ \eta_1^2 &\leq 2H(n-1) - \eta_1^2(n-1) \\ \eta_1^2 + \eta_1^2(n-1) &\leq 2H(n-1) \\ \eta_1^2(n) &\leq 2H(n-1) \\ \eta_1 &\leq \sqrt{\frac{2H(n-1)}{n}} \end{aligned}$$

**Theorem 3.3** *If  $G$  be a graph with  $n$  –vertices then*

$$\sqrt{2H} \leq E_{PE}(G) \leq \sqrt{2nH}$$

**Proof.** Consider a graph  $G$  with  $n$  –vertices. Let  $PE(G)$  be the product eccentricity matrix of a graph  $G$  and  $\eta_1, \eta_2, \dots, \eta_n$  are the eigen values obtained from the  $PE(G)$  matrix. By the idea of Cauchy-Schwartz inequality the theorem is proved

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right) \left(\sum_{i=1}^n b_i^2\right)$$

Let us assume that  $a_i = 1$  and  $b_i = \eta_i \forall i = 1, 2, 3, \dots, n$ .

$$\left(\sum_{i=1}^n 1 \cdot \eta_i\right)^2 \leq \left(\sum_{i=1}^n 1^2\right) \left(\sum_{i=1}^n \eta_i^2\right)$$

$$\left(\sum_{i=1}^n \eta_i\right)^2 \leq n \left(\sum_{i=1}^n \eta_i^2\right)$$

$$(E_{PE}(G))^2 \leq 2nH$$

$$E_{PE}(G) \leq \sqrt{2nH}$$

This is an upper bound,

We have,

$$(E_{PE}(G))^2 = (\sum_{i=1}^n |\eta_i|)^2 \geq \sum_{i=1}^n |\eta_i|^2 = 2H.$$

Thus, we obtain  $E_{PE}(G) \geq \sqrt{2H}$  which is the lower bound.

Hence the inequality holds.

Thus we have,

$$\sqrt{2H} \leq E_{PE}(G) \leq \sqrt{2nH}$$

**Theorem 4.4:** Let  $G$  be a graph with  $n$  –vertices and  $m$  – edges. If  $|\eta_1| \geq |\eta_2| \geq \dots \geq |\eta_n|$  are the eigen values of  $PE(G)$ , then the following inequality holds.

$$E_{PE}(G) \geq \frac{2\sqrt{2nH|\eta_1||\eta_n|}}{|\eta_1| + |\eta_n|}$$

**Proof.** Consider a graph  $G$  with  $n$  –vertices and  $|\eta_1| \geq |\eta_2| \geq \dots \geq |\eta_n|$  are the eigen values of  $PE(G)$ , where  $|\eta_1|$  and  $|\eta_n|$  are the maximum and minimum eigen values of  $|\eta_i|$  respectively. From theorem 2.1

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \leq \frac{1}{4} \left( \sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2 \left( \sum_{i=1}^n a_i b_i \right)^2$$

Assume  $a_i = 1$  and  $b_i = |\eta_i|$ ,  $M_1 M_2 = |\eta_1|$  and  $m_1 m_2 = |\eta_n|$  then,

$$\sum_{i=1}^n 1^2 \sum_{i=1}^n |\eta_i|^2 \leq \frac{1}{4} \left( \sqrt{\frac{|\eta_1|}{|\eta_n|}} + \sqrt{\frac{|\eta_n|}{|\eta_1|}} \right)^2 \left( \sum_{i=1}^n 1 |\eta_i| \right)^2$$

From lemma 3.1 and using the idea of Arithmetic-Geometric inequality we obtain,

$$\begin{aligned} 2nH &\leq \frac{1}{4} \left[ \frac{(|\eta_1| + |\eta_n|)^2}{|\eta_1||\eta_n|} \right] (E_{PE}(G))^2 \\ (E_{PE}(G))^2 &\geq \frac{8nH |\eta_1||\eta_n|}{(|\eta_1| + |\eta_n|)^2} \\ (E_{PE}(G))^2 &\geq \frac{2\sqrt{2nH|\eta_1||\eta_n|}}{|\eta_1| + |\eta_n|} \end{aligned}$$

**Theorem 3.5** Let  $G$  be a graph with  $n$  –vertices, then the following inequalities holds

$$E_{PE}(G) \geq \frac{2H_n |\eta_1||\eta_n|}{|\eta_1| + |\eta_n|}$$

**Proof.** Consider a graph  $G$  with order  $n$  and size  $m$ . Let  $|\eta_1| \geq |\eta_2| \geq \dots \geq |\eta_n|$  be the eigen values of the product eccentricity matrix, arranged in non- increasing order, where  $|\eta_1|$  and  $|\eta_n|$  are the maximum and minimum eigen values respectively.

Using the inequality from theorem 2.4

$$\sum_{i=1}^n b_i^2 + rR \sum_{i=1}^n a_i^2 \leq (r + R) \left( \sum_{i=1}^n a_i b_i \right)$$

Assume  $b_i = |\eta_i|$ ,  $a_i = 1$ ,  $r = |\eta_n|$  and  $R = |\eta_1|$ , then the inequality implies to

$$\sum_{i=1}^n |\eta_i|^2 + |\eta_n||\eta_1| \sum_{i=1}^n 1^2 \leq (|\eta_n| + |\eta_1|) \left( \sum_{i=1}^n 1|\eta_i| \right)$$

Using lemma 3.1

$$2H + (|\eta_n||\eta_1|)n \leq (|\eta_n| + |\eta_1|)E_{PE}(G)$$

$$E_{PE}(G) \geq \frac{2H + |\eta_n||\eta_1|}{|\eta_n| + |\eta_1|}$$

**Theorem 3.6:** Let  $G$  be a graph with  $n$  – vertices, then the following inequality holds

$$E_{PE}(G) \geq \sqrt{2nH - \frac{n^2}{4} (|\eta_1| - |\eta_n|)^2}$$

**Proof.** Consider a graph  $G$  with order  $n$  and size  $m$ . Let  $|\eta_1| \geq |\eta_2| \geq \dots \geq |\eta_n|$  be the eigen values of  $PE(G)$  matrix, where  $|\eta_1|$  and  $|\eta_n|$  are the maximum and minimum eigen values respectively.

From theorem 2.2 we have the inequality

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left( \sum_{i=1}^n a_i b_i \right)^2 \leq \frac{n^2}{4} (M_1 M_2 - m_1 m_2)^2$$

Assume  $a_i = 1, b_i = |\eta_i|, M_1 M_2 = |\eta_1|$  and  $m_1 m_2 = |\eta_n|$  then

$$\sum_{i=1}^n 1^2 \sum_{i=1}^n |\eta_i|^2 - \left( \sum_{i=1}^n 1 |\eta_i| \right)^2 \leq \frac{n^2}{4} (|\eta_1| + |\eta_n|)^2$$

From lemma 3.1

$$2nH - (E_{PE}(G))^2 \leq \frac{n^2}{4} (|\eta_1| - |\eta_n|)^2$$

$$(E_{PE}(G))^2 \geq \sqrt{2nH - \frac{n^2}{4} (|\eta_1| - |\eta_n|)^2}$$

Hence the inequality holds true.

**Theorem 3.7:** Let  $G$  be a graph with  $n$  – vertices, then the following inequality holds

$$E_{PE}(G) \geq \sqrt{2nH - \mu(n)(|\eta_1| - |\eta_n|)^2}$$

**Proof.** Consider a graph  $G$  with order  $n$  and size  $m$ . Let  $|\eta_1| \geq |\eta_2| \geq \dots \geq |\eta_n|$  be the eigen values of  $PE(G)$  matrix, where  $|\eta_1|$  and  $|\eta_n|$  are the maximum and minimum eigen values respectively.

Consider the inequality from the theorem 2.3

$$\left| n \sum_{i=1}^n a_i b_i - \sum_{i=1}^n a_i \sum_{i=1}^n b_i \right| \leq \mu(n)(A - a)(B - b)$$

Now assume that  $a_i = b_i = |\eta_i|$ ,  $A = B = |\eta_1|$  and  $a = b = |\eta_n|$ , then the inequality reduces to

$$\left| n \sum_{i=1}^n |\eta_i|^2 - \left( \sum_{i=1}^n |\eta_i| \right)^2 \right| \leq \mu(n)(|\eta_1| - |\eta_n|)(|\eta_1| + |\eta_n|)$$

From lemma 3.1

$$\left| 2nH - (E_{PE}(G))^2 \right| \leq \mu(n)(|\eta_1| - |\eta_n|)^2$$

$$E_{PE}(G) \geq \sqrt{2nH - \mu(n)(|\eta_1| - |\eta_n|)^2}$$

Hence the inequality holds true.

## References

- [1] Adiga.C and Smitha,M, On maximum degree energy of a graph, *Int. J. Contempt. Math. Sci.*, 4(2009), 385-396.
- [2] Ahmed Naji.M and Soner N.D , *The maximum eccentricity energy of a graph*, *Int. J. Sci. Engin. Research*, 7(2016), 5-13
- [3] M. Biernacki, H. Pidek and C. Ryll-Nardzewsk, *Sur une iné galité entre des intégrales définies*, Maria Curie SkÅĆodowska University, A4, (2009), 1-4.
- [4] J. B. Diaz and F. T. Metcalf, *Stronger forms of a class of inequalities of G. Pólya-G. Szegò and L. V. Kantorovich*, *Bulletin of the AMS-American Mathematical Society*, 69(1963), 415-418.
- [5] I.Gutman, *The energy of graph*, *Ber. Math.-Stat. Sect. Forshungsz. Graz 103* (1978) 1–22.
- [6] Li.X, Y. Shi and I. Gutman, *Graph Energy*, Springer, 2012.
- [7] Mohammad Issa Sowaity and B.Sharada, *The Sum-Eccentricity Energy of a Graph*, *International Jorunal on Recent and Innovation Trends in Computing and Communication*, 5(2017), 293-304.
- [8] N. Ozeki, *On the estimation of inequalities by maximum and minimum values*, *Journal of College Arts and Science, Chiba University*, 5(1968), 199-203.
- [9] G. Pólya and G. Szegò, *Problems and theorems in analysis, Series, Integral calculus, Theory of functions*, Springer, Berlin, (1972).
- [10] Priya Karen S and Arokia Lancy A, *On Product eccentricity energy of graphs*. (Communicated)