

CONNECTED EDGE AND ENTIRE DOMINATION OF INVOLUTORY ADDITION CAYLEY GRAPH

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ABSTRACT

Graph theory is one of the most advanced branches of discrete mathematics with variety of applications to different branches of Science and Technology. For a positive integer $n > 1$, the involutory addition Cayley graph $G^+(Z_n, I_v)$, is the graph whose vertex set is $Z_n = \{0, 1, 2, 3, \dots, n-1\}$ and edge set $E(G_n) = \{xy \mid x, y \in Z_n, x + y \in I_v\}$, where $I_v = \{x \in Z_n : x^2 \equiv 1 \pmod{n}\}$ is the set of involutory elements of Z_n . By taking Involutory Addition Cayley Graphs $G^+(Z_n, I_v)$ the author's evaluated graph related Connected edge domination numbers and Entire domination numbers. In this paper, Connected edge domination number, Entire domination number of the Involutory addition cayley graphs $G^+(Z_n, I_v)$ were discussed.

Keywords: Involutory addition cayley graphs, Connected edge dominating sets, Entire dominating sets, Connected edge Domination number, Entire domination number.

AMS Subject Classification: 05C40, 05C19.

1. INTRODUCTION

A graph $G(V, E)$ is a mathematical object that may be thought of as a collection of edges and a set of vertices that connect any or all of the vertices. In a graph G , two vertices are considered neighboring if an edge joins them; otherwise, the edge is considered non-adjacent. We indicate that a graph G has $V(G)$ vertices and $E(G)$ edges, accordingly. The cardinality of $V(G)$ is the definition of the order of G . The cardinality of $E(G)$ is represented by $|E|$, and that of $V(G)$ by $|V|$. The number of edges that occur with a vertex v in a graph G is known as its degree, or $\deg(v)$.

Involutory cayley graphs were introduced by Venkata Anusha et al. [8], who also looked into some of its characteristics. Involutory addition Cayley graphs were first shown by Shanmuga Priya et al. [7], who also examined several of their characteristics. In addition to studying the characterization of the set of involutory elements of (Z_n, \oplus, \odot) , Prameela Rani et al. [3,4]

investigated certain properties of the domatic number and dominance of the involutory cayley graph.

Lavanya et al. [6] investigated the involutory addition cayley graphs edge domination number and total edge domination number.

For a positive integer $n > 1$, the involutory addition Cayley graph $G^+(Z_n, I_v)$, is the graph whose vertex set is $Z_n = \{0, 1, 2, 3, \dots, n - 1\}$ and edge set $E(G_n) = \{xy / x, y \in Z_n, x + y \in I_v\}$, where $I_v = \{x \in Z_n : x^2 \equiv 1 \pmod{n}\}$ is the set of involutory elements of Z_n .

The theory of domination in graphs was introduced in 1958 by Claude Berge[1] in which he used the concept ‘coefficient of external stability’ to refer the domination number of a graph. In 1962, Dominating sets of edges were studied by Mitchell and Hedetniemi[3]. Connected edge domination was introduced by Kulli and Sigarkanti[5]. Entire domination was introduced by Kulli[6]. Kulli Sigarkanti and Soner established a relationship between the domination, edge domination and entire domination number.

2 CONNECTED EDGE DOMINATION

The theory of domination in graphs was introduced in 1958 by Claude Berge[1] in which he used the concept ‘coefficient of external stability’ to refer the domination number of a graph. In 1962, Oystein Ore[4] wrote another book on graph theory, in which he studied the concept of domination using the terms ‘dominating set’ and ‘domination number’ with notation $d(G)$ for the first time. Cockayne et al[2] discussed the review of results and applications concerning dominating sets in graphs. Dominating sets of edges were studied by Mitchell and Hedetniemi[3]. Connected edge domination was introduced by Kulli and Sigarkanti[5].

An edge domination set F of a graph G is a connected edge domination set if the induced subgraph $\langle F \rangle$ is connected. The connected edge domination number $\gamma'_c(G)$ of G is the minimum cardinality of a connected edge dominating set.

Theorem 2.1: For the involutory addition cayley graph $G^+(Z_n, I_v)$, if n is even, $n > 2$ then the connected edge domination number is

$$\gamma'_c(G^+(Z_n, I_v)) = n - 2$$

Proof: Consider a graph $G^+(Z_n, I_v)$ with vertex set $Z_n = \{0, 1, 2, 3, \dots, n - 1\}$ where I_v denotes the set of involutory elements in Z_n .

Let n be even and $n > 2$. $|E| = n$ or $2n$

If $|E| = n$, graph is a Hamilton cycle. If $|E| = 2n$, graph contains two Hamilton cycles, each contains n number of edges. From first Hamilton cycle, consider $F = \{e_i/e_i \neq e_1, e_2\}$.

Now every edge in $E - F$ is adjacent to atleast one edge in F , F is induced subgraph of $G^+(Z_n, I_v)$ and connected. Then F becomes connected edge dominating set of $G^+(Z_n, I_v)$ and it is minimum. Therefore $|F| = |E| - 2$ ($E =$ set of edges in one path) $= n - 2$.

Hence, $\gamma'_c(G^+(Z_n, I_v)) = n - 2$.

Theorem 2.2: For the involutory addition cayley graph $G^+(Z_n, I_v)$, if n is odd, $n > 3$ then the connected edge domination number is

$$\gamma'_c(G^+(Z_n, I_v)) = n - 3$$

Proof: Consider a graph $G^+(Z_n, I_v)$ with vertex set $Z_n = \{0, 1, 2, 3, \dots, n - 1\}$ where I_v denotes the set of involutory elements in Z_n .

Let n be odd, $n > 3$. $|E| = n - 1$ or $n - 2$.

If $|E| = n - 1$, graph is a path. If $|E| = 2n - 2$, graph contains two paths and each path contains $n - 1$ number of edges. From first path, consider $F = \{e_i/e_i \neq e_1, e_n\}$.

Now every edge in $E - F$ is adjacent to at least one edge in F , F is induced subgraph of $G^+(Z_n, I_v)$ and connected. Then F becomes connected edge dominating set of $G^+(Z_n, I_v)$ and it is minimum. Therefore $|F| = |E| - 2$ ($E =$ set of edges in one path) $= n - 3$.

Hence $\gamma'_c(G^+(Z_n, I_v)) = n - 3$

3. ENTIRE DOMINATION

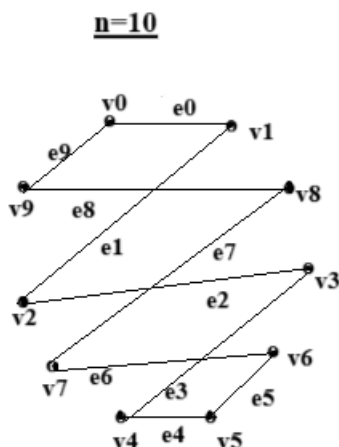
The term domination was first used by Ore[4]. Cockayne et al[2] discussed the review of results and applications concerning dominating sets in graphs. Dominating sets of edges were studied by Mitchell and Hedetniemi[3]. Entire domination was introduced by Kulli[6]. Kulli Singarkanti and Soner established a relationship between the domination, edge domination and entire domination number. The vertices and edges of a graph G are called the elements of G . Let x and y be two elements of G . We say x dominates y if $x = y$ or if x and y are adjacent or incident. Thus a vertex v of G is said to dominate itself, all vertices adjacent to v

and all edges incident to it. Similarly, an edge e of G dominates itself, the two end vertices of e and all edges adjacent to e .

A set S of elements of G is an entire dominating set if every element not in S is either adjacent or incident to atleast one element in S . Equivalently, a set S of elements of G is an entire dominating set if each element in G is dominated by some element in S . The entire domination number $\gamma_{en}(G)$ is the cardinality of a smallest entire dominating set.

Example:

**Involuntary
addition cayley
graph**



Dominating set = $S_1 \cup S_2$

= $\{e_0, e_5\} \cup \{v_3, v_8\}$

Theorem 3.1: For the involuntary addition cayley graph $G^+(Z_n, I_v)$, if n is even $|I_v| = 2, n > 2$, then the Entire domination number is

$$\gamma_{en}(G^+(Z_n, I_v)) = \begin{cases} \frac{2n}{5} & \text{if } 2n \text{ divisible by } 5 \\ \frac{2n-r}{5} + 1 & \text{if } 2n \text{ not divisible by } 5 \end{cases}$$

where r is the remainder when $2n$ divided by 5.

Proof: Consider a graph $G^+(Z_n, I_v)$ with vertex set $Z_n = \{0,1,2,3, \dots, n-1\}$ where I_v denotes the set of involuntary elements in Z_n and n is even, $n > 2, |e| = n$.

Case 1: if $2n$ divisible by 5 (start Entire dominating set with edge)

Then consider $S = \{e_i, v_j/i = 0,5,10, \dots \text{ and } j = 3,8,13, \dots\}$

Now every edge in $G^+(Z_n, I_v)$ adjacent to atleast one edge in S or incident with atleast one vertex in S also every vertex in $G^+(Z_n, I_v)$ either incident with atleast one edge or adjacent to atleast one vertex in S . Therefore S becomes entire domination set of $G^+(Z_n, I_v)$ and it is minimum.

$$|S| = \frac{|E| + |V|}{5} = \frac{n + n}{5} = \frac{2n}{5}$$

Therefore $\gamma_{en}(G^+(Z_n, I_v)) = \frac{2n}{5}$

Case 2: If $2n$ not divisible by 5

Then consider $S_1 = \{e_i, v_j/i = 0,5,10, \dots \text{ and } j = 3,8,13, \dots\}$

Take $S = S_1 \cup (G - S_1)$. Now every edge in $G^+(Z_n, I_v)$ adjacent to atleast one edge in S or incident with atleast one vertex in S also every vertex in $G^+(Z_n, I_v)$ either incident with atleast one edge or adjacent to atleast one vertex in S . Therefore S becomes entire domination set of $G^+(Z_n, I_v)$ and it is minimum.

Consider r is the remainder when $2n$ divisible by 5

$$|S| = \frac{|E| + |V| - r}{5} + 1 = \frac{n + n - r}{5} + 1 = \frac{2n - r}{5} + 1.$$

Therefore $\gamma_{en}(G^+(Z_n, I_v)) = \frac{2n-r}{5} + 1$

Theorem 3.2: For the involutory addition cayley graph $G^+(Z_n, I_v)$, if n is odd $|I_v| = 2$,

$n > 2$ then the entire domination number

$$\gamma_{en}(G^+(Z_n, I_v)) = \begin{cases} \frac{2n - 1}{5} & \text{if } 2n - 1 \text{ divisible by } 5 \\ \frac{2n - r + 4}{5} & \text{if } 2n - 1 \text{ not divisible by } 5 \end{cases}$$

where r is the remainder when $2n - 1$ divided by 5.

Proof: Consider a graph $G^+(Z_n, I_v)$ with vertex set $Z_n = \{0,1,2,3, \dots, n - 1\}$ where I_v denotes the set of involutory elements in Z_n and n is odd, $n > 2$, $|e| = n - 1$

Case 1: If $2n - 1$ divisible by 5. Consider $S = \{e_i, v_j/i = 0,5,10, \dots \text{ and } j = 3,8,13, \dots\}$.

Now every edge in $G^+(Z_n, I_v)$ adjacent to atleast one edge in S or incident with atleast one vertex in S also every vertex in $G^+(Z_n, I_v)$ either incident with atleast one edge or adjacent to atleast one vertex in S . Therefore S becomes entire domination set of $G^+(Z_n, I_v)$ and it is minimum.

$$|S| = \frac{|E| + |V|}{5} = \frac{n + n - 1}{5} = \frac{2n - 1}{5}.$$

Therefore $\gamma_{en}(G^+(Z_n, I_v)) = \frac{2n-1}{5}$

Case 2: If $2n - 1$ not divisible by 5. Then consider $S_1 = \{e_i, v_j / i = 0, 5, 10, \dots \text{ and } j = 3, 8, 13, \dots\}$. Take $S = S_1 \cup (G - S_1)$. Now every edge in $G^+(Z_n, I_v)$ adjacent to atleast one edge in S or incident with atleast one vertex in S also every vertex in $G^+(Z_n, I_v)$ either incident with atleast one edge or adjacent to atleast one vertex in S

Therefore S becomes entire domination set of $G^+(Z_n, I_v)$ and it is minimum.

$$|S| = \frac{|E| + |V|}{5} + 1 = \frac{(n + n - 1) - r}{5} + 1 = \frac{2n - r - 1}{5} + 1 = \frac{2n - r + 4}{5}.$$

Therefore $\gamma_{en}(G^+(Z_n, I_v)) = \frac{2n-r+4}{5}$

Theorem 3.3: For the involutory addition cayley graph $G^+(Z_n, I_v)$, if n is even and $n > 2, |I_v| = 4, n = 2^q, q = 3, 4, 5, \dots$ then the entire domination number is

$$\gamma_{en}(G^+(Z_n, I_v)) = \frac{n}{2}.$$

Proof: Consider a graph $G^+(Z_n, I_v)$ with vertex set $Z_n = \{0, 1, 2, 3, \dots, n - 1\}$ where I_v denotes the set of involutory elements in Z_n and n is even, $n > 2, |e| = 2n$.

$G^+(Z_n, I_v)$ contains two Hamiltonian cycles. From first Hamiltonian cycle.

Then consider $S = \{e_{ij} / 0 \leq i \leq \frac{n}{2} - 1 \text{ and } \frac{n}{2} \leq j \leq n - 1\}$. Now every edge in $G^+(Z_n, I_v)$ adjacent to atleast one edge in S or incident with atleast one vertex in S also every vertex in $G^+(Z_n, I_v)$ either incident with atleast one edge or adjacent to atleast one vertex in S .

Therefore S becomes entire domination set of $G^+(Z_n, I_v)$ and it is minimum, $|S| = \frac{|V|}{2} = \frac{n}{2}$

Therefore $\gamma_{en}(G^+(Z_n, I_v)) = \frac{n}{2}$.

Theorem 3.4: For the involutory addition cayley graph $G^+(Z_n, I_v)$, if n is even and $n > 2, |I_v| = 4, n = 4p, p > 2, p$ is prime then the entire domination number is

$$\gamma_{en}(G^+(Z_n, I_v)) = \begin{cases} \frac{n}{2} & \text{for } p = 3 \\ \frac{2n}{5} & \text{if } 2n \text{ divisible by } 5 (p > 3) \\ \frac{2n - r + 5}{5} & \text{if } 2n \text{ not divisible by } 5 (p > 3) \end{cases}$$

where r is the remainder when $2n$ divided by 5.

Proof: Consider a graph $G^+(Z_n, I_v)$ with vertex set $Z_n = \{0, 1, 2, 3, \dots, n - 1\}$ where I_v denotes the set of involutory elements in Z_n and n is even, $n > 2, |I_v| = 4, n = 4p, p > 2, p$ is prime $|E| = 2n$. $G^+(Z_n, I_v)$ contains two Hamiltonian cycles, each Hamiltonian cycle contains n number of elements.

Case 1: From first Hamiltonian cycle, for $p = 3$. Then consider $S = \left\{ \begin{matrix} e_{ij} \leq i \leq \frac{n}{2} - 1 \\ \text{and } \frac{n}{2} \leq j \leq n - 1 \end{matrix} \right\}$. Now every edge in $G^+(Z_n, I_v)$ adjacent to atleast one edge in S or incident with atleast one vertex in S also every vertex in $G^+(Z_n, I_v)$ either incident with atleast one edge or adjacent to atleast one vertex in S . Therefore S becomes entire domination set of $G^+(Z_n, I_v)$ and it is minimum, $|S| = \frac{|V|}{2} = \frac{n}{2}$. Therefore $\gamma_{en}(G^+(Z_n, I_v)) = \frac{n}{2}$

Case 2: From first Hamiltonian cycle, for $p > 3$, If $2n$ divisible by 5. Then consider $S = \{e_i, v_j / i = 0, 5, 10, \dots \text{ and } j = 3, 8, 13, \dots\}$. Now every edge in $G^+(Z_n, I_v)$ adjacent to atleast one edge in S or incident with atleast one vertex in S also every vertex in $G^+(Z_n, I_v)$ either incident with atleast one edge or adjacent to atleast one vertex in S . Therefore S becomes entire domination set of $G^+(Z_n, I_v)$ and it is minimum,

$$|S| = \frac{|E|}{2} + |V| = \frac{2n}{2} + n = \frac{n + n}{5} = \frac{2n}{5}$$

Therefore $\gamma_{en}(G^+(Z_n, I_v)) = \frac{2n}{5}$

Case 3: From first Hamiltonian cycle, for $p > 3$, If $2n$ not divisible by 5. Then consider

$$S_1 = \{e_i, v_j / i = 0, 5, 10, \dots \text{ and } j = 3, 8, 13, \dots\}$$

Take $S = S_1 \cup (G - S_1)$. Now every edge in $G^+(Z_n, I_v)$ adjacent to atleast one edge in S or incident with atleast one vertex in S also every vertex in $G^+(Z_n, I_v)$ either incident with atleast one edge or adjacent to atleast one vertex in S . Therefore S becomes entire domination set of $G^+(Z_n, I_v)$ and it is minimum,

$$|S| = \frac{\left(\frac{|E|}{2} + |V|\right) - r}{5} + 1 = \frac{\left(\frac{2n}{2} + n\right) - r}{5} + 1 = \frac{(n + n) - r}{5} + 1 = \frac{2n - r + 5}{5}.$$

Therefore $\gamma_{en}(G^+(Z_n, I_v)) = \frac{2n-r+5}{5}$

Theorem 3.5: For the involutory addition cayley graph $G^+(Z_n, I_v)$, if n is odd and $n > 2, |I_v| = 4, n = 3p, p > 3$ then the entire domination number is

$$\gamma_{en}(G^+(Z_n, I_v)) = \frac{n + 1}{2}.$$

Proof: Consider a graph $G^+(Z_n, I_v)$ with vertex set $Z_n = \{0,1,2,3, \dots, n - 1\}$ where I_v denotes the set of involutory elements in Z_n and n is odd, $n > 2, |I_v| = 4, n = 3p, p > 3, p$ is prime $|e| = 2n - 2$. $G^+(Z_n, I_v)$ contains two paths and each path contains $n - 1$ number of edges. From first Hamiltonian cycle, consider $S = \left\{e_{ij}, \frac{v_k}{0} \leq i \leq \frac{n-3}{2}, \frac{n+1}{2} \leq j \leq n - 1 \text{ and } k = \frac{n-1}{2}\right\}$. Now every edge in $G^+(Z_n, I_v)$ adjacent to atleast one edge in S or incident with atleast one vertex in S also every vertex in $G^+(Z_n, I_v)$ either incident with atleast one edge or adjacent to atleast one vertex in S . Therefore S becomes entire domination set of $G^+(Z_n, I_v)$ and it is minimum.

$$|S| = \frac{|V| - 1}{2} + 1 = \frac{n - 1}{2} + 1 = \frac{n + 1}{2}.$$

Therefore $\gamma_{en}(G^+(Z_n, I_v)) = \frac{n+1}{2}$

CONCLUSION

Using involutory addition cayley graphs, it is interesting to find the Connected edge domination number and Entire domination number of involutory addition cayley graphs and the authors have also studied this aspect.

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