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ALGEBRAIC STRUCTURE OF RELATIONS ON FUZZY SOFT SETS

Anju S Mattam

Assistant Professor
Little Flower College
Guruvayoor, Thrissur, Kerala.
anju@littleflowercollege.edu.in;

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Abstract:
Algebraic properties of operations on fuzzy soft relations are investigated and the lattice structure associated with fuzzy soft relations and fuzzy soft equivalence relations are established.

Keywords: Fuzzy soft set, Fuzzy soft Relation, Fuzzy soft equivalence relation.

1. **Introduction.** The concept of soft set [1] is gaining popularity among the researchers operating in multidisciplinary areas. Embedded with recent developments theory of soft sets is getting richer and richer everyday[2, 3]. Fuzzification of soft sets [4, 5] also play a very important role in fuzzy logic and it has the potential of hybridization. In this aspect fuzzy soft set among with its application [6, 7]have been probed many authors. Relations on collection of fuzzy soft sets [8] are structured using minimum function. It provides a broad and flexible technique for molding any decision making process. In section II adequate concepts related to fuzzy soft sets are given. In the next section different types of relations that can be defined on the collection of fuzzy soft sets are investigated with a detailed study on its properties. Section 4 is entirely devoted to the study of fuzzy soft equivalence relations containing the concepts of fuzzy soft reflexive, fuzzy soft symmetric and fuzzy soft transitive relations. In the last section the lattice structure of fuzzy soft relations and fuzzy soft equivalence relations are also studied.

2. **Preliminaries.** Let Z and $\wp$ be the universal set and the parameter set respectively and let the collection of all fuzzy subsets of Z be denoted as I^Z. A fuzzy soft set (fs-set) over Z is a pair $(\gamma,\wp)$ where function γ is defined from $\wp$ to I^Z.

Definition 2.1. The fuzzy set in $(\gamma,\wp)$ corresponding to the parameter t $\in \wp$ is called the fs-element denoted by γ_t, where γ_t is a function from Z to [0,1]. The collection of all fs-sets over the universal set Z and parameter set $\wp$ is denoted by *fss*(Z,$\wp$). The fs-set $(\gamma,\wp)$ is called a null fs-set , denoted by $\tilde{0}_\wp$, if $\gamma_t(\varepsilon)$=0, $\forall$ t $\in \wp$ and $\forall\ \varepsilon \in$ Z. The fs-set $(\gamma,\wp)$ is called the whole fs-set , denoted by $\tilde{1}_\wp$, if $\gamma_t(\varepsilon)$= 1, $\forall$ t $\in \wp$ and $\forall\ \varepsilon \in$ Z.

Complement of fs-set $(\gamma,\wp)$ over Z is given by $(\gamma^c,\wp)$ where ${\gamma^c}_t(\varepsilon)$= 1-$\gamma_t(\varepsilon)$.
Note that ${\tilde{0}_\wp}^c = \tilde{1}_\wp$ and ${\tilde{1}_\wp}^c = \tilde{0}_\wp$

Definition 2.2. Fs-union of two fs-sets $(\gamma,\wp)$ and $(\xi,\wp_\xi)$ over Z is defined as the fs-set $(\tau,U) = (\gamma,\wp) \cup (\xi,\wp_\xi)$ where U =$\wp \cup \wp_\xi$ and for all u $\in$ U

$$\tau_u(\varepsilon) = \begin{cases} \gamma_u(\varepsilon) & \text{if } u \in \wp - \wp_\xi \\ \xi_u(\varepsilon) & \text{if } u \in \wp_\xi - \wp \\ \gamma_u(\varepsilon) \vee \xi_u(\varepsilon) & \text{if } u \in \wp \cap \wp_\xi \end{cases}$$

Definition 2.3. Fs-intersection of fs-sets $(\gamma,\wp)$ and $(\xi,\wp_\xi)$ over Z is defined as the fs-set $(\tau,U) = (\gamma,\wp) \cap (\xi,\wp_\xi)$ where U = $\wp \cap$ S and for all u $\in$ U, $\tau_u(\varepsilon) = \gamma_u(\varepsilon) \wedge \xi_u(\varepsilon)$

Definition 2.4. Let $(\gamma,\wp)$ and $(\xi,\wp_\xi)$ be two fs-sets over Z. Then product of $(\gamma,\wp)$ and $(\xi,\wp_\xi)$ is defined as $(\gamma,\wp)$ X $(\xi,\wp_\xi)$ = (k, $\wp$ x $\wp_\xi$) where k: $\wp$ x $\wp_\xi \rightarrow I^X$ and $\forall$ (t,s) $\in \wp$ x $\wp_\xi$

$$k_{(t,s)}(\varepsilon) = \begin{cases} 1 & \text{, if t} = s \\ \min(\gamma_t(\varepsilon),\xi_s(\varepsilon)) & \text{, if t} \neq s \end{cases}$$

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3. Fuzzy soft Relation.

Definition 3.1. Let $(\gamma,\wp)$ and $(\xi$,S) be the fs-sets over Z. Fuzzy Soft Relation (fs-relation) from $(\gamma,\wp)$ to $(\xi$,S) is the fs-subset of $(\gamma,\wp)$ X $(\xi$,S) and usually denoted by $\Re$. If $\Re$ is a fs-subset of product $(\gamma,\wp)$ x $(\gamma,\wp)$ then $\Re$ is called a fs-relation on $(\gamma,\wp)$. Inverse of a fs-relation $\Re$ is defined by $\Re_{ts}^{-1} = \Re_{st}$, $\forall$ (t,s) $\in \wp$XS. Null fuzzy soft relation Õ on $(\gamma,\wp)$ is defined as $\tilde{O}_{ts}$(z)=0, $\forall\ \varepsilon \in$ X and t,s $\in \wp$.

Theorem 3.2. Let $\Re$ be a fs-relation from $(\gamma,\wp)$ to $(\xi$,S). Then the fs-relation $\Re^{-1}$ is from $(\xi$,S) to $(\gamma,\wp)$.
Proof
Fs-relation $\Re$ is from $(\gamma,\wp)$ to $(\xi$,S) $\Longrightarrow$ $\Re \subseteq (\gamma,\wp)$ X $(\xi$,S).
$\Re_{ts}^{-1} = \Re_{st} \leq \min(\xi_s,\ \gamma_t)$, $\forall$ (t,s) $\in \wp$xS.
Hence $\Re^{-1}$ is a fs-relation from $(\xi$,S) to $(\gamma,\wp)$.

Definition 3.3. Union and intersections of fs-relations $\Re_1$ and $\Re_2$ defined on the fs-set $(\gamma,\wp)$ are defined as follows.
$(\Re_1 \cup \Re_2)_{ts}(\varepsilon) = (\Re_1)_{ts}(\varepsilon) \vee (\Re_2)_{ts}(\varepsilon)$
$(\Re_1 \cap \Re_2)_{ts}(\varepsilon) = (\Re_1)_{ts}(\varepsilon) \wedge (\Re_2)_{ts}(\varepsilon)$,
$\forall\ \varepsilon \in$ Z and $\forall$ (t,s) $\in \wp$ x $\wp$.
The partial ordering $\leq$ in the set of all fs-relations on $(\gamma,\wp)$ denoted as $\Re(\gamma,\wp)$ is given by, if $\Re_1$ and $\Re_2 \in \Re(\gamma,\wp)$ then $\Re_1 \leq \Re_2$ iff
$(\Re_1)_{ts}\ (\varepsilon) \leq (\Re_2)_{ts}(\varepsilon)$, $\forall\ \varepsilon \in$ Z and $\forall$ (t,s) $\in \wp$ x $\wp$.
Identity fuzzy soft relation Ĩ on $(\gamma,\wp)$ is defined for any t,m $\in \wp$ as

$$\tilde{I}_{tm} = \begin{cases} 0 & \text{if t } \neq m \\ 1 & \text{if t = m} \end{cases}$$

Definition 3.4. Consider two fs-relations $\Re_1$ and $\Re_2$ from $(\gamma,\wp)$ to $(\xi$,S) and $(\xi$,S) to $(\zeta$,M) respectively. Composition of fs-relations $\Re_1$ and $\Re_2$ denoted by $\Re_1 \circ \Re_2$ is a fs-relation from $(\gamma,\wp)$ to $(\zeta$,M) defined as
$(\Re_1 \circ \Re_2)_{tm} = \bigvee_{s\in S} \min((\Re_1)_{ts},(\Re_2)_{sm})$ where (t,s) $\in \wp$xS and (s,m) $\in$ SxM.

Theorem 3.5. If $\tilde{I}$ and $\tilde{0}$ be the whole and null fuzzy soft relations respectively on $(\gamma,\wp)$ over Z then
1) $\tilde{I}^{-1} = \tilde{I}$
2) $\tilde{0}^{-1} = \tilde{0}$
3) $\tilde{I} \circ \tilde{I} = \tilde{I}$
4) $\tilde{0} \circ \tilde{0} = 0$

Proof
Proof of 1 and 2 follows obviously.
3) Consider the parameters t and s $\in \wp$ such that t = s, $(\tilde{I} \circ \tilde{I})_{tt} = \bigvee_{m\in\wp}(\min(\tilde{I}_{tm},\tilde{I}_{mt}))$
$= \bigvee_{m\neq t}(\min(\tilde{I}_{tm},\tilde{I}_{mt})) \vee \min(\tilde{I}_{tt},\tilde{I}_{tt}) = \max(0,1) = 1$
For the parameters t,s $\in \wp$, with t $\neq$ s, $(\tilde{I} \circ \tilde{I})_{ts} = \bigvee_{m\in\wp}(\min(\tilde{I}_{tm},\tilde{I}_{ms}))$

$= \underset{m\neq t,s}{\vee}(\min(\tilde{I}_{tm},\tilde{I}_{ms})) \vee \min(\tilde{I}_{tt},\tilde{I}_{ts}) \vee \min(\tilde{I}_{ts},\tilde{I}_{ss}) = 0 \vee \min(1,\tilde{I}_{ts}) \vee \min(\tilde{I}_{ts},1) = 0$

4) Let parameters t,s $\in \wp$, $(\tilde{0} \circ \tilde{0})_{ts} = \underset{m\in\wp}{\vee}(\min(\tilde{0}_{tm},\tilde{0}_{ms})) = \underset{m\in\wp}{\vee}\min(0,0) = 0 = \tilde{0}_{ts}$

Theorem 3.6. Let $\mathcal{S}$, $\mathcal{T}$ and $\Re$ be the fs-relations defined on the fs-set $(\gamma,\wp)$ then

1) $(\mathcal{S}^{-1})^{-1} = \mathcal{S}$
2) $(\mathcal{S}^{c})^{c} = \mathcal{S}$
3) $(\mathcal{S}^{c})^{-1} = (\mathcal{S}^{-1})^{c}$
4) $\mathcal{S} \subseteq \mathcal{T} \implies \mathcal{S}^{-1} \subseteq \mathcal{T}^{-1}$
5) $(\mathcal{S} \circ \mathcal{T})^{-1} = \mathcal{T}^{-1} \circ \mathcal{S}^{-1}$
6) $\mathcal{S} \subseteq \mathcal{T} \implies \mathcal{S} \circ \Re \subseteq \mathcal{T} \circ \Re$
7)$(\mathcal{S} \circ \mathcal{T}) \circ \Re = \mathcal{S} \circ (\mathcal{T} \circ \Re)$
8)$(\mathcal{S} \cup \mathcal{T})^{-1} = \mathcal{S}^{-1} \cup \mathcal{T}^{-1}$ and $(\mathcal{S} \cap \mathcal{T})^{-1} = \mathcal{S}^{-1} \cap \mathcal{T}^{-1}$
9)$(\mathcal{S} \cup \mathcal{T})^{c} = \mathcal{S}^{c} \cap \mathcal{T}^{c}$ and $(\mathcal{S} \cap \mathcal{T})^{c} = \mathcal{S}^{c} \cup \mathcal{T}^{c}$.

Proof

1)Using the definition for inverse of fs-relation $(\mathcal{S}^{-1})^{-1}{}_{ts}(\varepsilon)$ is same as $(\mathcal{S}^{-1})_{st}(\varepsilon)$ which is equal to $(\mathcal{S})_{ts}(\varepsilon)$, $\forall\ \varepsilon \in$Z and t, s $\in \wp$

2)$(\mathcal{S}^{c})^{c}{}_{ts}(\varepsilon)$ = 1- $(\mathcal{S}^{c})_{ts}(\varepsilon)$ = 1-(1- $\mathcal{S}_{ts}(\varepsilon)$)
= $\mathcal{S}_{ts}(\varepsilon)$, $\forall\ \varepsilon \in$ Z and t, s $\in \wp$

3) $\forall\ \varepsilon \in$ Z and t, s $\in \wp$, $(\mathcal{S}^{c})^{-1}{}_{ts}(\varepsilon) = (\mathcal{S}^{c})_{st}(\varepsilon)$ = 1 - $\mathcal{S}_{st}(\varepsilon)$,
$(\mathcal{S}^{-1})^{c}{}_{ts}(\varepsilon)$ = 1 - $(\mathcal{S}^{-1})_{ts}(\varepsilon)$ = 1- $\mathcal{S}_{st}(\varepsilon)$,
$\implies (\mathcal{S}^{c})^{-1} = (\mathcal{S}^{-1})^{c}$

4)Using definition 3.1 we have $(\mathcal{S}^{-1})_{ts}(\varepsilon)$ is equal to $(\mathcal{S})_{ts}(\varepsilon)$ which is less than $(\mathcal{T})_{ts}(\varepsilon)$. Using the similarity property $(\mathcal{T})_{ts}(\varepsilon) = (\mathcal{T}^{-1})_{ts}(\varepsilon)$, $\forall\ \varepsilon \in$ Z and t,s $\in \wp$
$\implies \mathcal{S}^{-1} \subseteq \mathcal{T}^{-1}$

5) $(\mathcal{T}^{-1} \circ \mathcal{S}^{-1})_{ts}(\varepsilon) = \underset{m\in\wp}{\vee} (\min(\mathcal{T}^{-1}{}_{tm}(\varepsilon),\ \mathcal{S}^{-1}{}_{ms}(\varepsilon))) = \underset{m\in\wp}{\vee} (\min(\mathcal{T}_{mt}(\varepsilon),\ \mathcal{S}_{sm}(\varepsilon))) = \underset{m\in\wp}{\vee} (\min(\mathcal{S}_{sm}(\varepsilon),\ \mathcal{T}_{mt}(\varepsilon))) = (\mathcal{S} \circ \mathcal{T})_{st}(\varepsilon)$
$= (\mathcal{S} \circ \mathcal{T})^{-1}{}_{ts}(\varepsilon)$, $\forall\ \varepsilon \in$ Z and t, s $\in \wp$

6) $(\mathcal{S} \circ \Re)_{ts} = \underset{m\in\wp}{\vee} (\min((\mathcal{S})_{tm},\ (\Re)_{ms})) \leq \underset{m\in\wp}{\vee}(\ \min((\mathcal{T})_{tm},\ (\Re)_{ms})) \leq (\mathcal{T} \circ \Re)_{ts}$

7) $((\mathcal{S} \circ \mathcal{T}) \circ \Re)_{ts} = \underset{m\in\wp}{\vee} (\min((\mathcal{S} \circ \mathcal{T})_{tm},(\Re)_{ms}))$
$= \underset{m\in\wp}{\vee} (\min(\underset{u\in T}{\vee} (\min((\mathcal{S})_{tu},(\mathcal{T})_{um}),(\Re)_{ms})))$
$= \underset{m\in\wp}{\vee}\ \underset{u\in T}{\vee} (\min((\mathcal{S})_{tu},\ \min((\mathcal{T})_{um},\ (\Re)_{ms})))$
$= \underset{u\in T}{\vee} \min(\mathcal{S}_{tu}, \underset{m\in\wp}{\vee} (\min\ ((\mathcal{T})_{um},\ (\Re)_{ms})\ ))$
$= \underset{u\in T}{\vee} (\min\ (\mathcal{S}_{tu},\ (\mathcal{T}) \circ (\Re)_{us})) = (\mathcal{S} \circ ((\mathcal{T}) \circ (\Re)))_{ts}$

8) $(\mathcal{S} \cup \mathcal{T})^{-1}{}_{ts}(\varepsilon) = (\mathcal{S} \cup \mathcal{T})_{st}(\varepsilon) = \mathcal{S}_{st}(\varepsilon) \vee \mathcal{T}_{\ st}(\varepsilon) = \mathcal{S}^{-1}{}_{ts}(\varepsilon) \vee \mathcal{T}^{\ -1}{}_{ts}(\varepsilon) = (\mathcal{S}^{-1} \cup \mathcal{T}^{-1})_{ts}(\varepsilon)$, $\forall\ \varepsilon \in$ Z and t, s $\in \wp$.
$\implies (\mathcal{S} \cup \mathcal{T})^{-1} = \mathcal{S}^{-1} \cup \mathcal{T}^{-1}$
$(\mathcal{S} \cap \mathcal{T})^{-1}{}_{ts}(\varepsilon) = (\mathcal{S} \cap \mathcal{T})_{st}(\varepsilon) = \mathcal{S}_{st}(\varepsilon) \wedge \mathcal{T}_{st}(\varepsilon) = \mathcal{S}^{-1}{}_{ts}(\varepsilon) \wedge \mathcal{S}^{-1}{}_{ts}(\varepsilon) = (\mathcal{S}^{-1} \cap \mathcal{T}^{-1})_{ts}(\varepsilon)$, $\forall\ \varepsilon \in$ Z and t, s $\in \wp$
$\implies (\mathcal{S} \cap \mathcal{T})^{-1} = \mathcal{S}^{-1} \cap \mathcal{T}^{-1}$

9)$(\mathcal{S} \cup \mathcal{T})^{c}{}_{ts}(\varepsilon)$ = 1- $(\mathcal{S} \cup \mathcal{T})_{ts}(\varepsilon)$ = 1- $(\mathcal{S}_{ts}(\varepsilon)\vee(\mathcal{T})_{ts}(\varepsilon))$
= (1- $\mathcal{S}_{ts}(\varepsilon)$) $\wedge$ (1- $(\mathcal{T})_{ts}(\varepsilon)$) = $\mathcal{S}^{c}{}_{ts}(\varepsilon)$) $\wedge$ $(\mathcal{T})^{c}{}_{ts}(\varepsilon)$
= $(\mathcal{S}^{c} \cap \mathcal{T}^{c})_{ts}(\varepsilon)$, $\forall\ \varepsilon \in$ Z and t, s $\in \wp$
$\implies (\mathcal{S} \cup \mathcal{T})^{c} = \mathcal{S}^{c} \cap \mathcal{T}^{c}$
$(\mathcal{S} \cap \mathcal{T})^{c}{}_{ts}(\varepsilon)$ = 1- $(\mathcal{S} \cap \mathcal{T})_{ts}(\varepsilon)$ = 1- $(\mathcal{S}_{ts}(\mathrm{z})\wedge(\mathcal{T})_{ts}(\varepsilon))$
= (1- $\mathcal{S}_{ts}(\varepsilon)$) $\vee$ (1- $(\mathcal{T})_{ts}(\varepsilon)$) = $\mathcal{S}^{c}{}_{ts}(\varepsilon)$) $\vee$ $(\mathcal{T})^{c}{}_{ts}(\varepsilon)$

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$= (\mathcal{S}^c \cup (\mathcal{T})^c)_{ts}(\varepsilon)$, $\forall\ \varepsilon \in$ Z and t, s $\in \wp$
$\implies (\mathcal{S} \cap \mathcal{T})^c = \mathcal{S}^c \cup \mathcal{T}^c$

4. **FS-Equivalence Relation.**

Definition 4.1. Let $\Re$ be a fs- relation defined on fs-set $(\gamma,\wp)$ over Z.
1) If $\Re_{ts}(\varepsilon) \leq \Re_{tt}(\varepsilon)$ and $\Re_{st}(\varepsilon) \leq \Re_{tt}(\varepsilon)$, $\forall$ t,s $\in \wp$ with t $\neq$ s and $\forall\ \varepsilon \in$ Z, then the fs-relation $\Re$ is called fs-reflexive.
2) If the fs-reflexive relation $\Re$ is such that $\Re_{tt}(\varepsilon) = 1$, $\forall\ \varepsilon \in$ Z and $\forall$ t $\in \wp$, then $\Re$ is called Identically fs-reflexive relation
3) If $\Re = \Re^{-1}$, $\Re$ is called a fs-symmetric relation
4) If $\Re \circ \Re \subseteq \Re$, then $\Re$ is called a fs-transitive relation
5) If the relation $\Re$ satisfies the conditions 1,3,4 then $\Re$ is called a fs-equivalence relation.

Theorem 4.2. Let $\Re$ be a fs-relation on fs-set $(\gamma,\wp)$, then $\Re^{-1}$ and $\Re \circ \Re$ are fs-equivalence relations on $(\gamma,\wp)$

Proof
For t,s $\in \wp$ with t $\neq$ s and for $\varepsilon \in$ Z, $\Re^{-1}_{ts}(\varepsilon) = \Re_{st}(\varepsilon) \leq \Re_{tt}(\varepsilon)$
Similarly $\Re^{-1}_{st}(\varepsilon) \leq \Re_{tt}(\varepsilon) \implies \Re^{-1}$ is fs-reflexive.
Since $\Re$ is fs-symmetric we have $\Re = \Re^{-1}$
Hence $\Re^{-1} = (\Re^{-1})^{-1} = \Re$
Therefore the fs-relation $\Re^{-1}$ is fs-symmetric.
The fs-relation $\Re$ is fs-transitive $\implies \Re \circ \Re \subseteq \Re$
which implies $(\Re \circ \Re)^{-1} \subseteq \Re^{-1}$. By theorem 3.6 we have $\Re^{-1} \circ \Re^{-1} \subseteq \Re^{-1}$
Hence the fs-relation $\Re^{-1}$ is fs-transitive.
Therefore $\Re^{-1}$ is a fs-equivalence relation.
Let t and s be the parameters in $\wp$ such that t $\neq$ s, then for all ε belongs to Z we have,
$(\Re \circ \Re)_{ts}(\varepsilon) \leq \Re_{ts}(\varepsilon) \leq \Re_{tt}(\varepsilon)$
$\implies \Re \circ \Re$ is fs-reflexive relation.
From theorem 3.6 $(\Re \circ \Re)^{-1}_{ts} = (\Re^{-1} \circ \Re^{-1})_{ts} = (\Re \circ \Re)_{ts}$
Thereby implies that $\Re \circ \Re$ is fs-symmetric.
Since $\Re$ is fs-transitive, from the definition we have $\Re^2 = \Re \circ \Re$ which is contained in $\Re$
$\implies \Re^2_{tm}(\varepsilon) \leq \Re_{tm}(\varepsilon)$ and $\Re^2_{ms}(\varepsilon) \leq \Re_{ms}(\varepsilon)$
By monotonicity of min function it follows that min($\Re^2$(tm),$\Re^2$(ms) $\leq$ min($\Re_{ts}$, $\Re_{ms}$))
The inequality holds for all m $\in \wp$ and it is true for supremum also.
Hence for all t,s in $\wp$ we have $\Re^2_{ts} = (\Re \circ \Re)_{ts} = \underset{t\in\wp}{\forall} \min(\Re_{ts}, \Re_{ms})$
$\geq \underset{m\in\wp}{\forall} \min(\Re^2_{tm}, \Re^2_{ms}) \geq (\Re^2 \circ \Re^2)_{ts}$.
Therefore $\Re \circ \Re$ contains $(\Re \circ \Re)\circ(\Re \circ \Re)$. Thus we have proved that $\Re \circ \Re$ is fs-transitive relation.

Theorem 4.3. If $\Re_1$ and $\Re_2$ are two fs-equivalence relations on fs-set $(\gamma,\wp)$ then $\Re_1 \circ \Re_2$ is a fs-equivalence relation on fs-set $(\gamma,\wp)$ iff $\Re_1 \circ \Re_2 = \Re_2 \circ \Re_1$.

Proof
Suppose that $\Re_1 \circ \Re_2 = \Re_2 \circ \Re_1$
Since $\Re_1$ and $\Re_2$ are fs-reflexive relation on $(\gamma,\wp)$ and $\forall\ \varepsilon \in$ Z we have $(\Re_1)_{ts}$(z) $\leq (\Re_1)_{tt}$(z) and $(\Re_2)_{ts}$(z) $\leq (\Re_2)_{tt}$(z)
$\implies (\Re_1 \circ \Re_2)_{ts} \leq (\Re_1 \circ \Re_2)_{tt}$
Similarly we can prove $(\Re_1 \circ \Re_2)_{st} \leq (\Re_1 \circ \Re_2)_{tt}$
$\implies \Re_1 \circ \Re_2$ is a fs-reflexive relation on fs-set (γ,T)
$(\Re_1 \circ \Re_2)_{ts} = \underset{m\in\wp}{\forall} \min((\Re_1)_{tm},(\Re_2)_{ms})$

$= \underset{m\in\wp}{\forall}\min((\Re_1)_{tm},(\Re_2)_{ms}) = \underset{m\in\wp}{\forall}\min((\Re_1)^{-1}_{tm},(\Re_2)^{-1}_{ms})$
$= ({\Re_1}^{-1}\circ{\Re_2}^{-1})_{ts} = (\Re_2\circ\Re_1)^{-1}_{ts}$ which is equal to $(\Re_1\circ\Re_2)^{-1}_{ts}$
Now consider the composition of $(\Re_1\circ\Re_2)$ with itself
so that $\Re_1\circ(\Re_2\circ\Re_1)\circ\Re_2$ and composition of fs-relations $\Re_1\circ(\Re_1\circ\Re_2)\circ\Re_2$ are equal.
But this is less than or equal to $\Re_1\circ\Re_2$
$\Longrightarrow \Re_1\circ\Re_2$ is a fs-transitive relation on (γ,T).
conversely let $\Re_1\circ\Re_2$ is a fs-equivalence relation
then by fs-symmetric property, $(\Re_1\circ\Re_2)^{-1} = \Re_1\circ\Re_2$
Thereby implies $\Re_2^{-1}\circ\Re_1^{-1} = \Re_1\circ\Re_2$. Thus we have the result $\Re_2\circ\Re_1 = \Re_1\circ\Re_2$.

Theorem 4.4. Let (γ,$\wp$) be a fs-set over Z then the fs-relation $\Re$ = (γ,$\wp$) X (γ,$\wp$) is a fs-equivalence relation.

Proof
$\Re_{ts}(\varepsilon)$ = minimum of $\gamma_t(\varepsilon)$ and $\gamma_s(\varepsilon) \leq \gamma_t(\varepsilon) \leq 1$
Thereby implies that $\Re_{ts}(\varepsilon) \leq \Re_{tt}(\varepsilon)$, $\forall$ t,s $\in\wp$ and $\forall$ $\varepsilon \in$ Z, hence $\Re$ is a fs-reflexive relation.
$\Re_{ts}(\varepsilon) = \min(\gamma_t(\varepsilon), \gamma_s(\varepsilon))$. This is same as $\min(\gamma_s(\varepsilon), \gamma_t(\varepsilon)) = \Re_{st}(\varepsilon) = \Re^{-1}_{ts}(\varepsilon) \Longrightarrow \Re$ is a fs-symmetric relation.
$(\Re\circ\Re)_{ts}(\varepsilon) = \underset{m\in\wp}{\forall}(\min(\Re_{tm}(\varepsilon), \Re_{ms}(\varepsilon)))$
$= \underset{m\neq t,s}{\forall}(\min(\Re_{tm}(\varepsilon), \Re_{ms}(\varepsilon)) \vee \min(\Re_{ttp}(\varepsilon),\Re_{ts}(\varepsilon)) \vee \min(\Re_{ts}(\varepsilon),\Re_{ss}(\varepsilon))$
$= \underset{m\neq t,s}{\forall}(\min(\gamma_t(\varepsilon)\wedge\gamma_m(\varepsilon), \gamma_m(\varepsilon)\wedge\gamma_s(\varepsilon))) \vee \min(\gamma_t(\varepsilon),\gamma_s(\varepsilon))$
$= \underset{m\neq t,s}{\forall}\min(\gamma_t(\varepsilon), \gamma_m(\varepsilon), \gamma_s(\varepsilon)) \vee \min(\gamma_t(\varepsilon),\gamma_s(\varepsilon))$
$= (\gamma_t(\varepsilon) \wedge \underset{m\in\wp}{\forall}\gamma_m(\varepsilon) \wedge \gamma_s(\varepsilon)) \vee ((\gamma_t(\varepsilon) \wedge \gamma_s(\varepsilon)))$
$\leq (\gamma_t(\varepsilon) \wedge \gamma_s(\varepsilon)) = \Re_{ts}(\varepsilon)$
Thus the fs-relation $\Re$ is fs-transitive.
Therefore $\Re$ is a fs-equivalence relation.

5. **Lattice structure of fs-relation.**

Theorem 5.1. The collection of all fs-relations **R**(γ,$\wp$) form a complete lattice under the ordering $\leq$ with the bounds $\tilde{0}_\wp$ and (γ,$\wp$) X (γ,$\wp$).
Proof
Let $\Re$ and S be the fs-relation defined on (γ,$\wp$). Then fs-union of $\Re$ and S is the supremum and fs-intersection of $\Re$ and S is the infimum of the fs-relations $\Re$ and S. This is also true if we replace $\Re$ and S with an arbitrary family of **R**(γ,$\wp$). So ($\Re$(γ,$\wp$), $\cup$, $\cap$) is a complete lattice.

Example 5.2. Let (γ,$\wp$) be a fs-set over Z ={ε, φ, ω, ψ} and let $\wp$ ={t, s, m} be the parameter set. Consider two fs-equivalence relations Q and S on (γ,Z) as in following tables.

Table 1. Q.

	ε	φ	ω	ψ
Q_{tt}	0.76	0.5	0.82	0.64
Q_{ts}	0.58	0.075	0.6	0.56
Q_{st}	0.58	0.075	0.6	0.56
Q_{ss}	0.6	0.7	0.82	0.9

Table 2. S.

	ε	φ	ω	ψ
S_{tt}	0.45	0.55	0.7	0.6
S_{tm}	0.2	0.3	0.67	0.5
S_{mt}	0.2	0.3	0.67	0.5
S_{mm}	0.5	0.6	0.8	0.7
S_{ss}	0.4	0.5	0.8	0.5

Table 3. Q ∪ S.

	ε	φ	ω	ψ
$(Q \cup S)_{tt}$	0.76	0.5	0.82	0.64
$(Q \cup S)_{ts}$	0.58	0.075	0.6	0.56
$(Q \cup S)_{st}$	0.58	0.075	0.6	0.56
$(Q \cup S)_{ss}$	0.6	0.7	0.82	0.9
$(Q \cup S)_{tm}$	0.2	0.3	0.67	0.5
$(Q \cup S)_{mm}$	0.5	0.6	0.8	0.7

Union of two relations is computed as in above tables. Composition of relation $\Re \cup S$ corresponding to the parameter sm is given by
$[(Q \cup S)\circ(Q \cup S)]_{sm}(\varepsilon) = 0.2$,
$[(Q \cup S)\circ(Q \cup S)]_{sm}(\varphi) = 0.075$,
$[(Q \cup S)\circ(Q \cup S)]_{sm}(\omega) = 0.6$,
$[(Q \cup S)\circ(Q \cup S)]_{sm}(\psi) = 0.5$
But $(Q \cup S)_{sm}(z) = 0, \forall \varepsilon \in Z$. Therefore $[(Q \cup S)\circ(Q \cup S)]$ is not a fs-subset of $(Q \cup S)$. Hence the relation $(Q \cup S)$ is not fs-tansitive and hence it is not a fs-equivalence relation.

Consider the collection of all fs-equivalence relation on $(\gamma,\wp)$ denoted by $\mathbf{R}_E(\gamma,\wp)$ and partial ordering $\leq$ on $\mathbf{R}_E(\gamma,\wp)$ induced by the ordering on $\mathbf{R}(\gamma,\wp)$. Since the collection $\mathbf{R}_E(\gamma,\wp)$ is not closed under the binary operation union, $(\mathbf{R}_E(\gamma,\wp), \leq)$ is not a sublattice of $(\mathbf{R}(\gamma,\wp), \leq)$.
To explain the lattice structure of $\mathbf{R}_E(\gamma,\wp)$ we define a new operation Ü as join on $\mathbf{R}_E(\gamma,\wp)$ in the next theorem.

Theorem 5.3. $(\mathbf{R}_E(\gamma,\wp), \leq)$ is a bounded complete lattice.
Proof:
The fs-relation $\tilde{0}_\wp$ is the least element and the fs-relation $(\gamma,\wp)$ X $(\gamma,\wp)$ is greatest element in $\mathbf{R}_E(\gamma,\wp)$.
Let $\Re$, P $\in \mathbf{R}_E(\gamma,\wp)$.
To prove that $\Re \cap P \in \mathbf{R}_E(\gamma,\wp)$.
$(\Re \cap P)_{ts}(\varepsilon) = \Re_{ts}(\varepsilon) \wedge P_{ts}(\varepsilon) \leq \Re_{tt}(\varepsilon) \wedge P_{tt}(\varepsilon)$ which is less than or equal to $(\Re \cap P)_{tt}(\varepsilon)$. From this we have $\Re \cap P$ is a fs-reflexive relation.
$(\Re \cap P)^{-1}_{ts}(\varepsilon) = \Re^{-1}_{ts}(\varepsilon) \wedge P^{-1}_{ts}(\varepsilon) = \Re_{ts}(\varepsilon) \wedge P_{ts}(\varepsilon)$ which is equal to $(\Re \cap P)_{ts}(\varepsilon)$

Hence $\Re \cap P$ is a fs-symmetic relation.
$(\Re \cap P) \circ (\Re \cap P)$ is a fs-subset of both the relations $\Re \circ \Re$ and $P \circ P$
Therefore $(\Re \cap P) \circ (\Re \cap P)$ is a fs-subset of the fs-intersection of $(\Re \circ \Re)$ and $(P \circ P)$ which is a subset of $\Re \cap P$
Hence $\Re \cap P$ is a fs-transitive relation.
Define union on $\mathbf{R}_E(\gamma, T)$ as follows
$\Re \ddot{U} P = \wedge \{ U \in \mathbf{R}_E(\gamma, \wp) : \Re \subseteq U \text{ and } P \subseteq U \}$
Next we prove that $\Re \ddot{U} P$ belongs to $\mathbf{R}_E(\gamma, \wp)$.
Let J be an indexed family and $\{U_j : j \in J\} \subseteq \mathbf{R}_E(\gamma, \wp)$ such that $\Re \subseteq U_j$ and $P \subseteq U_j$, $\forall j \in J$
$(\Re \ddot{U} P)_{ts}(\varepsilon) \leq (U_j)_{ts}(\varepsilon) \leq (U_j)_{tt}(\varepsilon), \forall j \in J$
$(\Re \ddot{U} P)_{ts}(\varepsilon) \leq \underset{j \in J}{\wedge} (U_j)_{pp}(\varepsilon) = (\Re \ddot{U} P)_{tt}(\varepsilon)$,
hence $(\Re \ddot{U} P)_{ts}$ is a fs-subset of $(\Re \ddot{U} P)_{tt}$.
Similarly we have $(\Re \ddot{U} P)_{st}$ fs-subset of $(\Re \ddot{U} P)_{tt}$.
$\Longrightarrow (\Re \ddot{U} P)$ is a fs-reflexive relation on given fs-set.
$(\Re \ddot{U} P)^{-1}_{ts}(\varepsilon) = \wedge \{ U^{-1}_{ts}(\varepsilon) : U \in \mathbf{R}_E(\gamma, \wp) \text{ with } \Re_{ts}(\varepsilon) \leq U_{ts}(\varepsilon)$
and $P_{ts}(\varepsilon) \leq U_{ts}(\varepsilon) \}$
$= \wedge \{ U_{ts}(\varepsilon) : U \in \mathbf{R}_E(\gamma, \wp) \text{ with } \Re_{ts}(\varepsilon) \leq U_{ts}(\varepsilon) \text{ and } P_{ts}(\varepsilon) \leq U_{ts}(\varepsilon) \}$ which is equal to $(\Re \ddot{U} P)_{ts}(\varepsilon)$.
This implies that $(\Re \ddot{U} P)$ is fs-symmetric relation.
The fs-equivalence relation $(\Re \ddot{U} P)$ is a subset of U_j, for every $j \in J$.
By the property 6 of theorem 3.6 $(\Re \ddot{U} P \circ (\Re \ddot{U} P)$ is a subset of $U_j \circ U_j$ which is contained in U_j, for every $j \in J$.
This implies that the composition of fs-relation$(\Re \ddot{U} P)$ with itself is contained in $\underset{j \in J}{\wedge} U_j$
Therefore composition of $(\Re \ddot{U} P)$ and $(\Re \ddot{U} P)$ is a fs-subset of fs-relation $(\Re \ddot{U} P)$.
This implies that $(\Re \ddot{U} P)$ is a fs-transitive relation.
Hence $(\Re \ddot{U} P) \in \mathbf{R}_E(\gamma, \wp)$
There is no difficulty in replacing $\Re$ and P with an arbitary family of $\mathbf{R}_E(\gamma, \wp)$, hence $(\mathbf{R}_E(\gamma, \wp), \ddot{U}, \cap)$ is a complete lattice.

6. **Conclusions.** The concept of relations on fuzzy soft sets and the properties of fuzzy soft equivalence relations are studied in detail. Finally, we characterize the lattice structure of fs-relations and fs-equivalence relations.

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