

A Sustainable Two Warehouse Capacities based Inventory Model with Exponential Demand Pattern for Deteriorating Items Under Learning Effect

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Abstract

Present paper deals a sustainable two warehouse based inventory mathematical model with exponential type of demand pattern for deteriorating items under effect of the learning where the deterioration rate follows the linear pattern and some carbon units emit from the electricity for the preserving of the deteriorating items.

Two warehouse inventory models for degrading products are covered in this study. Physical products deterioration is a crucial component of every manufacturing and inventory system. Its influence can be disregarded if the pace of deterioration is relatively modest. On the other hand, degradation is a significant factor in many real-world scenarios. For the contemporary organisation, it is critical to manage and preserve the inventories of deteriorating assets. When degradation is time-dependent, the product's value decreases proportionately with time. Modellers were inspired to take the degradation factor into account as one of the modelling features upon understanding this factor. Deterioration rate is considered time dependent in this work. We take exponential demand rate into consideration in order to examine actual business scenarios and the learning effect involves in the holding cost, deterioration cost and ordering cost. The carbon emission cost also included in this study. We presented the numerical example for the justification of the model in this study, the cost reduction approach is applied.

Keywords: Inventory, Two warehouses, exponential demand, learning effect.

1. INTRODUCTION

The assumption of a constant demand rate, which served as the foundation for the 1915 development of the conventional economic order quantity (EOQ) model, was widely seen as being highly restricted and impractical. Actually, the continual demand assumption is only true up until the point of maturity. Inventory modellers gradually began to grasp the idea of time-varying needs. A uniform variation in the demand rate per unit of time is implied by a linear time changing demand rate. The EOQ model was only partially expanded by researchers to time-varying demand patterns. Donaldson (1977) developed a computer approach for

determining the best reorder times with a linear demand trend over a finite planning horizon using elementary mathematics. Using the Silver–Meal heuristic (1969), Silver (1979) created an approximate solution approach for a linearly time-varying demand. A deterministic lot-size inventory model with shortages and a linear demand trend was put out by Dave in 1989. Goswami and Chaudhuri (1991) talked about a variety of inventory models that included a demand curve. Hill (1995) initially addressed the lack of discipline in the time-dependent demand pattern by seeing each order cycle as the result of the fusion of two distinct time periods, during which the demand is disciplined. Hariga (1995) examined how an inventory model with a time-dependent demand rate and shortages was affected by inflation and the time value of money.

Deterministic models of perishable inventory with stock-dependent demand rate and nonlinear holding cost were presented by Bhunia A. K. and Maiti M. in 1998. A damageable item inventory with a variable replenishment rate and deterministic demand was examined by Mandal and Maiti (1997). One of the most pressing problems in the manufacturing and sale of goods is the control of inventory. Many companies fail each year due to the lack of adequate control of the inventories. Whether it is raw materials used to manufacture a product or products waiting to be sold, problems arise when too few or too many items are held in inventory. Inventory control techniques are very helpful in the development of inventory policies and in activity down the investment cost of inventories. Inventory control techniques not only help in minimizing the capital tied up in inventory but also provide the required service level. Study of deteriorating inventory model began with Ghare and Schrader (1963) who established the classical no shortage inventory model for constant rate of decay. Covert and Philip (1973) extended the above inventory model with a two-parameter Weibull distribution.

An early discussion on an inventory model with two storage facilities is given by Hartely (1976). It is generally assumed that the holding cost in the RW is greater than the same in the OW. Hence, the items are stored first in the OW, and only excess of stock is stored in the RW. Further, the items of the RW are released first, and then the items of the OW. Dave and Patel (1981) considered an inventory model for deteriorating items with time proportional demand. Sarma (1983) developed a deterministic inventory model with finite replenishment rate. Sachan (1984) extended the model of Dave and Patel (1981) by allowing shortages. Dave (1988) further discussed the cases of bulk release pattern for both finite and infinite replenishments. He rectified the errors in Murdeshwar and Sathe (1985) and gives a complete solution for the model given by Sarma (1983). In the above literature, deterioration phenomenon was not taken into account. Assuming the deterioration in both warehouses, Sarma (1987), extended his earlier model to the case of infinite replenishment rate with shortages.

A declining inventory model with stock-dependent demand and partial backlog was created by Dye (2002). An order level decaying inventory model with such time dependent quadratic demand was examined by Khanra and Chaudhuri (2003). An inventory model for degrading products with stock-dependent and time-varying demand rates over a constrained

planning horizon was created by Balkhi & Benkherouf (2004). Stochastic lead times were taken into account in a manufacturing/remanufacturing system with predictable requests and returns by Zhou Y. W. and Yang S. (2005). Hou (2006) examined an inflation-adjusted inventory model for degrading goods with stock-dependent demand. There was a full backorder for the shortage. But in many real-world scenarios, the longer the waiting period is during a shortage period, the lower the backlog rate would be. Fuzzy-based optimization methods combine fuzzy logic with optimization processes. This helps handle uncertainty, vague information, and imprecise data in decision-making. These techniques use fuzzy sets and fuzzy numbers to represent uncertain data, which makes decision-making more flexible and effective in uncertain situations Kumar et al. (2023), Malik et al. (2012), Singh et al. (2014), Tyagi et al. (2023), Verma et al. (2022), Yadav and Malik (2014), Yadav et al. (2022).

An EOQ model with fluctuating demand and deteriorating Weibull distribution was examined by Panda et al. in 2007. An inventory problem for non-instantaneously degrading commodities with stock-dependent demand was described by Ouyang L. Y. (2008) when the provider offered an all-unit quantity discount. Jayaswal et al. (2022) described a learning based inventory model for imperfect quality based inventory model under fuzzy environment. Alsaedi et al. (2023) presented carbon emissions based inventory model with fuzzy environment under learning effect. Jayaswal et al. (2021) generalised an inventory model with the effect of fuzzy under human learning. Deterministic developed an inventory model in this study with two warehousing facilities, degrading products under shortages, and deterministic and exponential demand. The two distinct warehouses—one that is owned and the other that is rented—are thought to contain a single degrading object. The surplus units beyond the fixed capacity of W units of the OW are stored in the rented warehouse. Deterioration rates for both owned and rented warehouses are considered to be discrete, variable functions of time.

2. NOTATION AND ASSUMPTIONS

2.1. Assumptions

The following assumption forms the basis of the single period inventory problem's mathematical model:

1. There is a set, restricted capacity of W units in the owned warehouse.
2. There is no limit to the size of the rented warehouse.
3. When RW is empty, OW things begin to be consumed.
4. Demand is time-dependent and predictable, involving D units per unit of time. evenly during the cycle duration T .
5. The expenses associated with inventory in RW, including storage and degradation costs, are greater than those seen in OW

6. There is no limit to the pace of replenishment, and it happens instantly.
7. If there are any backorders for excess demand at OW, shortages are allowed.
8. There is no lead time (zero).
9. In this scenario, the taxation policy is allowed.
10. The exchange policies of perishable items are not permitted.
11. The repair of the items is not allowed.
12. It is considered that carbon emissions come from the burning of the fuel.
13. It is also assumed, the deteriorating cost, holding cost (OW and RW) and ordering cost follow the effect of learning.
14. Shortages are allowed.
15. Preservation technology is also considered.

2.2 Notations

In this paper, the notation used is as follows:

$D = Ae^{Bt}$ Any given time t , the demand rate, where A and B are positive constants.

$\alpha + at$, Inventory item degradation rate in OW where $0 < \alpha < 1, a > 0$

$\beta + bt$, Rate of decline of inventory item in RW where $b > 0$ and $0 < \beta < 1$.

W , The OW's maximum fixed storage capacity is $W < S$.

T , whole preparation period.

SL_1 , $\{i: \text{Cycle } i\}$ uses both OW and RW.

SL_2 , $\{j: \text{In cycle } j\}$, just OW is utilised.

C_{ow} , carrying cost in OW for each inventory unit.

C_{RW} , carrying cost in RW for each inventory unit.

S , The highest possible stock level at OW and RW.

C_o , The cost of replenishment for each order.

C_d , decline Price per unit.

$I_o(t)$, At time t , the inventory level in OW.

$I_r(t)$, The RW value of the inventory at time t .

C_s , Cost of shortage per unit of time.

$$K = K_o + \frac{K_1}{n^s}, \quad \text{Ordering cost for the retailer;}$$

K_o, K_1 = fixed and variable ordering cost;

s = Learning factor;

n = Number of shipment

$$D_d = D_1 + \frac{D_2}{n^s}, \text{The cost per unit item for the deterioration in OW/RW;}$$

D_o, D_2 , Fixed and variable ordering cost;

$$H_r = H_1 + \frac{H_2}{n^s} \text{The cost per unit item for the storage of items in RW;}$$

H_1, H_2 , Fixed and variable storage cost in RW;

$$H_o = H_3 + \frac{H_4}{n^s}, \quad \text{The cost per unit item for the storage of items in OW;}$$

H_3, H_4 , Fixed and variable storage cost in OW;

c_1 Average vehicle fuel consumption when it is empty

F_e Carbon emissions due to electricity

T_x Carbon taxation

3. MODEL DISCRIPTION

We talk about the Deterministic Inventory Model for SL2 (system), at time $t = 0$, a lot size of S units, decaying with two warehouses where shortages happen at the conclusion of cycle. SL2, where W and $S-W$ units are stored in OW and RW, respectively. The products of OW are only used up when RW runs out. The inventory $S-W$ in RW declines as a result of degradation and demand throughout the time interval $[0, t_1]$, and it disappears at $t = t_1$ in OW. Only degradation causes the inventory W to drop, but only during $[t_1, t_2]$. Deterioration and demand both contribute to the inventory's depletion. The inventory in OW approaches zero at time $t = t_2$. At the start of the following cycle, the client is meant to receive the shortfall quantity.

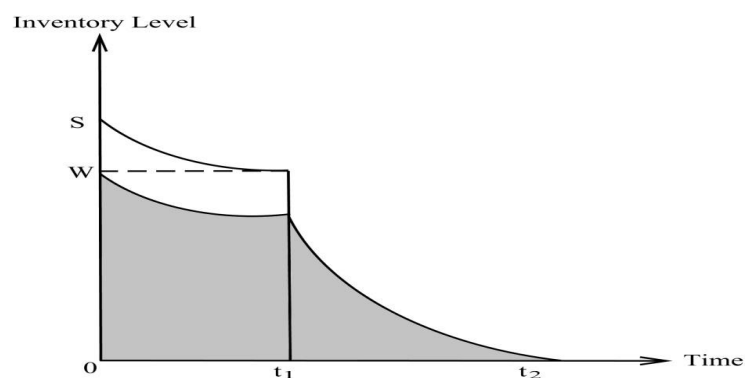


Figure 1: Representation of the inventory system under warehouse system

Determining the timings of t_1, t_2 , in order to maintain the lowest feasible inventory level in terms of total relevant cost per unit of time is the aim of the suggested inventory system I2.

$$\frac{dI_r(t)}{dt} + I_r(t)(\beta + bt) = -Ae^{Bt}, \quad 0 \leq t \leq t_1 \quad \dots(1)$$

$$\frac{dI_o(t)}{dt} + I_o(t)(\alpha + at) = 0, \quad 0 \leq t \leq t_1 \quad \dots(2)$$

$$\frac{dI_o(t)}{dt} + I_o(t)(\alpha + at) = -Pe^{Qt}, \quad t_1 \leq t \leq t_2 \quad \dots(3)$$

$$I_r(t_1) = 0, \quad I_o(t_1) = W, \quad I_o(t_2) = 0 \quad \{\text{Boundary Condition}\}$$

$$I_r(t) = A \left[t_1 - t + t_1^2 \left(\frac{B + \alpha}{2} \right) + \frac{bt_1^3}{2} + t^2 \left(\frac{\alpha - B}{2} \right) + t^3 \left(\frac{2b + 3B\alpha + 3\alpha^2}{6} \right) \right. \\ \left. + t^4 \left(\frac{3b\alpha + 5b\alpha}{12} \right) - \left(\frac{b^2t^5}{12} \right) - btt_1 - tt_1^2 \left(\frac{B\alpha + \alpha^2}{2} \right) - t^2t_1^2 \left(\frac{bB + b\alpha}{4} \right) \right. \\ \left. - \frac{b}{2}t^2t_1 - \frac{b^2t^2t_1^3}{12} - \frac{batt_1^3}{6} \right]$$

In the RW, the commutative inventory from 0 to t_1 is provided by

$$I_r(0, t_1) = P \int_0^{t_1} \left[t_1 + t + t_1^2 \left(\frac{B + \alpha}{2} \right) + \frac{bt_1^3}{2} + t^2 \left(\frac{\alpha - B}{2} \right) + t^3 \left(\frac{2b + 3B\alpha + 3\alpha^2}{6} \right) \right. \\ \left. + t^4 \left(\frac{3b\alpha + 5b\alpha}{12} \right) - \left(\frac{b^2t^5}{12} \right) - btt_1 - tt_1^2 \left(\frac{B\alpha + \alpha^2}{2} \right) - t^2t_1^2 \left(\frac{bB + b\alpha}{4} \right) \right. \\ \left. - \frac{b}{2}t^2t_1 - \frac{b^2t^2t_1^3}{12} - \frac{batt_1^3}{6} \right] dt$$

$$I_r(0, t_1) = P \left[\frac{t_1^2}{2} + t_1^3 \left(\frac{2B + \alpha}{6} \right) + t_1^4 \left(\frac{2b + 3B\alpha + 3\alpha^2}{24} \right) - t_1^5 \left(\frac{2b\alpha + 5b\alpha}{60} \right) - \frac{b^2t_1^6}{72} \right] \quad \dots(4)$$

From the equation (2), we calculated the inventory stock for the rented warehouse

$$I_r(t) = We^{-\gamma t - \frac{ct^2}{2}} \quad \dots(5)$$

Similarly, we calculated the inventory stock, from the equation (3)

$$I_o(t) = A \left[t_2 - t + t_2^2 \left(\frac{\gamma + B}{2} \right) + \frac{ct_2^3}{6} + t^2 \left(\frac{\gamma - B}{2} \right) + t^3 \left(\frac{2c + 3B\gamma + 3\gamma^2}{6} \right) + t^4 \left(\frac{3cB + 5c\gamma}{12} \right) \right. \\ \left. + \left(\frac{c^2t^5}{12} \right) - \gamma tt_2 - tt_2^2 \left(\frac{\gamma B + \gamma^2}{2} \right) - t^2t_2^2 \left(\frac{cB + c\gamma}{4} \right) - \frac{c}{2}t^2t_2 - \frac{c^2t^2t_2^3}{12} - \frac{c\gamma tt_2^3}{6} \right] \quad \dots(6)$$

For the time interval $[0, t_2]$, the commutative inventory in OW is provided by

$$I_o(0, t_2) = \int_0^{t_1} I_o(t) dt + \int_{t_1}^{t_2} I_o(t) dt \quad (\text{From the equation (6)})$$

$$\begin{aligned}
 I_0(0, t_2) = & W \left(t_1 - \frac{\gamma t_1^2}{2} - \frac{ct_1^3}{6} \right) + A \left[-t_1 t_2 + \left(\frac{t_1^2 + t_2^2}{2} \right) + \left(\frac{\gamma + B}{2} \right) (t_2^3 - t_1 t_2^2) + \right. \\
 & \frac{c}{6} (t_2^4 - t_1 t_2^3) + \left(\frac{\gamma - B}{6} \right) (t_2^3 - t_1^3) + \left(\frac{2c + 3cB + 3\gamma^2}{24} \right) (t_2^4 - t_1^4) + \left(\frac{3cB + 5c\gamma}{60} \right) (t_2^5 - t_1^5) + \\
 & \frac{c^2}{72} (t_2^6 - t_1^6) - \frac{\gamma}{2} (t_2^3 - t_2 t_1^2) - \left(\frac{\gamma B + \gamma^2}{4} \right) (t_2^4 - t_1^2 t_2^2) - \frac{\gamma c}{12} (t_2^5 - t_1^2 t_2^3) - \frac{c}{6} (t_2^4 - t_2 t_1^3) - \\
 & \left. \frac{c^2}{36} (t_2^6 - t_1^3 t_2^3) - \left(\frac{\gamma c + cB}{12} \right) (t_2^5 - t_1^3 t_2^2) \right] \dots (7)
 \end{aligned}$$

Continuity of $I_0(t)$ at time $t = t_1$, if follows that:

$$\begin{aligned}
 I_0(t_1) = & W e^{-\gamma t_1 - \frac{ct_1^2}{2}} \\
 = & A \left[t_2 - t_1 + t_2^2 \left(\frac{\gamma + B}{2} \right) + \frac{ct_2^3}{6} + t_1^2 \left(\frac{\gamma - B}{2} \right) + t_1^3 \left(\frac{2c + 3B\gamma + 3\gamma^2}{6} \right) \right. \\
 & + t_1^4 \left(\frac{3cB + 5c\gamma}{12} \right) + \left(\frac{c^2 t_1^5}{12} \right) - \gamma t_1 t_2 - t_1 t_2^2 \left(\frac{\gamma B + \gamma^2}{2} \right) \\
 & \left. - t_1^2 t_2^2 \left(\frac{cB + c\gamma}{4} \right) - \frac{a}{2} t_1^2 t_2 - \frac{c^2 t_1^2 t_2^3}{12} - \frac{c\gamma t_1 t_2^3}{6} \right] \\
 W \left(1 - \gamma t_1 - \frac{ct_1^2}{2} \right) = & A \left[t_2 - t_1 + t_2^2 \left(\frac{\gamma + B}{2} \right) + \frac{ct_2^3}{6} + t_1^2 \left(\frac{\gamma - B}{2} \right) + t_1^3 \left(\frac{2c + 3B\gamma + 3\gamma^2}{6} \right) \right. \\
 & + t_1^4 \left(\frac{3cB + 5c\gamma}{12} \right) + \left(\frac{c^2 t_1^5}{12} \right) - \gamma t_1 t_2 - t_1 t_2^2 \left(\frac{\gamma B + \gamma^2}{2} \right) \\
 & \left. - t_1^2 t_2^2 \left(\frac{cB + c\gamma}{4} \right) - \frac{c}{2} t_1^2 t_2 - \frac{c^2 t_1^2 t_2^3}{12} - \frac{c\gamma t_1 t_2^3}{6} \right]
 \end{aligned}$$

...(8)

The following items have deteriorated in both RW and OW:

$$D_r = I_r(0) - Dt_1$$

$$\begin{aligned}
 D_r = & A \left[\frac{t_1^2}{2} + t_1^3 \left(\frac{2B + \alpha}{6} \right) + t_1^4 \left(\frac{2b - 3\alpha^2 - 3B\alpha}{24} \right) - t_1^5 \left(\frac{2bB + 5b\alpha}{60} \right) - \frac{b^2 t_1^6}{72} \right] \\
 & - A e^{Bt_1} t_1
 \end{aligned}$$

$$\begin{aligned}
 = & A \left[\frac{t_1^2}{2} + t_1^3 \left(\frac{2B + \beta}{6} \right) + t_1^4 \left(\frac{2b - 3\alpha^2 - 3B\alpha}{24} \right) - t_1^5 \left(\frac{2bB + 5b\alpha}{60} \right) - \frac{b^2 t_1^6}{72} \right. \\
 & \left. - t_1 (1 + Bt_1) \right]
 \end{aligned}$$

$$= A \left[-t_1 + t_1^2 \left(\frac{1 - 2B}{2} \right) + t_1^3 \left(\frac{2B + \alpha}{6} \right) + t_1^4 \left(\frac{2b - 3\alpha^2 - 3B\alpha}{24} \right) - t_1^5 \left(\frac{2bB + 5b\alpha}{60} \right) - \frac{b^2 t_1^6}{72} \right] \dots (9)$$

$$D_0 = I_0(0) - D(t_2 - t_1)$$

$$\begin{aligned}
 &= W - Ae^{B(t_2-t_1)}(t_2-t_1) \\
 &= I_0(t_1)e^{\gamma t_1 + \frac{ct_1^2}{2}} - A(t_2-t_1)(1 + Bt_2 - Bt_1) \\
 &= A \left(1 + \gamma t_1 + \frac{ct_1^2}{2} \right) \left[t_2 - t_1 + t_2^2 \left(\frac{\gamma+B}{2} \right) + \frac{ct_2^3}{6} + t_1^2 \left(\frac{\gamma-B}{2} \right) + t_1^3 \left(\frac{2c+3B\gamma+3\gamma^2}{6} \right) + \right. \\
 &t_1^4 \left(\frac{3cB+5c\gamma}{12} \right) + \left. \left(\frac{c^2 t_1^5}{12} \right) - \gamma t_1 t_2 - t_1 t_2^2 \left(\frac{\gamma+B+\gamma^2}{2} \right) - t_1^2 t_2^2 \left(\frac{cB+c\gamma}{4} \right) - \frac{c}{2} t_1^2 t_2 - \frac{c^2 t_1^2 t_2^3}{12} - \right. \\
 &\left. \frac{c\gamma t_1 t_2^3}{6} \right] - A(t_2-t_1)(1 + Bt_2 - Bt_1) \quad \dots(10)
 \end{aligned}$$

Total carbon emission cost $CE = c_1 F_x T_x \quad \dots(11)$

Total ordering cost for this scenario $K = K_0 + \frac{K_1}{n^s} \quad \dots (12)$

As a result, the inventory system's overall average relevant cost per unit of time (including holding and deterioration costs) is as follows.

$$TC = \frac{1}{T} [\text{Replenishment Cost} + \text{Holding Cost} + \text{Deterioration Cost} + \text{carbon emission cost}]$$

$$= \frac{1}{T} [K + H_o I_0(o, t_2) + H_R I_r(o, t_1) + C_d(D_r + D_o) + CE]$$

$$\begin{aligned}
 TC &= \frac{1}{T} \left(\left(K_o + \frac{K_1}{n^s} \right) + \left(H_2 + \frac{H_3}{n^s} \right) \left\{ w \left(t_1 - \frac{\gamma t_1^2}{2} - \frac{ct_1^3}{6} \right) + A \left[-t_1 t_2 + \left(\frac{t_1^2 + t_2^2}{2} \right) + \right. \right. \\
 &\left. \left(\frac{\gamma+B}{2} \right) (t_2^3 - t_1 t_2^2) + \frac{c}{6} (t_2^4 - t_1 t_2^3) + \left(\frac{\gamma-B}{6} \right) (t_2^3 - t_1^3) + \left(\frac{2c+3B\gamma+3\gamma^2}{24} \right) (t_2^4 - t_1^4) + \right. \\
 &\left. \left(\frac{3cB+5c\gamma}{60} \right) (t_2^5 - t_1^5) + \left(\frac{c^2}{72} \right) (t_2^6 - t_1^6) - \frac{\gamma}{2} (t_2^3 - t_1^2 t_2) - \left(\frac{\gamma B + \gamma^2}{4} \right) (t_2^4 - t_1^2 t_2^2) - \right. \\
 &\left. \frac{\gamma c}{12} (t_2^5 - t_1^2 t_2^3) - \frac{c}{6} (t_2^4 - t_1^3 t_2) - \frac{c^2}{36} (t_2^6 - t_1^3 t_2^3) - \left(\frac{cB+c\gamma}{12} \right) (t_2^5 - t_1^3 t_2^2) \right\} + \\
 &\left(H_1 + \frac{H_2}{n^s} \right) A \left\{ \frac{t_1^2}{2} + t_1^3 \left(\frac{2B+\alpha}{6} \right) + t_1^4 \left(\frac{2b-3\alpha^2-3B\alpha}{24} \right) - t_1^5 \left(\frac{2bB+5b\alpha}{60} \right) - \frac{b^2 t_1^6}{72} \right\} + \\
 &\left(D_o + \frac{D_1}{n^s} \right) A \left\{ -t_1 + t_1^2 \left(\frac{1-2B}{2} \right) + t_1^3 \left(\frac{2B+\alpha}{6} \right) + t_1^4 \left(\frac{2B-3\alpha^2-3B\alpha}{24} \right) - t_1^5 \left(\frac{2bB+5b\alpha}{60} \right) - \right. \\
 &\left. \frac{b^2 t_1^6}{72} \right\} + C_d \left\{ P \left(1 + \gamma t_1 + \frac{\gamma t_1^2}{2} \right) \left[t_2 - t_1 + t_2^2 \left(\frac{\gamma+B}{2} \right) + \frac{ct_2^3}{6} + t_1^2 \left(\frac{\gamma-B}{2} \right) + t_1^3 \left(\frac{2c+3B\gamma+3\gamma^2}{6} \right) + \right. \right. \\
 &t_1^4 \left(\frac{3cB+5c\gamma}{12} \right) + \left. \left(\frac{c^2 t_1^5}{12} \right) - \gamma t_1 t_2 - t_1 t_2^2 \left(\frac{\gamma+B+\gamma^2}{2} \right) - t_1^2 t_2^2 \left(\frac{cB+c\gamma}{4} \right) - \frac{c}{2} t_1^2 t_2 - \frac{c^2 t_1^2 t_2^3}{12} - \right. \\
 &\left. \left. \frac{c\gamma t_1 t_2^3}{6} \right] - A(t_2-t_1) - AB(t_2-t_1)^2 \right\} + c_1 F_x T_x \quad \dots(13)
 \end{aligned}$$

3.1 Approximate Solution Procedure:

In order to minimize the total relevant cost per unit time, the following equation can be solved to determine the approximate optimal values of t_1 and t_2 , represented by t_1^* and t_2^* .

$$\frac{\partial TC}{\partial t_1} = 0 \text{ and } \frac{\partial TC}{\partial t_2} = 0, \text{ and at } T = 1, \frac{\partial^2 TC}{\partial t_1^2} \Big|_{(t_1^*, t_2^*)} > 0, \frac{\partial^2 TC}{\partial t_2^2} \Big|_{(t_1^*, t_2^*)} > 0 \text{ and also we}$$

have to show that for the convexity of the total cost

$$\left[\left(\frac{\partial^2 TC}{\partial t_1^2} \right) \left(\frac{\partial^2 TC}{\partial t_2^2} \right) - \frac{\partial^4 TC}{\partial t_1^2 \partial t_2^2} \right] > 0 \text{ at } (t_1^*, t_2^*).$$

Now, $\frac{\partial TC}{\partial t_1} = 0$ reduces to $T = 1$. From the equation (13)

$$\begin{aligned} \frac{\partial TC}{\partial t_1} = & \left| \left(H_2 + \frac{H_3}{n^s} \right) \left\{ W \left(1 - \gamma t_1 - \frac{c t_1^2}{2} \right) \right. \right. \\ & - A \left[t_2 - t_1 + t_2^2 \left(\frac{\gamma + B}{2} \right) + \frac{c t_2^3}{6} + t_1^2 \left(\frac{\gamma - B}{2} \right) + t_1^3 \left(\frac{2c + 3B\gamma + 3\gamma^2}{6} \right) \right. \\ & + t_1^4 \left(\frac{3cB + 5c\gamma}{12} \right) + \left(\frac{c^2 t_1^5}{12} \right) - \gamma t_1 t_2 - t_1 t_2^2 \left(\frac{\gamma B + \gamma^2}{2} \right) \\ & \left. \left. - t_1^2 t_2^2 \left(\frac{cB + c\gamma}{4} \right) - \frac{c}{2} t_1^2 t_2 - \frac{c^2 t_1^2 t_2^3}{12} - \frac{c\gamma t_1 t_2^3}{6} \right] \right\} \\ & + \left(H_1 + \frac{H_2}{n^s} \right) P \left\{ t_1 + t_1^2 \left(\frac{2B + \alpha}{2} \right) + t_1^3 \left(\frac{2b - 3\alpha^2 - 3B\alpha}{6} \right) \right. \\ & \left. - t_1^4 \left(\frac{2bB + 5b\alpha}{12} \right) - \frac{b^2 t_1^5}{12} \right\} \\ & + \left(D_o + \frac{D_1}{n^s} \right) \left\{ -1 + t_1(1 - 2B) + t_1^2 \left(\frac{2B + \alpha}{2} \right) + t_1^3 \left(\frac{2b - 3\alpha^2 - 3B\alpha}{6} \right) \right. \\ & \left. - t_1^4 \left(\frac{2bB + 5b\alpha}{12} \right) - \frac{b^2 t_1^5}{12} \right\} \\ & + C_d \left\{ P \left(1 + \gamma t_1 + \frac{c t_1^2}{2} \right) \left[-1 + t_1(\gamma - B) + t_1^2 \left(\frac{2c + 3B\gamma + 3\gamma^2}{2} \right) + t_1^3 \left(\frac{3cB + 5c\gamma}{3} \right) \right. \right. \\ & + \left(\frac{5c^2 t_1^4}{12} \right) - \alpha t_2 - t_2^2 \left(\frac{\gamma B + \gamma^2}{2} \right) - t_1 t_2^2 \left(\frac{cB + c\gamma}{2} \right) - \frac{c}{2} t_1^2 t_2 - \frac{c^2 t_1^2 t_2^3}{6} \\ & \left. \left. - \frac{c\gamma t_2^3}{6} \right] \right\} \\ & + P(\gamma \\ & + c t_1) \left[t_2 - t_1 + t_2^2 \left(\frac{\gamma + B}{2} \right) + \frac{c t_2^3}{6} + t_1^2 \left(\frac{\gamma - B}{2} \right) + t_1^3 \left(\frac{2c + 3B\gamma + 3\gamma^2}{6} \right) \right. \\ & + t_1^4 \left(\frac{3cB + 5c\gamma}{12} \right) + \left(\frac{c^2 t_1^5}{12} \right) - \gamma t_1 t_2 - t_1 t_2^2 \left(\frac{\gamma B + \gamma^2}{2} \right) - t_1^2 t_2^2 \left(\frac{cB + c\gamma}{4} \right) \\ & \left. \left. - \frac{c}{2} t_1^2 t_2 - \frac{c^2 t_1^2 t_2^3}{12} - \frac{c\gamma t_1 t_2^3}{6} \right] + P + 2PB(t_2 - t_1) \right| = 0 \end{aligned}$$

We can determine the values of t_1 denoted by t_1^* , now, we have

$$\begin{aligned} \frac{\partial TC}{\partial t_1} = 0 = & A \left(H_2 + \frac{H_3}{n^s} \right) \left\{ -t_1 + t_2 + \left(\frac{\gamma + B}{2} \right) (3t_2^2 - 2t_1 t_2) + \frac{c}{6} (4t_2^3 - 3t_1 t_2^3) + \right. \\ & \left(\frac{\gamma - B}{2} \right) t_2^2 + t_2^3 \left(\frac{2c + 3B\gamma + 3\gamma^2}{6} \right) + t_2^4 \left(\frac{3cB + 5c\gamma}{12} \right) + \left(\frac{c^2 t_2^5}{12} \right) - \frac{\gamma}{2} (3t_2^2 - t_1^2) - \left(\frac{\gamma B + \gamma^2}{2} \right) (4t_2^3 - \\ & 2t_1^2 t_2) - \frac{\gamma c}{12} (5t_2^4 - 3t_1^2 t_2^2) - \frac{c}{6} (4t_2^3 - t_1^3) - \frac{c^2}{36} (6t_2^5 - 3t_1^3 t_2^2) - \left(\frac{cB + c\gamma}{12} \right) (5t_2^4 - \end{aligned}$$

$$2t_1^3 t_2) \} + C_d \left\{ P \left(1 + \gamma t_1 + \frac{ct_1^2}{2} \right) \left[1 + t_2(\gamma + B) + \frac{ct_2^2}{2} - \gamma t_1 - t_1 t_2(\gamma B + \gamma^2) - \frac{c\gamma t_1 t_2^2}{2} - \frac{ct_1^2}{2} - \frac{c^2 t_1^2 t_2^2}{4} - t_1^2 t_2 \left(\frac{cB + c\gamma}{2} \right) \right] - P - 2AB(t_2 + t_1) \right\} = 0$$

3.2 Numerical Example

Table 1: Model's inventory parameters and decision variable

Inventory parameters	Numerical values of the inventory parameters
K_o	400\$ per order
k_1	100\$ per order
H_3	0.70 \$ per unit item
H_4	0.05 \$ per unit item
H_1	0.35 per unit
H_2	0.05 per unit\$,
n	10
s	0.23
D_o	0.5 \$ per unit item
D_1	0.04 per unit\$,
α	0.06
β	0.07
A	80
B	90
T_2	1.09 year
c_1	0.27
T_x	\$75 per ton CO2
F_x	2.6×10^{-3} ton CO2
t_1^*	0.79 year
t_2^*	0.89 year
$TC(t_1^*, t_2^*)$	5672\$

3.3 Sensitivity Analysis

In this part, we discussed the behaviour of the inventory input on the decision variables for the future purpose in the field of the industrial sector;

Table 2: Impact of the shipment on the total inventory cost

n	t_1^* (Year)	t_2^* (Year)	$TC(t_1^*, t_2^*)$ (\$)
1	0.63	0.74	6503

2	0.64	0.73	6302
3	0.68	0.76	6154
4	0.75	0.84	5934
5	0.79	0.89	5672
6	0.79	0.89	5672

Table3: Impact of the learning rate on the total inventory cost

s	t_1^* (Year)	t_2^* (Year)	$TC(t_1^*, t_2^*)$ (\$)
0.62	0.59	0.67	6504
0.64	0.62	0.74	6309
0.67	0.65	0.78	6266
0.69	0.66	0.83	5906
0.75	0.76	0.89	5672
0.79	0.79	0.89	5672

Table4: Impact of the carbon taxation on the total inventory cost

T_x	t_1^* (Year)	t_2^* (Year)	$TC(t_1^*, t_2^*)$ (\$)
75	0.79	0.89	5672
76	0.75	0.83	6508
77	0.73	0.80	6876
78	0.70	0.78	7143

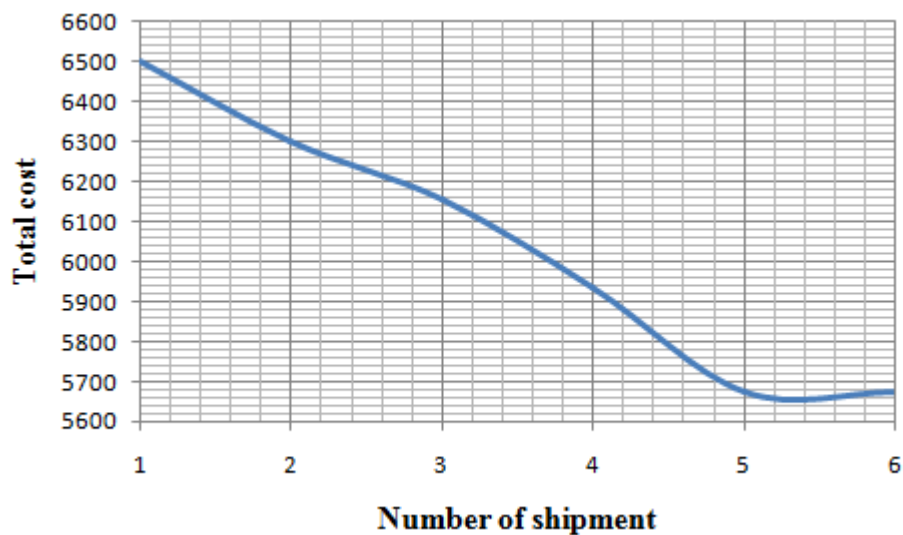


Figure 2: Effect of the shipment on the total cost

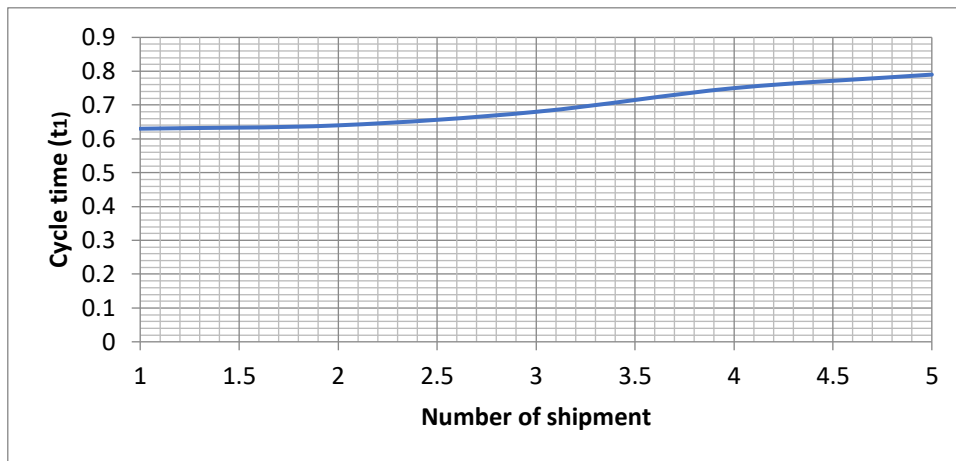


Figure 3: Effect of the shipment on the cycle time (t_1)

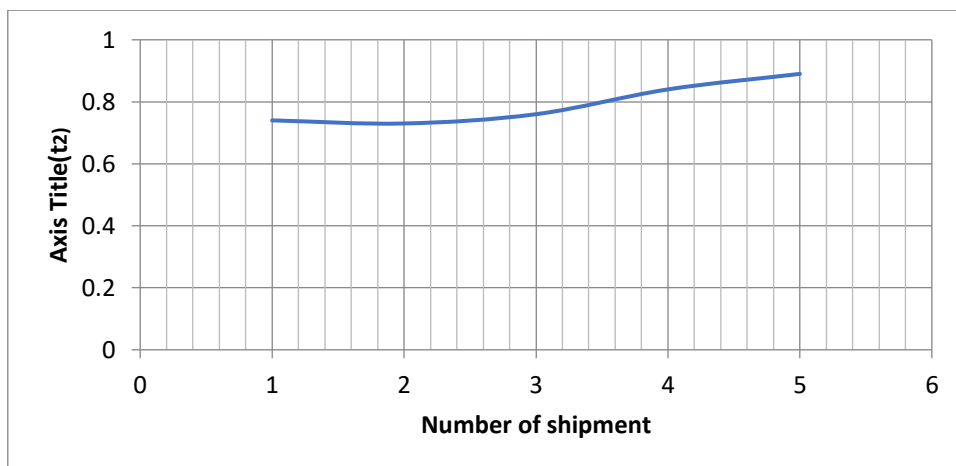


Figure 4: Effect of the shipment on the cycle time (t_2)

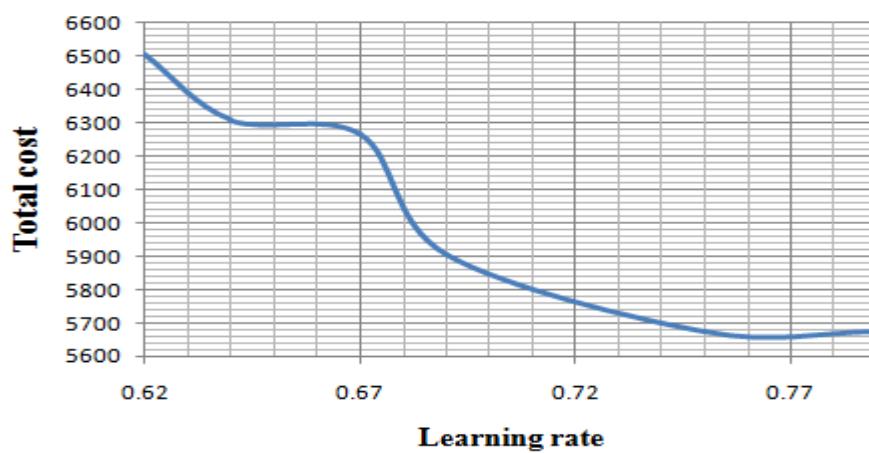


Figure 5: Effect of the learning rate on the total cost

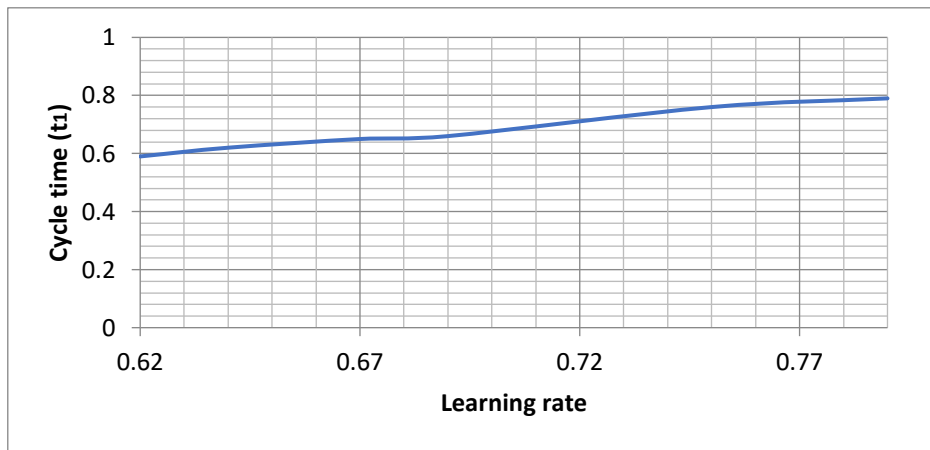


Figure 6: Effect of the learning rate on the cycle time(t_1)

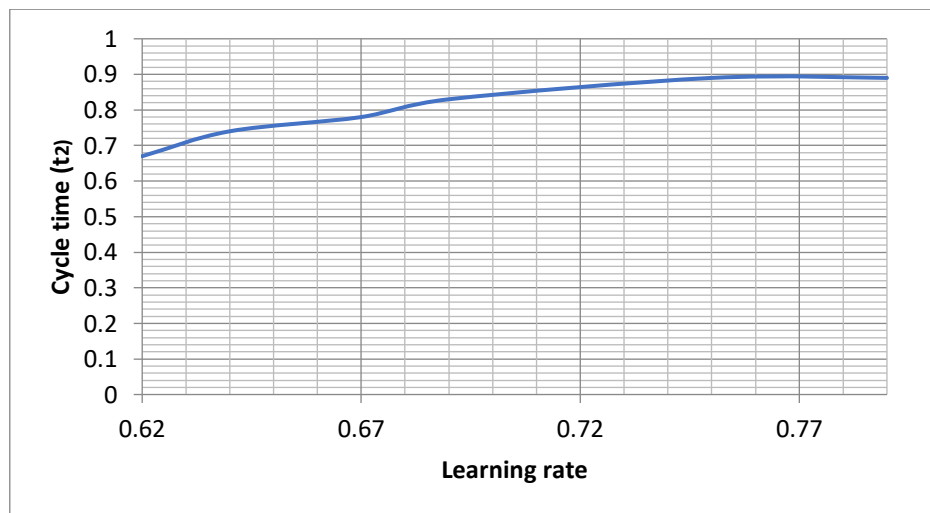


Figure 7: Effect of the learning rate on the cycle time(t_2)

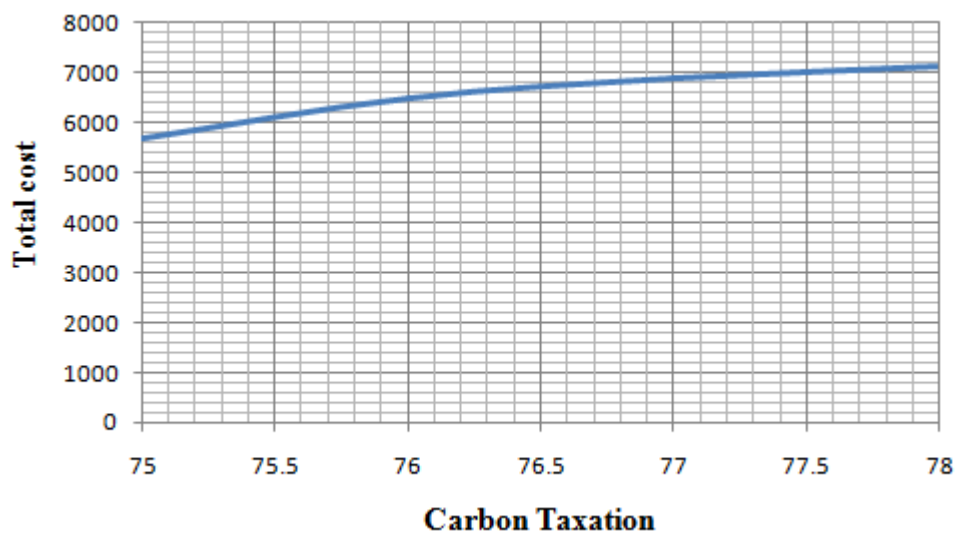


Figure 8: Effect of the carbon taxation on the total cost

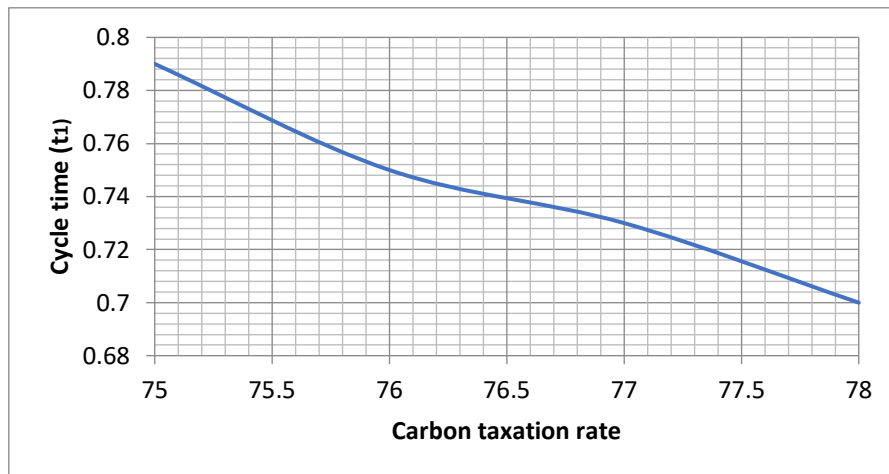


Figure 9: Effect of the carbon taxation on the cycle time (t_1)

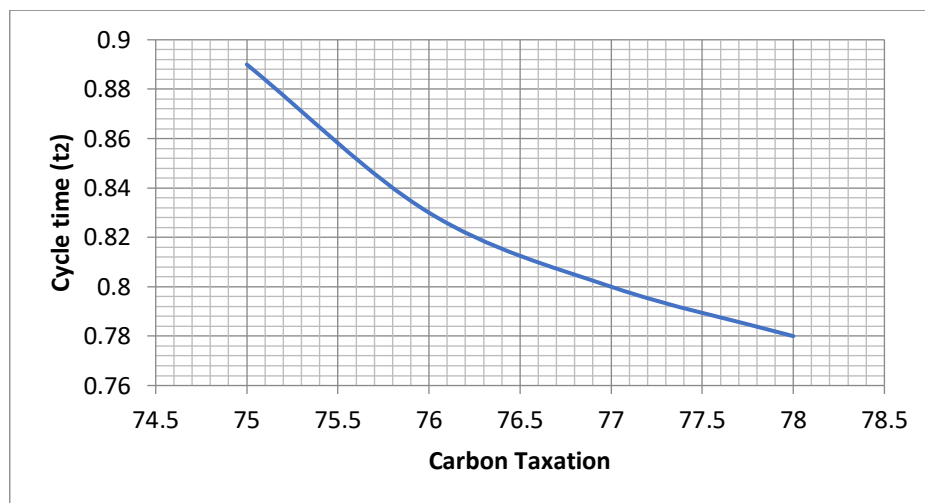


Figure 10: Effect of the carbon taxation on the cycle time (t_2)

Observation and managerial insights

- **Effect of the shipment**

We observed that from the table-2, if the number of shipment increases then the values of t_1^* and t_2^* increase but total cost decreases from 1 to 5, consequently, the buyer gets more information for the exercise of shipment and also presented the Figure 2, 3 and 4.

- **Effect of learning effect**

The role of the learning is more effective in this scenario and it minimizes the total inventory cost. From the table -3, if the learning rate increases then t_1^* and t_2^* increase but total inventory cost decreases and after some more shipment, it is constant due to learning in holding cost, ordering cost and deteriorating cost. The property of the deterioration of the product need to know to the seller and buyer during the leading of the business. The graphical represents in the Figure 5,6 and 7.

- **Effect of carbon tax**

From the table -4, if the carbon taxation increases then t_1^* and t_2^* decrease but total inventory cost increase because carbon taxation is also one type of the penalty for more carbon emits and represents in the Figure 8,9 and 10.

4. Conclusion

In actuality, items that are maintained in stock may lose value or function over time. Therefore, the impact of inventory model deterioration cannot be disregarded. Demand has been considered as an exponentially growing function of time when discussing the inventory problem. The exponentially growing demand determines the replenishment rate. We first build two potential storage models in the suggested model, assuming that the demand rate is predictable. Production does not take the inventory level in OW into consideration, which is W and S of RW. After the RW level is depleted, the consumer is served by OW. Considering that deterioration is also taken into account in both warehouses. The current approach works with recently launched products because, if clients are happy with the product's quality and pricing, demand for these items rises quickly. Finally examined the effect of learning and carbon emission on the variable decisions and total inventory cost and other input variable gave positive effect on the decision variable. We might expand the models created in this paper for next study by adding more practical scenarios like multiple goods and quantity discount regulations as well as fuzzy environment.

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