

Almost $G\Omega_\alpha$ -Closed Functions and Separation Axioms

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Abstract: In this article explores the concept of almost $G\Omega_\alpha$ -closed sets and almost $G\Omega_\alpha$ -closed functions and their relationship with separation axioms in topology. Almost $G\Omega_\alpha$ -closed functions serve as a generalization of closed functions and play a crucial role in the study of topological properties. We investigate various characterizations and fundamental properties of almost $G\Omega_\alpha$ -closed functions, analyzing their interaction with different separation axioms.

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1. INTRODUCTION

In topological spaces, it is well known that normality is preserved under closed continuous surjections. Many authors have tried to weaken the condition “closed” in this theorem. In 1978, Long and Herrington [5] used almost closedness due to Singal [17]. In 1982, Malghan [7] used g-closedness. In 1986, Greenwood and Reilly [4] used \square -closedness due to Mashhour et al. [8]. In 1995, Yoshimura et al. [19] used almost g-closedness which is a generalization of both almost closedness and g-closedness. In 1999, Noiri [9] introduced almost ag-closedness using ag-closed sets [35]. Ravi et. al. [14] introduced almost ags-closedness using ags-closed sets [13]. We use $G\Omega_\alpha$ -closed sets to define a new class of functions called almost $G\Omega_\alpha$ -closed functions. The purpose of the present article is to improve preservation theorems of separation axioms, that is, normality, weak normality, mild normality, almost normality, regularity, almost regularity, quasi-regularity and strong s-regularity.

Theorem A

Normality and weak normality are preserved under almost $G\Omega_\alpha$ -closed continuous surjections.

Theorem B

Regularity and strong s-regularity are preserved under almost a-open almost $G\Omega_\alpha$ -closed continuous surjections.

2. PRELIMINARIES

The family of regular open (resp. regular closed) sets of a space (X, τ) is denoted by $RO(X, \tau)$ (resp. $RC(X, \tau)$) or simply by $RO(X)$ (resp. $RC(X)$).

Definition 2.1

A subset A of a space (X, τ) is called a rag -closed [9] if $\alpha\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in (X, τ) . The complement of rag -closed set is called rag -open set.

Definition 2.2

A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be

- (i) α -closed [40] (resp. \hat{g} -closed [14], αgs -closed [66], g''' -closed) if for each closed set F of X , $f(F)$ is α -closed (resp. \hat{g} -closed, αgs -closed, g''' -closed);
- (ii) almost α -closed [9] (resp. almost \hat{g} -closed [14], almost αgs -closed [14], almost g''' -closed, almost g''' -closed) if for each $F \in \text{RC}(X, \tau)$, $f(F)$ is α -closed (resp. \hat{g} -closed, αgs -closed, g''' -closed, g''' -closed).

Definition 2.3

A space X is said to be

- (i) weakly normal [20] if for each decreasing sequence $\{F_n\}$ of closed sets of X such that $\bigcap \{F_n : n \in \mathbb{N}\} = \emptyset$ and each closed set H of X with $H \cap F_1 = \emptyset$, there exist $n \in \mathbb{N}$ and an open set U of X such that $F_n \subset U$ and $\text{cl}(U) \cap H = \emptyset$;
- (ii) mildly normal [18] if for any disjoint regular closed sets A and B , there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$;
- (iii) almost normal [15] if for every pair of disjoint sets A and B , one of which is closed and the other is regular closed, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

Lemma 2.4 [9]

If A is an α -open set of a space X , then the following hold:

$$\alpha\text{cl}(A) = \text{cl}(A) = \text{cl}(\text{int}(A)).$$

Lemma 2.5 [10]

A space X is weakly normal if and only if for each decreasing sequence $\{F_n\}$ of closed sets of X such that $\bigcap \{F_n : n \in \mathbb{N}\} = \emptyset$ and each open set U of X such that $F_1 \subset U$, there exist $n \in \mathbb{N}$ and an open set G of X such that $F_n \subset G \subset \text{cl}(G) \subset U$.

Definition 2.6

A function $f : X \rightarrow Y$ is said to be

- (i) R -map [1] (resp. almost continuous [79]) if $f^{-1}(V)$ is regular open (resp. open) in X for every $V \in \text{RO}(Y)$;
- (ii) almost open [17] (resp. almost α -open [49]) if $f(U)$ is open (resp. α -open) in Y for every regular open set U of X ;
- (iii) α -open [8] if $f(U)$ is α -open in Y for every open set U of X ;
- (iv) almost αg -closed [9] if $f(U)$ is αg -closed in Y for every regular closed set U of X .

Lemma 2.7 [9]

If a function $f : X \rightarrow Y$ is almost continuous almost α -open and V is regular open in Y , then $f^{-1}(V)$ is regular open in X .

Lemma 2.8

- (i) A subset A of a space X is αg -open if and only if $F \subset \alpha \text{int}(A)$ whenever $F \in RC(X)$ and $F \subset A$ [9].
- (ii) Every $\alpha g s$ -closed set is αg -closed but not conversely [14].
- (iii) Every αg -closed set is αg -closed but not conversely [9].

Definition 2.9

A space X is said to be

- (i) almost regular [16] if for each $F \in RC(X)$ and each $x \in X - F$, there exist disjoint open sets U and V of X such that $x \in U$ and $F \subset V$;
- (ii) quasi-regular [12] if for every nonempty open set V of X , there exists a nonempty open set U in X such that $\text{cl}(U) \subset V$;
- (iii) strongly s -regular [3] if for any closed set A of X and any point $x \in X - A$ there exists an $F \in RC(X)$ such that $x \in F$ and $F \cap A = \phi$.

Definition 2.10

A function $f : X \rightarrow Y$ is said to be

- (i) feebly continuous [2] if $\text{int}(f^{-1}(V)) \neq \phi$ for every nonempty open set V of Y ;
- (ii) feebly open [2] if $\text{int}(f(U)) \neq \phi$ for every nonempty open set U of X ;
- (iii) almost feebly open [9] if $\text{int}(f(U)) \neq \phi$ for every nonempty $U \in RO(X)$.

Theorem 2.11[9]

The following are equivalent for a space (X, τ) :

- (i) (X, τ) is regular (resp. almost regular);
- (ii) for each closed (resp. regular closed) set F and each $x \in X - F$, there exist disjoint $U, V \in \tau^\alpha$ such that $x \in U$ and $F \subset V$;
- (iii) for each open (resp. regular open) set V and $x \in V$, there exists $U \in \tau^\alpha$ such that $x \in U \subset \alpha \text{cl}(U) \subset V$.

3. ALMOST $\mathcal{G}\Omega_\alpha$ -CLOSED FUNCTIONS

Definition 3.1

A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be almost $\mathcal{G}\Omega_\alpha$ -closed if for each $F \in RC(X, \tau)$, $f(F)$ is $\mathcal{G}\Omega_\alpha$ -closed.

Theorem: 3.2

Every almost closed function is almost $G\Omega$ closed function.

Proof

Let F be regular closed space and a function $f : (X, \tau) \rightarrow (Y, \sigma)$ be almost closed function . Then if for each regular closed set F , $f(F)$ is closed and every closed set is $G\Omega$ closed set. Hence f is almost $G\Omega$ closed function.

The following example shows that the converse of the above theorem is not true.

Example: 3.3

Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a, b\}, Y\}$.

Then $RC(X, \tau) = \{\phi, \{a, c\}, \{b, c\}, X\}$ and $G\Omega$ closed in $Y = \{\phi, \{c\}, \{b,c\}, \{a, c\}, Y\}$ Then the identity function $f : (X, \tau) \rightarrow (Y, \sigma)$ is almost $G\Omega$ -closed. However, it is not almost closed since there exists $\{b, c\} \in RC(X, \tau)$ such that $f(\{b, c\}) = \{b, c\}$ is not closed in (Y, σ) .

Theorem: 3.4

Every almost α -closed function is almost $G\Omega\alpha$ closed function.

Proof

Let F be regular closed space and a function $f : (X, \tau) \rightarrow (Y, \sigma)$ be almost α -closed function . Then if for each regular closed set F , $f(F)$ is α -closed and every α -closed set is $G\Omega$ closed set. Hence f is almost $G\Omega\alpha$ closed function.

The following example shows that the converse of the above theorem is not true.

Example: 3.5

Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a,c\}X\}$ and $\sigma = \{\phi, \{a, b\}, Y\}$. Then $RC(X, \tau) = \{\phi, \{a, c\}, \{b, c\}, X\}$, α - closed in $Y = \{\phi, \{c\}, Y\}$ and $G\Omega\alpha$ closed in $Y = \{\phi, \{c\}, \{a, b\}, Y\}$ Then the function $f : (X, \tau) \rightarrow (Y, \sigma)$ defined as $f(a)=c$, $f(b)=b$, $f(c)=c$ is almost $G\Omega\alpha$ -closed. However, it is not almost α - closed since there exists $\{b, c\} \in RC(X, \tau)$ such that $f(\{b, c\}) = \{a, b\}$ is α - closed in (Y, σ) .

Theorem: 3.6

Every $G\Omega$ closed function is almost $G\Omega$ closed function but not conversely

Proof

Let F be regular closed space. Then every regular closed set is closed set and Let a function $f : (X, \tau) \rightarrow (Y, \sigma)$ be $G\Omega$ closed function . Then if for each closed set F , $f(F)$ is $G\Omega$ -closed. Hence f is almost $G\Omega$ closed function.

The following example shows that the converse of the above theorem is not true.

Example 3.7

Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a,c\}X\}$ and $\sigma = \{\phi, \{a, b\}, Y\}$. Then $RC(X, \tau) = \{\phi, \{a, c\}, \{b, c\}, X\}$ and $G\Omega$ closed in $Y = \{\phi, \{c\}, \{b,c\}, \{a, c\}, Y\}$ Then the function $f : (X, \tau) \rightarrow (Y, \sigma)$ defined as $f(a)=c$, $f(b)=b$, $f(c)=c$ is almost $G\Omega$ -closed. However, it is not $G\Omega$ - closed since there exists $\{b\} \in \tau^c$ such that $f(\{b\}) = \{b\}$ is not $G\Omega$ - closed in (Y, σ) .

Theorem: 3.8

Every $G\Omega\alpha$ closed function is almost $G\Omega\alpha$ closed function.

Proof

Let F be regular closed space. Then every regular closed set is closed set and Let a function $f : (X, \tau) \rightarrow (Y, \sigma)$ be $G\Omega\alpha$ closed function . Then if for each closed set F , $f(F)$ is $G\Omega\alpha$ -closed. Hence f is almost $G\Omega\alpha$ closed function.

The following example shows that the converse of the above theorem is not true.

Example 3.9

Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a,c\}X\}$ and $\sigma = \{\phi, \{a, b\}, Y\}$. Then $RC(X, \tau) = \{\phi, \{a, c\}, \{b, c\}, X\}$ and $G\Omega\alpha$ closed in $Y = \{\phi, \{c\}, \{a, b\}, Y\}$ Then the function $f : (X, \tau) \rightarrow (Y, \sigma)$ defined as $f(a)=c, f(b)=b, f(c)=c$ is almost $G\Omega\alpha$ -closed. However, it is not $G\Omega$ - closed since there exists $\{b\} \in \tau^c$ such that $f(\{b\}) = \{b\}$ is not $G\Omega\alpha$ - closed in (Y, σ) .

Theorem: 3.10

Every almost $G\Omega$ closed function is almost $G\Omega\alpha$ closed function.

Proof

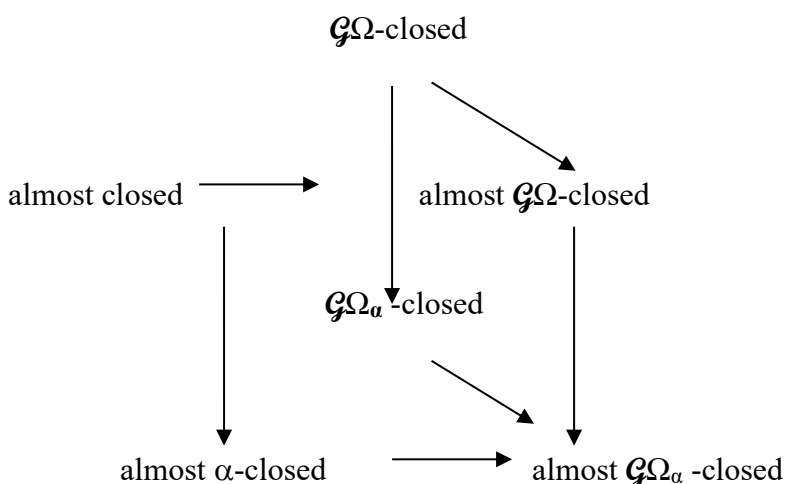
Let F be regular closed space and a function $f : (X, \tau) \rightarrow (Y, \sigma)$ be $G\Omega$ closed function . Then if for each regular closed set F , $f(F)$ is $G\Omega$ -closed and every $G\Omega$ closed set is $G\Omega\alpha$ closed set. Hence f is almost $G\Omega\alpha$ closed function.

Example 3.11

Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a,c\}X\}$ and $\sigma = \{\phi, \{a, b\}, Y\}$. Then $RC(X, \tau) = \{\phi, \{a, c\}, \{b, c\}, X\}$ and $G\Omega\alpha$ closed in $Y = \{\phi, \{c\}, \{a, b\}, Y\}$ Then the function $f : (X, \tau) \rightarrow (Y, \sigma)$ defined as $f(a)=c, f(b)=b, f(c)=c$ is almost $G\Omega\alpha$ -closed. However, it is not almost $G\Omega$ - closed since there exists $\{b, c\} \in RC(X, \tau)$ such that $f(\{b, c\}) = \{a, b\}$ is not $G\Omega$ - closed in (Y, σ) .

Remark 3.12

We have the following diagram for properties of functions:



Theorem 3.13

A surjection $f : X \rightarrow Y$ is almost $G\Omega\alpha$ -closed if and only if for each subset S of Y and each $U \in RO(X)$ containing $f^{-1}(S)$ there exists an $G\Omega\alpha$ -open set V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Proof

Necessity. Suppose that f is almost $\mathcal{G}\Omega_\alpha$ -closed. Let S be a subset of Y and $U \in \text{RO}(X)$ containing $f^{-1}(S)$. Put $V = Y - f(X - U)$, then V is an $\mathcal{G}\Omega_\alpha$ -open set of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Sufficiency. Let F be any regular closed set of X . Then $f^{-1}(Y - f(F)) \subset X - F$ and $X - F \in \text{RO}(X)$. There exists an $\mathcal{G}\Omega_\alpha$ -open set V of Y such that $Y - f(F) \subset V$ and $f^{-1}(V) \subset X - F$. Therefore, we have $f(F) \supset Y - V$ and $F \subset f^{-1}(Y - V)$. Hence, we obtain $f(F) = Y - V$ and $f(F)$ is $\mathcal{G}\Omega_\alpha$ -closed in Y . This shows that f is almost $\mathcal{G}\Omega_\alpha$ -closed.

Corollary 3.14

If $f : X \rightarrow Y$ is an almost $\mathcal{G}\Omega_\alpha$ -closed surjection, then for each sg-closed set F of Y and each $U \in \text{RO}(X)$ containing $f^{-1}(F)$ there exists an α -open set V of Y such that $F \subset V$ and $f^{-1}(V) \subset U$.

Proof

Let F be a sg-closed set of Y and $U \in \text{RO}(X)$ containing $f^{-1}(F)$. By Theorem 1.3.13, there exists an $\mathcal{G}\Omega_\alpha$ -open set W of Y such that $F \subset W$ and $f^{-1}(W) \subset U$. Since W is $\mathcal{G}\Omega_\alpha$ -open, we have $F \subset \alpha\text{int}(W)$. Put $V = \alpha\text{int}(W)$, then V is α -open in Y and $f^{-1}(V) \subset U$.

4. NORMAL SPACES

In this section, we make use of $\mathcal{G}\Omega_\alpha$ -closed sets to obtain further characterizations and preservation theorems of normal spaces.

Theorem 4.1

The following are equivalent for a space X :

- (i) X is normal;
- (ii) For any disjoint closed sets A and B , there exist disjoint $\mathcal{G}\Omega_\alpha$ -open sets U, V such that $A \subset U$ and $B \subset V$;
- (iii) For any closed set A and any open set V containing A , there exists an $\mathcal{G}\Omega_\alpha$ -open set U of X such that $A \subset U \subset \alpha\text{cl}(U) \subset V$.

Proof

(i) \Rightarrow (ii). This is obvious since every open set is $\mathcal{G}\Omega_\alpha$ -open.

(ii) \Rightarrow (iii). Let A be a closed set and V an open set containing A . Then A and $X - V$ are disjoint closed sets. There exist disjoint $\mathcal{G}\Omega_\alpha$ -open sets U and W such that $A \subset U$ and $X - V \subset W$. Since $X - V$ is closed and hence sg-closed, we have $X - V \subset \alpha\text{int}(W)$ and $U \cap \alpha\text{int}(W) = \emptyset$. Therefore, we obtain $\alpha\text{cl}(U) \cap \alpha\text{int}(W) = \emptyset$ and hence $A \subset U \subset \alpha\text{cl}(U) \subset X - \alpha\text{int}(W) \subset V$.

(iii) \Rightarrow (i). Let A, B be disjoint closed sets of X . Then $A \subset X - B$ and $X - B$ is open. There exists an $\mathcal{G}\Omega_\alpha$ -open set G of X such that $A \subset G \subset \alpha\text{cl}(G) \subset X - B$. Since A is closed, we have $A \subset \alpha\text{int}(G)$. Put $U = \text{int}(\text{cl}(\text{int}(\alpha\text{int}(G))))$ and $V = \text{int}(\text{cl}(\text{int}(X - \alpha\text{cl}(G))))$. Then U and V are disjoint open sets of X such that $A \subset U$ and $B \subset V$. Therefore, X is normal.

Theorem 4.2

If $f : X \rightarrow Y$ is a continuous almost $\mathcal{G}\Omega_\alpha$ -closed surjection and X is a normal space, then Y is normal.

Proof

Let A and B be any disjoint closed sets of Y . Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of X . Since X is normal, there exist disjoint open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Let $G = \text{int}(\text{cl}(U))$ and $H = \text{int}(\text{cl}(V))$, then G and H are disjoint regular open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By Theorem 1.3.13, there exists $\mathcal{G}\Omega_\alpha$ -open sets K and L of Y such that $A \subset K$, $B \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, so are K and L . It follows from Theorem 1.4.1 that Y is normal.

Theorem 4.3

If $f: X \rightarrow Y$ is an almost $\mathcal{G}\Omega_\alpha$ -closed continuous surjection and X is a weakly normal space, then Y is weakly normal.

Proof

Let $\{F_n\}$ be any decreasing sequence of closed sets of Y with no common point and any open set V of Y such that $F_1 \subset V$. Then $\{f^{-1}(F_n)\}$ is a decreasing sequence of closed sets of X with no common point and $f^{-1}(V)$ is an open set of X such that $f^{-1}(F_1) \subset f^{-1}(V)$. Since X is weakly normal, by Lemma 1.4.7, there exist $n \in \mathbb{N}$ and an open set U of X such that $f^{-1}(F_n) \subset U \subset \text{cl}(U) \subset f^{-1}(V)$. Therefore, $f^{-1}(F_n) \subset \text{int}(\text{cl}(U))$ and by Corollary 1.3.14, there exists an α -open set G of Y such that $F_n \subset G$ and $f^{-1}(G) \subset \text{int}(\text{cl}(U))$. Since $\text{cl}(U)$ is regular closed and f is almost $\mathcal{G}\Omega_\alpha$ -closed, $f(\text{cl}(U))$ is $\mathcal{G}\Omega_\alpha$ -closed in Y . Thus, we obtain $F_n \subset G \subset \alpha\text{cl}(G) \subset \alpha\text{cl}(f(\text{cl}(U))) \subset V$. Let $H = \text{int}(\text{cl}(\text{int}(G)))$, then by Lemma 7.2.4 we have $F_n \subset H \subset \text{cl}(H) = \alpha\text{cl}(G) \subset V$. It follows from Lemma 1.2.5 that Y is weakly normal.

Theorem 4.4

The following are equivalent for a space X :

- (i) X is mildly normal;
- (ii) for any disjoint $H, K \in \text{RC}(X)$, there exist disjoint $\mathcal{G}\Omega_\alpha$ -open sets U, V such that $H \subset U$ and $K \subset V$;
- (iii) for any disjoint $H, K \in \text{RC}(X)$, there exist disjoint α gs-open sets U, V such that $H \subset U$ and $K \subset V$;
- (iv) for any disjoint $H, K \in \text{RC}(X)$, there exist disjoint α g-open sets U, V such that $H \subset U$ and $K \subset V$;
- (v) for any $H \in \text{RC}(X)$ and any $V \in \text{RO}(X)$ containing H , there exists an α g-open set U of X such that $H \subset U \subset \alpha\text{cl}(U) \subset V$;
- (vi) for any $H \in \text{RC}(X)$ and any $V \in \text{RO}(X)$ containing H , there exists an α -open set U of X such that $H \subset U \subset \alpha\text{cl}(U) \subset V$;
- (vii) for any disjoint $H, K \in \text{RC}(X)$, there exist disjoint α -open sets U, V such that $H \subset U$ and $K \subset V$.

Proof

It is obvious that (i) \Rightarrow (ii), (ii) \Rightarrow (iii) and (iii) \Rightarrow (iv).

(iv) \Rightarrow (v). Let $H \in RC(X)$ and $V \in RO(X)$ containing H . There exist disjoint α g-open sets U, W such that $H \subset U$ and $X - V \subset W$. By Lemma 7.2.8, we have $X - V \subset \alpha\text{int}(W)$ and $U \cap \alpha\text{int}(W) = \emptyset$. Therefore, we obtain $\alpha\text{cl}(U) \cap \alpha\text{int}(W) = \emptyset$ and hence $H \subset U \subset \alpha\text{cl}(U) \subset X - \alpha\text{int}(W) \subset V$.

(v) \Rightarrow (vi). Let $H \in RC(X)$ and $V \in RO(X)$ containing H . There exists an α g-open set G of X such that $H \subset G \subset \alpha\text{cl}(G) \subset V$. Since $H \in RC(X)$, by Lemma 1.2.8, we have $H \subset \alpha\text{int}(G)$. Put $U = \alpha\text{int}(G)$, then U is α -open in X and $H \subset U \subset \alpha\text{cl}(U) \subset V$.

(vi) \Rightarrow (vii). Let H and K be any disjoint regular closed sets of X . Then, since $H \subset X - K$ and $X - K \in RO(X)$, there exists an α -open set U of X such that $H \subset U \subset \alpha\text{cl}(U) \subset X - K$. Put $V = X - \alpha\text{cl}(U)$, then U and V are disjoint α -open sets of X such that $H \subset U$ and $K \subset V$.

(vii) \Rightarrow (i). Let H and K be any disjoint regular closed sets of X . Then there exist disjoint α -open sets A and B of X such that $H \subset A$ and $K \subset B$. Since A and B are disjoint, we have $\text{int}(\text{cl}(\text{int}(A))) \cap \text{int}(\text{cl}(\text{int}(B))) = \emptyset$. Now, Put $U = \text{int}(\text{cl}(\text{int}(A)))$ and $V = \text{int}(\text{cl}(\text{int}(B)))$, then U and V are disjoint open sets of X such that $H \subset U$ and $K \subset V$. Therefore, X is mildly normal.

Theorem 4.5

Let $f : X \rightarrow Y$ be an R -map and an almost $\mathcal{G}\Omega_\alpha$ -closed surjection and X is mildly normal then Y is mildly normal.

Proof

Let A and B be any disjoint regular closed sets of Y . Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint regular closed sets of X . Since X is mildly normal, there exist disjoint open sets U and V of X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Put $G = \text{int}(\text{cl}(U))$ and $H = \text{int}(\text{cl}(V))$, then G and H are disjoint regular open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By Theorem 3.13 [14], there exist $\mathcal{G}\Omega_\alpha$ -open sets K and L of Y such that $A \subset K, B \subset L, f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, so are K and L . It follows from Theorem 1.4.4 that Y is mildly normal.

Theorem 4.6

If $f : X \rightarrow Y$ is an almost α -open almost α gs-closed continuous surjection and X is an almost normal space, then Y is almost normal.

Proof

Let B be any closed set of Y and $V \in RO(Y)$ containing B . Since f is continuous and almost α -open, $f^{-1}(B)$ is closed and $f^{-1}(V) \in RO(X)$ by Lemma 1.2.7. Since X is almost normal and $f^{-1}(B) \subset f^{-1}(V)$, there exists $U \in RO(X)$ such that $f^{-1}(B) \subset U \subset \text{cl}(U) \subset f^{-1}(V)$ [15, Theorem 2.1]. Since f is almost α -open and almost α gs-closed, $f(U)$ is α -open and $f(\text{cl}(U))$ is α gs-closed in Y . Therefore, we obtain $B \subset f(U) \subset \alpha\text{cl}(f(U)) \subset \alpha\text{cl}(f(\text{cl}(U))) \subset V$. Put $G = \text{int}(\text{cl}(\text{int}(f(U))))$. Then G is open in Y and $\alpha\text{cl}(f(U)) = \text{cl}(\text{int}(f(U))) = \text{cl}(G)$ by Lemma 1.2.4. Therefore, we obtain $B \subset f(U) \subset G \subset \text{cl}(G) \subset V$. It follows from [15, Theorem 2.1] that Y is almost normal.

5. REGULAR SPACES

In this section, we improve preservation theorems of regularity almost regularity and quasi-regularity.

Theorem 5.1

If $f : X \rightarrow Y$ is an almost α -open almost $\mathcal{G}\Omega_\alpha$ -closed continuous surjection and X is a regular space, then Y is regular.

Proof

Let y be any point of Y and V any open neighbourhood of y . There exists a point $x \in X$ with $f(x) = y$. Since X is regular and f is continuous, there exists an open set U of X such that $x \in U \subset \text{cl}(U) \subset f^{-1}(V)$. Therefore, we have $y \in f(U) \subset f(\text{int}(\text{cl}(U))) \subset f(\text{cl}(U)) \subset V$ and $f(\text{int}(\text{cl}(U)))$ is α -open because $\text{int}(\text{cl}(U)) \in \text{RO}(X)$ and f is almost α -open. Since $\text{cl}(U) \in \text{RC}(X)$ and f is almost $\mathcal{G}\Omega_\alpha$ -closed, $f(\text{cl}(U))$ is $\mathcal{G}\Omega_\alpha$ -closed and hence $y \in f(\text{int}(\text{cl}(U))) \subset \alpha\text{cl}(f(\text{int}(\text{cl}(U)))) \subset \alpha\text{cl}(f(\text{cl}(U))) \subset V$. It follows from Theorem 1.2.11 that Y is regular.

Theorem 5.2

If $f : X \rightarrow Y$ is an almost α -open almost α gs-closed almost continuous surjection and X is an almost regular space, then Y is almost regular.

Proof

Let y be any point of Y and $V \in \text{RO}(Y)$ containing y . Since f is almost α -open almost continuous, $f^{-1}(V) \in \text{RO}(X)$ by Lemma 1.2.7. Take a point $x \in f^{-1}(y)$. Since X is almost regular, there exists $U \in \text{RO}(X)$ such that $x \in U \subset \text{cl}(U) \subset f^{-1}(V)$ [16, Theorem 2.2]. Hence $y \in f(U) \subset f(\text{cl}(U)) \subset V$. Since f is almost α -open almost α gs-closed, $f(U)$ is α -open in Y and $f(\text{cl}(U))$ is α gs-closed in Y and hence we have $y \in f(U) \subset \alpha\text{cl}(f(U)) \subset \alpha\text{cl}(f(\text{cl}(U))) \subset V$. It follows from Theorem 1.2.11 that Y is almost regular.

Theorem 5.3

If $f : X \rightarrow Y$ is an almost feebly open feebly continuous almost $\mathcal{G}\Omega_\alpha$ -closed surjection and X is a quasi-regular space, then Y is quasi-regular.

Proof

Let V be any nonempty open set of Y . Since f is feebly continuous, $\text{int}(f^{-1}(V)) \neq \emptyset$ and by the quasi-regularity of X there exists a nonempty open set U of X such that $U \subset \text{cl}(U) \subset \text{int}(f^{-1}(V))$. We have $f(\text{int}(\text{cl}(U))) \subset f(\text{cl}(U)) \subset V$. Since f is almost feebly open, $\text{int}(f(\text{int}(\text{cl}(U)))) \neq \emptyset$. Since f is almost $\mathcal{G}\Omega_\alpha$ -closed, $f(\text{cl}(U))$ is $\mathcal{G}\Omega_\alpha$ -closed and hence $\alpha\text{cl}(f(\text{cl}(U))) \subset V$. Now, put $G = \text{int}(f(\text{int}(\text{cl}(U))))$, then by Lemma 1.2.4 we obtain $\emptyset \neq G \subset \text{cl}(G) = \alpha\text{cl}(G) \subset \alpha\text{cl}(f(\text{cl}(U))) \subset V$. This shows that Y is quasi-regular.

Theorem 5.4

If $f : X \rightarrow Y$ is an almost α -open almost $\mathcal{G}\Omega_\alpha$ -closed continuous surjection and X is a strongly s -regular space, then Y is strongly s -regular.

Proof

Let V be any open set of Y and y any point of V . Since f is continuous, $f^{-1}(V)$ is open in X . For a point $x \in f^{-1}(y)$, there exists $F \in \text{RC}(X)$ such that $x \in F \subset f^{-1}(V)$; hence $y = f(x) \in f(F) \subset V$. Since f is continuous, we have $f(F) = f(\text{cl}(\text{int}(F))) \subset \text{cl}(f(\text{int}(F)))$. Since f is almost $\mathcal{G}\Omega_\alpha$ -closed, $f(F)$ is $\mathcal{G}\Omega_\alpha$ -closed and $\alpha\text{cl}(f(F)) \subset V$. Moreover, f is almost α -open, $f(\text{int}(F))$ is α -open in Y and by Lemma 1.2.4 we have $\text{cl}(f(\text{int}(F))) = \text{cl}(\text{int}(f(\text{int}(F)))) = \alpha\text{cl}(f(\text{int}(F))) \subset \alpha\text{cl}(f(F))$. Therefore, we obtain $\text{cl}(\text{int}(f(\text{int}(F)))) \in \text{RC}(Y)$ and $y \in f(F) \subset \text{cl}(f(\text{int}(F))) = \text{cl}(\text{int}(f(\text{int}(F)))) \subset \alpha\text{cl}(f(F)) \subset V$. It follows from [3, Theorem 1] that Y is strongly s -regular.

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