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The Normalization of Complex Intuitionistic Fuzzy Matrices

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Abstract: The concept of intuitionistic fuzzy sets (IFS) provides a comprehensive framework for dealing with uncertainty, incorporating both membership and non-membership functions. Normalization of intuitionistic fuzzy matrices is an essential process in decision-making and data analysis where the matrix entries are expressed in terms of intuitionistic fuzzy numbers (IFNs). This paper explores the methods and techniques for normalizing intuitionistic fuzzy matrices to ensure that the degree of membership and non-membership values are consistent, thus enhancing the reliability of the matrix in decision analysis problems. We propose an efficient approach to normalize intuitionistic fuzzy matrices, ensuring the transformed values retain their essential characteristics while adhering to the constraints of the fuzzy set theory. The study also addresses the application of normalized intuitionistic fuzzy matrices in multi-criteria decision-making (MCDM) and other relevant areas such as image processing, pattern recognition, and system modelling. Several illustrative examples are provided to demonstrate the effectiveness of the proposed normalization techniques

Keywords: Complex Intuitionistic Fuzzy Sets, Fuzzy Matrices, Normalization Techniques, Membership and Non-membership Functions, Fuzzy Logic, Matrix Normalization, Intuitionistic Fuzzy Logic.

1. Introduction: Atanassov. K [1], discussed "Intuitionistic fuzzy sets", Fuzzy Sets and Systems in this year 1986. Ahmad B [2], discussed on fuzzy soft sets, Advances in Fuzzy Systems in this year 2009. Barbacioru I. C [3], discussed "Multiplication operation on intuitionistic fuzzy numbers" in this year 2018. Babitha K.V. [4], discussed Generalized Intuitionistic fuzzy soft sets and its applications in this year 2011. Chetia.B [5], discussed some result of intuitionistic fuzzy soft matrix theory in this year 2012.Cagman. N [6] discussed fuzzy soft matrix theory and its application in decision making, International Journal of Fuzzy Systems in this year 2012.Djatna. T [7] discussed "An intuitionistic fuzzy diagnosis analytics for stroke disease", Journal of Big Data, in this year 2018.Dubois. D [8], discussed Fuzzy Sets and Systems: Theory and Applications in this year1980. Emam E.G [9], discussed "Intuitionistic circular bifuzzy matrices "in this year 2017. 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Objective: The objective of normalization of Intuitionistic Fuzzy Matrices (IFMs) is to transform the matrix elements, which represent uncertain or vague information, into a standardized form while preserving the relationships and inherent structure of the fuzzy data. The purpose of normalization is to ensure that the elements in the matrix are comparable and consistent, allowing for more accurate and reliable decision-making or analysis

2.Preliminary

2.1Definition of Fuzzy: Fuzzy refers to something that is unclear, imprecise, or ambiguous. It often describes situations where boundaries or definitions are not strictly defined, leading to a certain level of vagueness. In various fields, "fuzzy" can have different connotations, but the core idea revolves around uncertainty or a lack of exact precision.

1.Physical/Visual Context: When something is blurry, soft, or lacking sharp details, it is often described as fuzzy. For example, "The picture is fuzzy" means the image isn't clear.

2.Metaphorical/Conceptual Context: It can refer to something vague, ambiguous, or hard to understand.

For example, "His explanation was fuzzy" means the explanation was unclear or imprecise. A fuzzy set can be represented in a matrix form, where the membership values of elements are arranged in rows and columns. Each element in the matrix corresponds to a specific item, and the values in the matrix represent the degree of membership of that item in the fuzzy set.

2.2 Examples 2.2.1. Temperature (Hot vs. Cold): Set Description: We can define a fuzzy set for temperature, where the degree of membership represents how hot or cold a temperature is,

Example: Consider the fuzzy set "Hot temperatures" where:
0°C → Membership degree of 0 (completely cold)
20°C → Membership degree of 0.2 (slightly hot)
30°C → Membership degree of 0.5 (moderately hot)
40°C → Membership degree of 0.8 (very hot)
50°C → Membership degree of 1 (completely hot)
2.2.2. Height (Tall vs. Short): Set Description: A fuzzy set can be defined for the height of a person, with the degree of membership indicating how tall someone is, Example: Consider the fuzzy set "Tall people"

where 150 cm → Membership degree of 0.2 (slightly tall)

160 cm → Membership degree of 0.4 (moderately tall)

170 cm → Membership degree of 0.7 (fairly tall)

180 cm → Membership degree of 1 (completely tall)

2.3 Application of Fuzzy

2.3.1. In mathematics and logic: In fuzzy logic, values are not limited to just true or false (1 or 0), but can take on any value between 0 and 1, reflecting degrees of truth. For example, instead of saying "a person is tall" as either true or false, fuzzy logic might say they are 0.7 tall (indicating they are somewhat tall but not extremely so).
2.3.2 In computing: A fuzzy search allows for approximate matches rather than exact ones. For example, searching for "appl" might also return "apple" or "applause" because the system is designed to tolerate small errors in spelling or typing.

2.3.3. Computer Science and Mathematics: In traditional binary logic, something is either true or false (1 or 0), but in fuzzy logic, truth values can be any number between 0 and 1, representing degrees of truth.

Example: A thermostat that adjusts the temperature gradually based on a range, not just switching between "on" and "off." If the target temperature is 70°F, and the room temperature is 68°F, the thermostat may turn on at 50% power instead of 100%.

2.3.4. Technology: In searching algorithms, fuzzy search allows for matching terms that are similar but not identical. This can be used when users make spelling errors or variations. Example: Searching for "apple" could also return results for "apple," "appl," or "apples."

2.3.5. Fuzzy Concepts (Philosophy and Linguistics): A fuzzy concept refers to something that doesn’t have clear-cut boundaries or definitions. It’s often subjective. Example: The term "tall" is fuzzy because what one person considers tall (e.g., 6 feet) might be different from another person's perception (e.g., 5'10").

2.3.6. Fuzzy Logic in Everyday Life: Using fuzzy reasoning in everyday decisions, where things aren't just black and white. Example: Deciding whether to go outside when the weather is "a little rainy" is a fuzzy decision it's not a simple "yes" or "no" answer, but something in between based on personal comfort.

2.4 Fuzzy Matrix

2.4.1Definition of fuzzy matrix: A fuzzy matrix is a mathematical structure used to represent data where the relationships between elements are not precisely defined, but instead are characterized by degrees of membership. This concept stems from fuzzy set theory, which allows for partial membership in a set rather than a strict binary classification (where something is either in the set or not).

In a fuzzy matrix, each element in the matrix represents a relationship or association between two entities, and the value in the matrix is a fuzzy number (typically ranging from 0 to 1) that indicates the degree of association. A value of 0 might mean "no association," while a value of 1 could represent "full association," and values in between indicate partial associations.

Membership Degrees: Instead of having binary values (0 or 1), fuzzy matrices contain values between 0 and 1, representing the degree of membership.

Symmetry: In some applications, the fuzzy matrix can be symmetric, meaning that if element A is related to element BBB to some degree, element BBB is related to element A to the same degree.

Application: Fuzzy matrices are widely used in decision-making processes, pattern recognition, image processing, and systems where uncertainty or vagueness in data exists.

2.5 Examples of Fuzzy matrix: A fuzzy matrix is a matrix in which the elements are fuzzy values, typically represented by fuzzy numbers or membership functions. Fuzzy matrices are used in various fields, such as fuzzy logic, fuzzy systems, and decision-making processes, to handle imprecision and uncertainty. Let's consider a fuzzy matrix representing the relationship between three factors (A, B, C) and their membership values. Here, each element in the matrix could represent a degree of relationship or membership function value ranging from 0 to 1.

Fuzzy Matrix: $\begin{bmatrix} 0.8 & 0.6 & 0.4 \\ 0.5 & 0.7 & 0.9 \\ 0.3 & 0.6 & 0.8 \end{bmatrix}$

In this matrix, the value 0.8 in the first row and first column indicates a high degree of membership or relationship between factor A and itself.

The value 0.6 in the first row and second column represents a medium degree of membership between factor A and B.

The value 0.4 in the first row and third column indicates a low degree of relationship between A and C. Each element in the matrix is a fuzzy value between 0 and 1, Where 0 means no membership or no relationship.
1 means full membership or a strong relationship.

2.5.1. Basic fuzzy Matrix (Binary Fuzzy Matrix): The matrix represents relationships between elements using values in [0,1]

$$A=\begin{bmatrix} 0.8 & 0.4 & 0.6 \\ 0.3 & 1.0 & 0.7 \\ 0.5 & 0.2 & 0.9 \end{bmatrix}$$

where a_{ij} represents the degree of membership or confidence in a relationship between elements. 2.5.2 Fuzzy Adjacency Matrix: In a fuzzy graph , has three nodes with fuzzy edges weights ,its adjacency matrix may look like:

$$M=\begin{bmatrix} 0 & 0.7 & 0.5 \\ 0.7 & 0 & 0.8 \\ 0.5 & 0.8 & 0 \end{bmatrix}$$

where M_{ij} represents the fuzzy weight of the edge between nodes i and j

2.5.3. Fuzzy Relation Matrix: A fuzzy matrix showing relationships between elements of two sets

$$R=\begin{bmatrix} 0.9 & 0.6 & 0.3 \\ 0.5 & 0.8 & 0.7 \\ 0.4 & 0.2 & 0.1 \end{bmatrix}$$

where represents the fuzzy relationships strength between elements of two sets

2.6 Fuzzy matrix application: A fuzzy matrix is a generalization of a traditional matrix that deals with fuzzy relations and fuzzy logic. In a fuzzy matrix, elements represent degrees of membership to certain sets, usually expressed as values in the range [0, 1], where 0 means no membership, 1 means full membership, and intermediate values represent partial membership. This concept is widely applied in fields where uncertainty, vagueness, or imprecision is present. Below are some applications of fuzzy matrices: 2.6.1 Fuzzy Logic Systems: Fuzzy matrices are used in fuzzy logic systems for decision-making, where each element in the matrix represents the degree of truth for a given fuzzy relationship between variables. For example, in a fuzzy control system for a thermostat, the fuzzy matrix can represent rules and conditions like "temperature is somewhat high," "humidity is low," etc., and the matrix helps to decide the appropriate output (e.g., turn on the air conditioning).

2.6.2 Fuzzy Graph Theory: In fuzzy graphs, the adjacency matrix can have values between 0 and 1, indicating the degree of connection between two nodes. This approach is useful in networks where relationships between nodes are not binary, such as in social networks, recommendation systems, or transportation networks. It allows the representation of relationships that aren't just "connected" or "not connected" but instead have varying strengths.

2.6.3 Fuzzy Relational Databases: Fuzzy matrices can be used in fuzzy relational databases where database relationships (such as queries or connections between tables) are not precise but instead have degrees of certainty. For instance, in a database of products, a fuzzy matrix could represent the degree of similarity between different items, which can help with fuzzy searching or recommendation engines. 2.6.4 Fuzzy Control Systems: In fuzzy control, fuzzy matrices are often used to represent the relationship between input and output variables in fuzzy inference systems (FIS). For example, in controlling the speed of a fan based on temperature readings, a fuzzy matrix might define how the degree of "high temperature" should correlate to the fan's speed in a fuzzy way.

2.6.5 Pattern Recognition and Classification: Fuzzy matrices are employed in fuzzy pattern recognition to classify patterns when there is ambiguity or overlap between classes. A fuzzy matrix can represent the degree to which a certain input belongs to multiple classes simultaneously. This is useful in applications like handwriting recognition, image segmentation, and speech recognition, where categories are not perfectly distinct.

2.6. 6 Fuzzy Decision Making: In multi-criteria decision analysis (MCDA), fuzzy matrices can represent the degrees of importance or preference for various criteria under uncertain conditions. For example, in selecting a vendor for a project, decision-makers might use a fuzzy matrix to rank multiple options based on subjective and vague criteria like "trustworthiness" or "reliability," which are not easily quantifiable.

2.6.7 Fuzzy Systems in AI: Fuzzy matrices can be used in AI-based decision support systems, where the relationships between inputs and outputs are not deterministic. They allow AI systems to handle uncertainty and vagueness, which is essential when processing real-world data, such as in natural language processing or autonomous driving.

2.6. 8 Fuzzy Optimization Problems: In optimization problems, fuzzy matrices can be used to represent uncertainty in constraints or objective functions. This is particularly useful in cases where the optimization criteria are vague or approximate, such as in supply chain management, where costs and demand are uncertain, and solutions need to be found that work well under a range of possible conditions

2.6.9 Image Processing: Fuzzy matrices are often applied in fuzzy image processing, where an image's pixel intensities are considered fuzzy rather than precise. This can help with tasks like image denoising, edge detection, or segmentation, where the boundaries between objects in an image are not perfectly clear. 2.6.10 Fuzzy Clustering: In fuzzy clustering algorithms (e.g., fuzzy c-means), fuzzy matrices represent the degree of membership of data points in different clusters. This allows for the classification of points that belong partially to multiple clusters, which is useful in situations like market segmentation, medical diagnosis, and customer behaviour analysis.

3.Intuitionistic Fuzzy Matrix

3.1Definition of intuitionistic fuzzy matrix: An intuitionistic fuzzy matrix A of order m × r is defined as: A=$[(\mu_{ij},\vartheta_{ij})]m$×n where, $\mu_{ij}$is the membership degree of the element (i, j), representing the degree of belonging to a set. ϑ_{ij} is the non - membership degree, representing the degree of not belonging to a set. For every elements (i ,j the sum of membership and non-membership degree satisfies: $0 \leq \mu_{ij} + \vartheta_{ij} \leq 1 for\ all\ i,$

The uncertainty degree (hesitancy)is given by($\mu_{ij} + \vartheta_{ij}$) which quantifies the hesitation or lack of complete knowledge about the membership status.

3.2 Examples of intuitionistic fuzzy matrix:

Intuitionistic fuzzy sets (IFS) extend traditional fuzzy sets by incorporating both membership and non-membership functions, as well as a degree of hesitation. In an intuitionistic fuzzy matrix, each element has three components:
Membership degree (μ): The degree to which an element belongs to the set.
Non-membership degree (ν): The degree to which an element does not belong to the set.

Hesitation degree (π): The degree of uncertainty or hesitation, calculated as π=1−μ−ν

1.A 2×2 Intuitionistic fuzzy matrix: A=$\begin{bmatrix} (0.7, & 0.2) & (0.5 & 0.3) \\ (0.6 & 0.1) & (0.4 & 0.4) \end{bmatrix}$

Membership=0.7, non –membership=0.2 and hesitation degree=1-(0.7+0.2) =0.1

2. A 2×3 Intuitionistic fuzzy matrix: B=$\begin{bmatrix} \langle 0.8 & 0.1\rangle & \langle 0.6 & 0.2\rangle & \langle 0.5 & 0.3\rangle \\ \langle 0.7 & 0.2\rangle & \langle 0.4 & 0.4\rangle & \langle 0.3 & 0.5\rangle \end{bmatrix}$

Membership=0.3, non –membership=0.5 and hesitation degree=1-(0.3+0.5) =0.2

3.A 3×3 Intuitionistic fuzzy matrix: C=$\begin{bmatrix} \langle 0.6 & 0.3\rangle & \langle 0.7 & 0.2\rangle & \langle 0.5 & 0.4\rangle \\ \langle 0.8 & 0.1\rangle & \langle 0.4 & 0.5\rangle & \langle 0.6 & 0.3\rangle \\ \langle 0.3 & 0.6\rangle & \langle 0.5 & 0.3\rangle & \langle 0.7 & 0.2\rangle \end{bmatrix}$

Membership=0.3, non–membership=0.5 and hesitation degree=1-(0.3+0.5) =0.2

3.3 Application of intuitionistic fuzzy matrix:

3.3.1 Decision-Making Problems (Multi-Criteria Decision Analysis): Intuitionistic fuzzy matrices are often used in decision-making problems where decisions depend on multiple criteria, and there is uncertainty in the assessment of each criterion.

Example: In a decision-making process for selecting the best supplier from a set of alternatives, each criterion (e.g., cost, quality, delivery time) may have uncertainty. Intuitionistic fuzzy matrices help incorporate both membership (degree of suitability) and non-membership (degree of unsuitability) values into the decision matrix.

3.3.2 Pattern Recognition: In pattern recognition, intuitionistic fuzzy matrices can be used to represent the degree of similarity and dissimilarity between patterns, making it easier to classify patterns with uncertainties. Example:
In image processing or speech recognition, patterns may not always be perfectly clear. Intuitionistic fuzzy matrices provide a way to handle imprecise or incomplete data, allowing more flexible and accurate classification.

3.3.3 Data Mining and Clustering: Intuitionistic fuzzy clustering algorithms, where each data point belongs to multiple clusters with varying degrees of membership and non-membership, can benefit from intuitionistic fuzzy matrices.
Example: In customer segmentation, instead of classifying customers strictly into one segment, intuitionistic fuzzy clustering allows a customer to belong to multiple segments with different degrees of membership and non-membership, representing their preferences or behaviours more flexibly.

3.3.4 Control Systems: In fuzzy control systems, where control rules are designed based on uncertain or imprecise data, intuitionistic fuzzy matrices help incorporate the uncertainty of the system's parameters. Example:
In temperature control of a system, intuitionistic fuzzy matrices can help model the uncertainty in sensor readings and adjust the control system's response accordingly.

3.3.5 Optimization Problem: In optimization problems with imprecise constraints or objective functions, intuitionistic fuzzy matrices can be used to represent the uncertainty in the data, allowing for more robust optimization results.

Example: In supply chain optimization, the cost or demand may have uncertainty. Using intuitionistic fuzzy matrices to model these uncertainties helps in obtaining optimal solutions under varying levels of confidence.

3.3.6 Fault Diagnosis and Reliability Analysis: Intuitionistic fuzzy matrices are useful in fault diagnosis systems where the exact cause of a fault is not immediately clear, and multiple possible faults need to be considered with varying levels of confidence.

Example: In industrial systems, intuitionistic fuzzy matrices can help in analysing machine faults by considering various possible failure modes, each with a degree of membership (how likely the fault is) and non-membership (how unlikely it is).

3.3.7 Expert Systems and Knowledge Representation: Intuitionistic fuzzy matrices can enhance expert systems by handling uncertainty in expert knowledge and decision rules.

Example: In medical diagnosis systems, intuitionistic fuzzy matrices can model the uncertainty in diagnosing a disease based on symptoms and test results, where the certainty of a diagnosis may not be absolute

3.3.8 Recommendation Systems: In recommendation systems, intuitionistic fuzzy matrices can be used to represent the preferences of users and the uncertainty in their preferences for different items or services.

Example: In an e-commerce platform, users may have different levels of preference for products, but these preferences may also be uncertain or ambiguous. Intuitionistic fuzzy matrices can help provide better personalized recommendations by taking into account both positive preferences and the lack of preference.

3.3. 9 Social Network Analysis: In analysing social networks, intuitionistic fuzzy matrices can model relationships between individuals, where relationships may have different levels of strength and uncertainty. Example:
In social media platforms, the relationship strength between two users may be uncertain (not just strong or weak) and could vary over time. Intuitionistic fuzzy matrices help represent this uncertainty in social network analysis.

3.3.10 Medical Diagnostics and Health Monitoring: Intuitionistic fuzzy matrices can be applied in the representation of medical data, where diagnostic results may have both degrees of certainty and uncertainty, especially when dealing with incomplete or ambiguous information.

Example: In health monitoring, an intuitionistic fuzzy matrix can represent various symptoms and their relationships to potential diseases, with both membership and non-membership values indicating the degree of certainty about the presence of a condition.

4 Overview on Normalization and Cartesian product of Intuitionistic Fuzzy Matrices:

4.1Definition:The normalization of an intuitionistic fuzzy matrix (IFM) refers to the process of transforming an intuitionistic fuzzy matrix into a standardized form while preserving its essential characteristics .An intuitionistic fuzzy matrix is a matrix in which each element is represented as an intuitionistic fuzzy number (IFM) ,typically denoted as a pair $(\mu_{ij}, \vartheta_{ij})$, where, $\mu_{ij}$is the membership degree of the element (i, j),representing the degree of belonging to a set. ϑ_{ij} is the non - membership degree, representing the degree of not belonging to a set. The hesitation degree is given by $\pi_{ij} = 1 - (\mu_{ij} + \vartheta_{ij})$

Normalization in this context ensures that the fuzzy matrix values are on the same scale, which is important for:
Reducing the impact of extreme values. Making the matrix more uniform for better comparison and analysis. Ensuring consistency when aggregating or combining intuitionistic fuzzy information across different decision criteria or alternatives.

4.2Normalization Methods for Intuitionistic Fuzzy Matrices: Normalization of IFMs helps to standardize the values of the matrix to a common range, typically between 0 and 1, to ensure that they are comparable across different criteria or decision-making factors. Here are some common normalization methods:
4.2.1 Linear Normalization:
In this method, the values of the membership, non-membership, and uncertainty degrees are normalized based on their maximum and minimum values across the entire matrix. The formula for normalization of an element aij = (μ(x), ν(x), π(x)) is given as

a_{ij}=$\left(\frac{\mu(x)-\mu_{min}}{\mu_{max}-\mu_{min}}, \frac{\nu(x)-\vartheta_{min}}{\vartheta_{max}-\vartheta_{min}}, \frac{\pi(x)-\pi_{min}}{\pi_{max}-\pi_{min}}\right)$Where

μ_{min}, μ_{max} are the minimum and maximum membership values across all elements.

$\vartheta_{min}, \vartheta_{max}$ are the minimum and maximum non-membership values across all elements. π_{min}, π_{max} are the minimum and maximum uncertainty values across all elements.

4.2.2Column-Wise Normalization: In this approach, normalization is done separately for each column of the Intuitionistic Fuzzy Matrix. Each element in a column is normalized by dividing by the sum of the membership, non-membership, and uncertainty values in that column. For an element aij = (μ(x), ν(x), π(x)) the normalization formula is

$a_{ij} = \left(\frac{\mu(x)}{\sum_{k=1}^{m}\mu_{kj}}, \frac{\vartheta(x)}{\sum_{k=1}^{m}\vartheta_{kj}}, \frac{\mu(x)}{\sum_{k=1}^{m}\pi_{kj}}\right)$ Where m is the number of rows in the matrix.

4.3.3 Max-Min Normalization: This method normalizes the elements of the Intuitionistic Fuzzy Matrix using the maximum and minimum values across each row. For an element aij=(μ(x), ν(x), π(x), the normalization is:

$$a_{ij} = \left(\frac{\mu(x)-\min(\mu)}{\max(\mu)-\min(\mu)}, \frac{\vartheta(x)-\min(\vartheta)}{\max(\vartheta)-\min(\vartheta)}, \frac{\pi(x)-\min(\pi)}{\max(\pi)-\min(\pi)}\right)$$

this method uses the maximum and minimum values for each degree across the matrix to scale the values.

4.3.4. Vector –Based Normalization: This method normalizes each element in the matrix by considering the vector magnitude of its components. Given that each element is a triplet, the normalization is based on the euclined norm or magnitude of the triplet vector (μ(x), ν(x), π(x)). The formula for normalization is:

$$a_{ij} = \left(\frac{\mu(x)}{\sqrt{\mu(x)^2+\vartheta(x)^2+\pi(x)^2}}, \frac{\vartheta(x)}{\sqrt{\mu(x)^2+\vartheta(x)^2+\pi(x)^2}}, \frac{\pi(x)}{\sqrt{\mu(x)^2+\vartheta(x)^2+\pi(x)^2}}\right)$$

this method ensures that the sum of the squares of the components of each element in the matrix equals1 effectively standardizing the values.

4.4 Examples: Steps for Normalization of Intuitionistic Fuzzy Matrices:

1.Identify the maximum and minimum values for each component (membership, non-membership, and indeterminacy) across the entire matrix.

2.Normalize each component by applying a normalization function. The most common normalization method is min-max normalization, where the values are mapped to a desired range, typically [0, 1].

For each component x_{ij} of the matrix (whether it's the membership, non-membership, or indeterminacy), the normalized value x'_{ij} can be computed as $= \frac{x_{ij-min(x)}}{\max(x)-\min(x)}$, where min(x) and max(x) are the minimum and maximum values of the respective component across the entire matrix.

3.Adjust the indeterminacy (if required). After normalizing the membership and non-membership components, you may need to adjust the indeterminacy to ensure that the sum of membership, non-membership, and indeterminacy still equals 1. you normalize the membership and non-membership components first; you can calculate the indeterminacy for each element as: $\pi'_{ij} = 1 - \mu'_{ij} - \vartheta'_{ij}$

This ensures that the sum of the normalized membership, non-membership, and indeterminacy remains 1 for each element.

Example of Normalization: Consider an intuitionistic fuzzy matrix with the following entries

$$A = \begin{pmatrix} (0.6,0.2,0.2) & (0.8,0.1,0.1) \\ (0.4,0.3,0.3) & (0.7,0.2,0.1) \end{pmatrix}$$

Step 1: Identify the max and min values for each component.

For membership (μ) : max(μ)=0.8 ,min(μ)=0.4
For non-membership (ϑ) : max(ϑ)=0.3 ,min(ϑ)=0.1
For indeterminacy (π) : max(π)=0.3 ,min(π)=0.1

Step 2: Normalize the membership and non-membership components using min-max normalization.

For each $a_{ij} = (\mu_{ij}, \vartheta_{ij}, \pi_{ij})$,we apply the normalization

for $\mu_{11} = 0.6$: $\mu'_{11} = \frac{0.6-0.4}{0.8-0.4} = \frac{0.2}{0.4} = 0.5$

For $\vartheta_{11} = 0.2$ $\vartheta'_{11} = \frac{0.2-0.1}{0.3-0.1} = \frac{0.1}{0.2} = 0.5$

For $\pi_{11} = 0.2$ $\pi'_{11} = 1 - 0.5 = 0.5 = 0$

Now, apply the same normalization for the other elements of the matrix.

Step 3: Adjust indeterminacy.

After normalizing the membership and non-membership values, the indeterminacy values are recalculated as:

$$\pi'_{ij} = 1 - \mu'_{ij} - \vartheta'_{ij}$$

The final normalized matrix will have the form $A' = \begin{pmatrix} (0.5,0.5\,,0) & (1\,,0\,\,,0) \\ (0,0) & (0.75\,,0.25) \end{pmatrix}$

2.Consider a 2.2 intuitionistic fuzzy matrix $A = \begin{bmatrix} (0.5\,,0.3) & (0.7\,,0.2) \\ (0.4\,,0.4) & (0.6\,,0.1) \end{bmatrix}$

using the Normalization:

$\mu'_{11} = \frac{0.5}{0.5+0.3} = 0.625$ $\vartheta'_{11} = \frac{0.3}{0.5+0.3} = 0.37$

$\mu'_{12} = \frac{0.7}{0.7+0.2} = 0.777$ $\vartheta'_{12} = \frac{0.2}{0.7+0.2} = 0.22$

$\mu'_{21} = \frac{0.4}{0.4+0.4} = 0.5$ $\vartheta'_{21} = \frac{0.4}{0.4+0.4} = 0$

$\mu'_{22} = \frac{0.6}{0.6+0.1} = 0.857$ $\vartheta'_{22} = \frac{0.1}{0.6+0.1} = 0.143$

The normalised matrix becomes $A' = \begin{pmatrix} (0.625\,, 0.375) & (0.777\,, 0.222) \\ (0.5\,, 0.5) & (0.857\,,\ 0.143) \end{pmatrix}$

4.5. Applications of Normalization of Intuitionistic Fuzzy Matrices

1. Multi-Criteria Decision Making (MCDM)
2. Pattern Recognition
3. Machine learning &AI

1. Multi-Criteria Decision Analysis (MCDA): When dealing with multi-criteria decision-making problems, intuitionistic fuzzy matrices allow capturing the hesitancy or uncertainty associated with expert judgments. Normalizing the IFM helps standardize the fuzzy decision criteria to a common scale, making it easier to compare alternatives across different criteria and achieve more reliable ranking. Fuzzy Topsis Method: In methods like TOPSIS (Technique for Order Preference by Similarity to Ideal Solution), normalization of intuitionistic fuzzy matrices allows a consistent comparison of alternatives by transforming the fuzzy values into a comparable scale, reducing inconsistencies and ambiguities in decision-making.

2.Image Processing: Fuzzy Image Segmentation: In image processing, intuitionistic fuzzy matrices can be used to represent uncertain or vague image boundaries. Normalization helps in adjusting pixel intensities and related fuzzy values (membership and non-membership) to ensure proper image segmentation, enhancing the performance of algorithms that rely on fuzzy logic.
3.Pattern Recognition and Classification: Feature Extraction and Classification: In pattern recognition, normalized intuitionistic fuzzy matrices can be used to extract features from data sets where uncertainty is present. The normalization ensures that the features, expressed in fuzzy terms, are on the same scale, allowing more effective classification and pattern recognition by algorithms such as fuzzy clustering or fuzzy decision trees.
4.Control Systems: Fuzzy Logic Controllers: In control systems, intuitionistic fuzzy matrices can represent uncertain inputs and outputs. Normalizing the IFM ensures that the fuzzy rules and control outputs are consistent, leading to better control decisions in real-time applications such as robotic navigation, industrial automation, and HVAC systems.
5.Risk Assessment and Reliability Analysis: Risk Modelling: In assessing the risk in uncertain environments, intuitionistic fuzzy matrices allow the modelling of both membership (degree of risk) and non-membership (degree of safety). Normalization of such matrices ensures that risk factors are represented proportionately, aiding in better decision-making for risk mitigation strategies.
6.Data Fusion: Sensor Data Fusion: In applications like sensor networks or multi-source information integration, normalization of intuitionistic fuzzy matrices can combine data from multiple sources while adjusting for different degrees of uncertainty across sensors. This leads to a more accurate and reliable fused data set. 7.Healthcare and Medical Diagnosis: Medical Decision Support Systems: In healthcare, intuitionistic fuzzy matrices are used to represent patient data, symptoms, and diagnostic criteria under uncertainty. Normalization helps to align various medical parameters and judgment values, ensuring that the final diagnosis or treatment plan is consistent with all available information.

4.6 Importance:
Consistent Comparison of Data: Normalization ensures that the elements in the intuitionistic fuzzy matrix are scaled to a standard range (typically [0, 1]), making it easier to compare and combine information from different sources. Without normalization, the data might be inconsistent and difficult to interpret, especially when the membership and non-membership values vary across different scales or ranges.

Improving Accuracy in Decision Making: In decision-making problems (like multi-criteria decision analysis or decision support systems), normalization helps to make sure that all the criteria or attributes are comparable on the same scale. This improves the accuracy and reliability of the decision-making process.

Eliminating Bias Due to Scale Differences:

Different elements of an intuitionistic fuzzy matrix can have different scales, leading to biases in the analysis. Normalization removes such biases by ensuring that all values contribute equally to the outcome. This is important when the matrix represents the relationship between various factors or alternatives in a decision-making problem.

Handling Uncertainty Effectively:

Intuitionistic fuzzy sets deal with both membership and non-membership, leaving room for uncertainty (indeterminacy). Normalization allows better handling and representation of this uncertainty, ensuring that the resulting matrix values still remain meaningful even when there is incomplete or imprecise data.

Facilitating Aggregation of Data:

When working with multiple intuitionistic fuzzy matrices, normalization allows for the proper aggregation of data from various sources. This is particularly important in fields such as multi-criteria decision analysis, where data from different criteria need to be aggregated into a final decision.

Simplifying Further Mathematical Operations: Normalized values in intuitionistic fuzzy matrices make it easier to apply further operations like addition, multiplication, or averaging, which are common in matrix-based decision-making methods (e.g., AHP, TOPSIS, etc.). Without normalization, the results of these operations may be skewed due to differences in scale between different elements of the matrix.

Ensuring Feasibility and Validity of Results:

For certain algorithms or applications, it's necessary that the values in the intuitionistic fuzzy matrix sum to a specific value (such as 1). Normalization ensures that the sum of membership and non-membership functions, along with the degree of indeterminacy, remains valid for further processing.

Enhancing Robustness of Algorithms

Many machines learning or optimization algorithms require data in a specific form for stable convergence. Normalized intuitionistic fuzzy matrices help improve the robustness and convergence speed of these algorithms, preventing issues caused by extreme or unbalanced values.

We define over the set of IFS, two modal operators which transform every IFS into fuzzy set. These operators are similar to the operator's 'necessity' and 'possibility' defined in some modal logics. This idea is drawn from the modal operators on IFS proposed by [3].

5. Overview on Normalization and Cartesian product of Intuitionistic Fuzzy Matrices

5.1 Definition: [Modal operators' necessity]: Let X be nonempty. If A is an IFS drawn from X,

then □A = {x, μ_A(x): x∈X} = {x, μ_A(x), $1 - \mu_A$(x): x∈X}

Definition: [Modal operators' possibility]: Let X be nonempty. If A is an IFS drawn from X,

then ◊A = {x, $1 - \vartheta_A$(x): x∈X} = {x, $1 - \vartheta_A$(x), ϑ_A(x): x∈X}

Definition: If A_E and B_F are two IFSs over different universes E and F gives as

A_E = {x, μ_A(x), ϑ_A(x): x∈E} and B_F= {y, μ_B(y), ϑ_B(y): y∈F}

Definition: If $\overline{A_E}$ and $\overline{B_F}$ are two IFSs over different universes E and F gives as

$\overline{A_E}$= {x, ϑ_A(x), μ_A(x): x∈E} and $\overline{B_F}$= {y, ϑ_B(y), μ_B(y): y∈F}

Definition: If A_E and B_F are two IFSs over different universes E and F gives

as A_E = {x, μ_A(x), ϑ_A(x): x∈E} and B_F= {y, μ_B(y), ϑ_B(y): y∈F}

Definition: If $\overline{A_E}$ and $\overline{B_F}$ are two IFSs over different universes E and F gives as

$\overline{A_E}$= {x, ϑ_A(x), μ_A(x): x∈E} and $\overline{B_F}$= {y, ϑ_B(y), μ_B(y): y∈F}

5.2 Definition: Let E be non-empty universe set. The normalization of intuitionistic fuzzy sets A denoted by Norm (A) is defined as Norm (A) = {x, $\mu_{Norm(A)}$(x), $\vartheta_{Norm(A)}$(x): x∈E},

where $\mu_{Norm(A)}(x) = \frac{\mu_A(x)}{\sup(\mu_A(x))}$ and $\vartheta_{Norm(A)}(x) = \frac{\vartheta_A(x) - \inf(\vartheta_A(x))}{1 - \inf(\vartheta_A(x))}$.

Definition5.2.1: If A_E and B_F are two IFSs over different universes sets E and F gives as

A_E = {x, $\mu_A(x)$, $\vartheta_A(x)$: x∈E} and B_F= {y, $\mu_B(y)$, $\vartheta_B(y)$: y∈F}.

1.$A_E \cap B_F$ = {(x, y), min ($\mu_A(x)$, $\mu_B(y)$), max ($\vartheta_A(x)$, $\vartheta_B(x)$): x∈E, y∈F}

2. $A_E \cup B_F$ = {(x, y), max ($\mu_A(x)$, $\mu_B(y)$), min ($\vartheta_A(x)$, $\vartheta_B(x)$): x∈E, y∈F}

3. $A_E \oplus B_F$ = {(x, y), ($\mu_A(x) + \mu_B(y)$) - $\mu_A(x)$. $\mu_B(y)$, $\vartheta_A(x).\vartheta_B(y)$: x∈E, y∈F}

4. $A_E \otimes B_F$ = {(x, y), $\mu_A(x)$. $\mu_B(y)$, $\vartheta_A(x) + \vartheta_B(y) - \vartheta_A(x).\vartheta_B(y)$: x∈E, y∈F}

5. $A_E @ B_F$ = {(x, y),$\frac{\mu_A(x) + \mu_B(y)}{2}, \frac{\vartheta_A(x) + \vartheta_B(y)}{2}$: x∈E, y∈F}

6. A#B_F = {(x, y), $\frac{2\mu_A(x).\mu_B(y)}{\mu_A(x) + \vartheta_B(y)}, \frac{2\vartheta_A(x).\vartheta_B(y)}{\vartheta_A(x) + \vartheta_B(y)}$: x∈E, y∈F}

7. $A_E \$ B_F$ = {(x, y),$\sqrt{\mu_A(x).\mu_B(y)}$, $\sqrt{\vartheta_A(x).\vartheta_B(y)}$x∈E, y∈F}

8. $A_E * B_F$ = {(x, y),$\frac{\mu_A(x) + \mu_B(y)}{2(\mu_A(x).\ \mu_B(y) + 1)}, \frac{\vartheta_A(x) + \vartheta_B(y)}{2(\vartheta_A(x).\vartheta_B(x) + 1)}$: x∈E, y∈F }

9. $A_E \Delta B_F$ = {(x, y),$\frac{\mu_A(x) + \mu_B(y)}{\mu_A(x) + \mu_B(y) + \vartheta_A(x) + \vartheta_B(y)}, \frac{\vartheta_A(x) + \vartheta_B(y)}{\mu_A(x) + \mu_B(y) + \vartheta_A(x) + \vartheta_B(y)}$: x∈E, y∈F}

Theorem 5.2.1: If E and F be two universal sets. For every normalization of complex intuitionistic fuzzy matrices A_E *and* B_F are in E and F then Norm $A_E \cap Norm\ B_F$ is also normalization of complex intuitionistic fuzzy matrices.

Proof: If Norm A_E and $Norm\ B_F$ are two normalization intuitionistic fuzzy matrices over different universes sets E and F. If we consider two 2 × 2 normalization intuitionistic fuzzy matrices, Norm A_E and $Norm\ B_F$.

Let Norm A_E =

$$\begin{pmatrix} (Norm_{\vartheta_{E11}}(\alpha_{11} + i\beta_{11}) & Norm_{\mu_{E11}}(\gamma_{11} + i\delta_{11})) & (Norm_{\vartheta_{E12}}(\alpha_{12} + i\beta_{12}) & Norm_{\mu_{E12}}(\gamma_{12} + i\delta_{12})) \\ (Norm_{\vartheta_{E21}}(\alpha_{21} + i\beta_{21}) & Norm_{\mu_{E21}}(\gamma_{21} + i\delta_{21})) & (Norm_{\vartheta_{E22}}(\alpha_{22} + i\beta_{22}) & Norm_{\mu_{E22}}(\gamma_{22} + i\delta_{22})) \end{pmatrix}$$

and $Norm\ B_F$ =

$$\begin{pmatrix} (Norm_{\delta_{F11}}(x_{11} + iy_{11}) & Norm_{\gamma_{F11}}(\mu_{11} + i\theta_{11})) & (Norm_{\delta_{F12}}(x_{12} + iy_{12}) & Norm_{\gamma_{F12}}(\mu_{12} + i\theta_{12})) \\ (Norm_{\delta_{F21}}(x_{21} + iy_{21}) & Norm_{\gamma_{F21}}(\mu_{21} + i\theta_{21})) & (Norm_{\delta_{F22}}(x_{22} + iy_{22}) & Norm_{\gamma_{F22}}(\mu_{22} + i\theta_{22})) \end{pmatrix}$$

are two normalization intuitionistic fuzzy matrices. Now applying both side on intersection, we have

Norm $A_E \cap Norm\ B_F$ =

$$\begin{pmatrix} (Norm_{\vartheta_{E11}}(\alpha_{11} + i\beta_{11}) & Norm_{\mu_{E11}}(\gamma_{11} + i\delta_{11})) & (Norm_{\vartheta_{E12}}(\alpha_{12} + i\beta_{12}) & Norm_{\mu_{E12}}(\gamma_{12} + i\delta_{12})) \\ (Norm_{\vartheta_{E21}}(\alpha_{21} + i\beta_{21}) & Norm_{\mu_{E21}}(\gamma_{21} + i\delta_{21})) & (Norm_{\vartheta_{E22}}(\alpha_{22} + i\beta_{22}) & Norm_{\mu_{E22}}(\gamma_{22} + i\delta_{22})) \end{pmatrix} \cap$$

$$\begin{pmatrix} (Norm_{\delta_{F11}}(x_{11} + iy_{11}) & Norm_{\gamma_{F11}}(\mu_{11} + i\theta_{11})) & (Norm_{\delta_{F12}}(x_{12} + iy_{12}) & Norm_{\gamma_{F12}}(\mu_{12} + i\theta_{12})) \\ (Norm_{\delta_{F21}}(x_{21} + iy_{21}) & Norm_{\gamma_{F21}}(\mu_{21} + i\theta_{21})) & (Norm_{\delta_{F22}}(x_{22} + iy_{22}) & Norm_{\gamma_{F22}}(\mu_{22} + i\theta_{22})) \end{pmatrix}$$

Calculate, X_{11}, X_{12}, $X_{21}$and X_{22}, we have

$X_{11} =\big(Norm_{\vartheta_{E11}}(\alpha_{11}+i\beta_{11}) \quad Norm_{\mu_{E11}}(\gamma_{11}+i\delta_{11})\big)$
$\quad \cap\big(Norm_{\delta_{F11}}(x_{11}+iy_{11}) \quad Norm_{\gamma_{F11}}(\mu_{11}+i\theta_{11})\big)$

$X_{12} =\big(Norm_{\vartheta_{E12}}(\alpha_{12}+i\beta_{12}) \quad Norm_{\mu_{E12}}(\gamma_{12}+i\delta_{12})\big)$
$\quad \cap \big(Norm_{\delta_{F12}}(x_{12}+iy_{12}) \quad Norm_{\gamma_{F12}}(\mu_{12}+i\theta_{12})\big)$

$X_{21} =(Norm_{\vartheta_{E21}}(\alpha_{21}+i\beta_{21}) \quad Norm_{\mu_{E21}}(\gamma_{21}+i\delta_{21}))$
$\quad \cap (Norm_{\delta_{F21}}(x_{21}+iy_{21}) \quad Norm_{\gamma_{F21}}(\mu_{21}+i\theta_{21}))$

$X_{22} =(Norm_{\vartheta_{E22}}(\alpha_{22}+i\beta_{22}) \quad Norm_{\mu_{E22}}(\gamma_{22}+i\delta_{22}))$
$\quad \cap (Norm_{\delta_{F22}}(x_{22}+iy_{22}) \quad Norm_{\gamma_{F22}}(\mu_{22}+i\theta_{22}))$

Applying Formula $A_E \cap B_F = \{(x, y), \min(\mu_A(x), \mu_B(y)), \max(\vartheta_A(x), \vartheta_B(x)): x\in E, y\in F\}$

$X_{11} =\{\min(Norm_{\vartheta_{E11}}(\alpha_{11}+i\beta_{11}),\ Norm_{\delta_{F11}}(x_{11}+iy_{11}),\ \max(Norm_{\mu_{E11}}(\gamma_{11}+i\delta_{11}),$
$\quad Norm_{\gamma_{E11}}(\mu_{11}+i\theta_{11})\}$

where $|\mu11|= (Norm_{\vartheta_{E11}}(\alpha_{11}))^2+(Norm_{\vartheta_{E11}}(\beta_{11}))^2 < 1$, $|v11|= (Norm_{\mu_{E11}}(\gamma_{11}))2 + (Norm_{\gamma_{E11}}(\delta_{11}))\,2 < 1$, $|\mu11|+|v11| \le 1$.

$X_{12} =\{\min(\ Norm_{\vartheta_{E12}}(\alpha_{12}+i\beta_{12}),\ Norm_{\delta_{F12}}(x_{12}+iy_{12}),\ \min(Norm_{\mu_{E12}}(\gamma_{12}+i\delta_{12}), Norm_{\gamma_{E12}}(\mu_{12}+i\theta_{12})\}$

where $|\mu12|= (Norm_{\vartheta_{E12}}(\alpha_{12}))^2+(Norm_{\vartheta_{E12}}(\beta_{12}))^2 < 1$, $|v11|= (Norm_{\mu_{E12}}(\gamma_{12}))2 + (Norm_{\gamma_{E12}}(\delta_{12}))\,2 < 1$, $|\mu12|+|v12| \le 1$.

$X_{21} =\{\max(\ Norm_{\vartheta_{E21}}(\alpha_{21}+i\beta_{21}),\ Norm_{\delta_{F21}}(x_{21}+iy_{21}),\ \min(Norm_{\mu_{E21}}(\gamma_{21}+i\delta_{21}), Norm_{\gamma_{E21}}(\mu_{21}+i\theta_{21})\}$

where $|\mu21|= (Norm_{\vartheta_{E21}}(\alpha_{21}))^2+(Norm_{\vartheta_{E21}}(\beta_{21}))^2 < 1$, $|v21|= (Norm_{\mu_{E21}}(\gamma_{21}))2 + (Norm_{\gamma_{E21}}(\delta_{21}))\,2 < 1$, $|\mu21|+|v21| \le 1$.

$X_{22} =\{\max(\ Norm_{\vartheta_{E22}}(\alpha_{22},\ x_{22})+i\beta_{22}),\ iNorm_{\delta_{F22}}(+iy_{22}),\ \min(Norm_{\mu_{E22}}(\gamma_{22}+i\delta_{22}), Norm_{\gamma_{E22}}(\mu_{22}+i\theta_{22})\}$

where $|\mu22|= (Norm_{\vartheta_{E22}}(\alpha_{22}))^2+(Norm_{\vartheta_{E22}}(\beta_{22}))^2 < 1$, $|v11|= (Norm_{\mu_{E22}}(\gamma_{22}))2 + (Norm_{\gamma_{E22}}(\delta_{22}))\,2 < 1$, $|\mu22|+|v22| \le 1$.

$X_{11} =(Norm_{\vartheta_{E11}}(\min(\alpha_{11},\ x_{11})+\ i\min(\beta_{11},y_{11}))),\ \ Norm_{\mu_{E11}}(\max(\gamma_{11},\ \mu_{11})+ i\max(\delta_{11},\theta_{11}))$

Where $|\mu11|= (Norm_{\vartheta_{E11}}(\min(\alpha_{11},\ x_{11}))^2+(Norm_{\vartheta_{E11}}\min(\beta_{11},\ y_{11}))^2 +(Norm_{\delta_{F11}}\,min\,(\gamma_{11}, \mu_{11}))^2$
$+(Norm_{\delta_{F11}}\,min\,(\gamma_{11},\ \mu_{11}))^2 < 1$, $|v11|= (Norm_{\mu_{E11}}max(\gamma_{11}, \mu_{11}))^2 +(Norm_{\gamma_{E11}}max(\delta_{11},\theta_{11}))^2 < 1$, $|\mu11|+|v11| \le 1$.

$X_{12} =(Norm_{\vartheta_{E12}}(\min(\alpha_{12},\ x_{12})+\ i\min(\beta_{12},y_{12})),\ \ Norm_{\mu_{E12}}(\max(\gamma_{12},\ \mu_{12})+i\max(\delta_{12}, \theta_{12}))$

where $|\mu 12| = (Norm_{\vartheta_{E12}} min\,(\alpha_{12},\ x_{12}))^2 + (Norm_{\delta_{F12}}\, min\,(\beta_{12}, y_{12}))^2 < 1$, $|\nu 12| = (Norm_{\mu_{E12}} max(\gamma_{12},\ \mu_{12}))^2 + (Norm_{\gamma_{E12}} max(\,(\delta_{12},\ \theta_{12}))^2 < 1$, $|\mu 12|+|\nu 12| \le 1$.

$X_{21} = \{(\ Norm_{\vartheta_{E21}} min\,(\alpha_{21}, x_{21}) + iNorm_{\delta_{F21}}(\beta_{21}, y_{21}),\ \max\,(Norm_{\mu_{E21}}(\gamma_{21}, \mu_{21}) + iNorm_{\gamma_{E21}}(\delta_{21},\ \theta_{21})\}$

where $|\mu 21| = (Norm_{\vartheta_{E21}} min\,(\alpha_{21},\ x_{21}))^2 + (Norm_{\delta_{F21}}\, min\,(\beta_{21}, y_{21}))^2 < 1$, $|\nu 21| = (Norm_{\mu_{E21}} max(\gamma_{21},\ \mu_{21}))^2 + (Norm_{\gamma_{E21}} max(\,(\delta_{21},\ \theta_{21}))^2 < 1$, $|\mu 21|+|\nu 21| \le 1$.

$X_{22} = \{\max(\ Norm_{\vartheta_{E22}}(\alpha_{22},\ x_{22})),\ iNorm_{\delta_{F22}} min\,(\beta_{22}, y_{22}),\ \max\,(Norm_{\mu_{E22}}(\gamma_{22,}\, \mu_{21}),\ Norm_{\gamma_{E22}}(\delta_{22}, \theta_{22})\}$

where $|\mu 22| = (Norm_{\vartheta_{E22}} min\,(\alpha_{22},\ x_{22}))^2 + (Norm_{\delta_{F22}}\, min\,(\beta_{22}, y_{22}))^2 < 1$, $|\nu 21| = (Norm_{\mu_{E22}} max(\gamma_{22},\ \mu_{22}))^2 + (Norm_{\gamma_{E22}} max(\,(\delta_{22},\ \theta_{22}))^2 < 1$, $|\mu 22|+|\nu 22| \le 1$.

We have Norm $A_E \cap Norm\ B_F = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ is also normalization of complex intuitionistic fuzzy matrix.

Example5.2.2: If $A_E = \begin{pmatrix} (0.6+0.2i,\ 0.3+0.1i)(0.4+0.1i,\ 0.5+0.2i) \\ (0.5+0.3i,\ 0.2+0.1i)(0.3+0.2i,\ 0.6+0.1i) \end{pmatrix}$ and

$$B_F = \begin{pmatrix} (0.4+0.3i,\ 0.5+0.1i)(0.3+0.4i,\ 0.4+0.2i) \\ (0.6+0.2i,\ 0.2+0.2i)(0.5+0.1i,\ 0.4+0.3i) \end{pmatrix}$$

are normalization of complex intuitionistic fuzzy matrix two intuitionistic fuzzy matrices over different universe E and F then Norm $A_E \cap Norm\ B_F$ is also normalization of complex intuitionistic fuzzy matrix.

Proof:

Given $\boldsymbol{A_E} = \begin{pmatrix} (0.6+0.2i,\ 0.3+0.1i)(0.4+0.1i,\ 0.5+0.2i) \\ (0.5+0.3i,\ 0.2+0.1i)(0.3+0.2i,\ 0.6+0.1i) \end{pmatrix}$ and

$$\boldsymbol{B_F} = \begin{pmatrix} (0.4+0.3i,\ 0.5+0.1i)(0.3+0.4i,\ 0.4+0.2i) \\ (0.6+0.2i,\ 0.2+0.2i)(0.5+0.1i,\ 0.4+0.3i) \end{pmatrix}$$

Norm $A_E = \begin{pmatrix} (1+0.67i \quad 0.125+0i) & (0.67+0.33i \quad 0.30+0.10i) \\ (0.83+1i \quad 0+0i) & (0.5+0.67i \quad 0.5+0i) \end{pmatrix}$

$Norm\ B_F = \begin{pmatrix} (0.67+0.75i \quad 0.375+0i) & (0.5+1i \quad 0.25+0.11i) \\ (1+0.5i \quad 0+0.11i) & (0.83+0.25i \quad 0.25+0.22i) \end{pmatrix}$

Norm $A_E \cap Norm\ B_F = \begin{pmatrix} (1+0.67i \quad 0.125+0i) & (0.67+0.33i \quad 0.30+0.10i) \\ (0.83+1i \quad 0+0i) & (0.5+0.67i \quad 0.5+0i) \end{pmatrix} \cap$

$$\begin{pmatrix} (0.67+0.75i \quad 0.375+0i) & (0.5+1i \quad 0.25+0.11i) \\ (1+0.5i \quad 0+0.11i) & (0.83+0.25i \quad 0.25+0.22i) \end{pmatrix}$$

$X_{11} = (1+0.67i \quad 0.125+0i) \cap (0.67+0.75i \quad 0.375+0i)$

$X_{12} = (0.67+0.33i \quad 0.30+0.10i) \cap (0.5+1i \quad 0.25+0.11i)$

$X_{21} = (0.83+1i \quad 0+0i) \cap (1+0.5i \quad 0+0.11i)$

$X_{22} = (0.5+0.67i \quad 0.5+0i) \cap (0.83+0.25i \quad 0.25+0.22i)$

Applying Formula $A_E \cap B_F = \{(x, y), \min\,(\mu_A(x), \mu_B(y)), \max\,(\vartheta_A(x), \vartheta_B(x)): x \in E, y \in F\}$

X_{11} ={$min(1 + 0.67i, 0.67 + 0.75i)$, $max(0.125 + 0i, 0.375 + 0i)$}

X_{12} ={$min(0.67 + 0.33i, 0.5 + 1i)$, $max(0.30 + 0.10i, 0.25 + 0.11i)$}

X_{21} ={$min(0.83 + 1i, 1 + 0.5i)$, $max(0 + 0i, 0 + 0.11i)$}

X_{22} ={$min(0.5 + 0.67i, 0.83 + 0.25i)$, $max(0.5 + 0i, 0.25 + 0.22i)$}

X_{11} ={0.67+0.67i, 0.375+0i}

X_{12} ={0.5+0.33i, 0.3+0.11i}

X_{21} ={0.83+0.5i, 0+0.11i}

X_{22} ={0.5+0.25i, 0.5+0.22i}

We have Norm $A_E \cap Norm\ B_F = \begin{pmatrix} (0.67 + 0.67\text{i}, 0.375 + 0\text{i}) & (0.5 + 0.33\text{i}, 0.3 + 0.11\text{i}) \\ (0.83 + 0.5\text{i}, 0 + 0.11\text{i}) & (0.5 + 0.25\text{i}, 0.5 + 0.22\text{i}) \end{pmatrix}$ is also normalization of complex intuitionistic fuzzy matrix.

Theorem 5.2.3: If E and F be two universal sets. For every normalization of complex intuitionistic fuzzy matrices A_E *and* B_F are in E and F then Norm $A_E \cup Norm\ B_F$ is also normalization of complex intuitionistic fuzzy matrices.

Proof: If Norm A_E and $Norm\ B_F$ are two normalization intuitionistic fuzzy matrices over different universes *sets E and F*. If we consider two 2×2 normalization intuitionistic fuzzy matrices, Norm A_E *and* $Norm\ B_F$.

Let Norm A_E =

$$\begin{pmatrix} \left(Norm_{\vartheta_{E11}}(\alpha_{11} + i\beta_{11}) & Norm_{\mu_{E11}}(\gamma_{11} + i\delta_{11})\right) & \left(Norm_{\vartheta_{E12}}(\alpha_{12} + i\beta_{12}) & Norm_{\mu_{E12}}(\gamma_{12} + i\delta_{12})\right) \\ \left(Norm_{\vartheta_{E21}}(\alpha_{21} + i\beta_{21}) & Norm_{\mu_{E21}}(\gamma_{21} + i\delta_{21})\right) & \left(Norm_{\vartheta_{E22}}(\alpha_{22} + i\beta_{22}) & Norm_{\mu_{E22}}(\gamma_{22} + i\delta_{22})\right) \end{pmatrix}$$

and $Norm\ B_F$ =

$$\begin{pmatrix} \left(Norm_{\delta_{F11}}(x_{11} + iy_{11}) & Norm_{\gamma_{F11}}(\mu_{11} + i\theta_{11})\right) & \left(Norm_{\delta_{F12}}(x_{12} + iy_{12}) & Norm_{\gamma_{F12}}(\mu_{12} + i\theta_{12})\right) \\ \left(Norm_{\delta_{F21}}(x_{21} + iy_{21}) & Norm_{\gamma_{F21}}(\mu_{21} + i\theta_{21})\right) & \left(Norm_{\delta_{F22}}(x_{22} + iy_{22}) & Norm_{\gamma_{F22}}(\mu_{22} + i\theta_{22})\right) \end{pmatrix}$$

are two normalization intuitionistic fuzzy matrices.

Norm$A_E \cup NormB_F$ =

$$\begin{pmatrix} \left(Norm_{\vartheta_{E11}}(\alpha_{11} + i\beta_{11}) & Norm_{\mu_{E11}}(\gamma_{11} + i\delta_{11})\right) & \left(Norm_{\vartheta_{E12}}(\alpha_{12} + i\beta_{12}) & Norm_{\mu_{E12}}(\gamma_{12} + i\delta_{12})\right) \\ \left(Norm_{\vartheta_{E21}}(\alpha_{21} + i\beta_{21}) & Norm_{\mu_{E21}}(\gamma_{21} + i\delta_{21})\right) & \left(Norm_{\vartheta_{E22}}(\alpha_{22} + i\beta_{22}) & Norm_{\mu_{E22}}(\gamma_{22} + i\delta_{22})\right) \end{pmatrix} \cup \begin{pmatrix} \left(Norm_{\delta_{F11}}(x_{11} + iy_{11}) & Norm_{\gamma_{F11}}(\mu_{11} + i\theta_{11})\right) & \left(Norm_{\delta_{F12}}(x_{12} + iy_{12}) & Norm_{\gamma_{F12}}(\mu_{12} + i\theta_{12})\right) \\ \left(Norm_{\delta_{F21}}(x_{21} + iy_{21}) & Norm_{\gamma_{F21}}(\mu_{21} + i\theta_{21})\right) & \left(Norm_{\delta_{F22}}(x_{22} + iy_{22}) & Norm_{\gamma_{F22}}(\mu_{22} + i\theta_{22})\right) \end{pmatrix}$$

Calculate, X_{11}, X_{12}, $X_{21}$and X_{22}, we have. Now applying both side on intersection, we have

Norm $A_E \cup Norm\ B_F$ =

$$\begin{pmatrix} \left(Norm_{\vartheta_{E11}}(\alpha_{11} + i\beta_{11}) & Norm_{\mu_{E11}}(\gamma_{11} + i\delta_{11})\right) & \left(Norm_{\vartheta_{E12}}(\alpha_{12} + i\beta_{12}) & Norm_{\mu_{E12}}(\gamma_{12} + i\delta_{12})\right) \\ \left(Norm_{\vartheta_{E21}}(\alpha_{21} + i\beta_{21}) & Norm_{\mu_{E21}}(\gamma_{21} + i\delta_{21})\right) & \left(Norm_{\vartheta_{E22}}(\alpha_{22} + i\beta_{22}) & Norm_{\mu_{E22}}(\gamma_{22} + i\delta_{22})\right) \end{pmatrix} \cup \begin{pmatrix} \left(Norm_{\delta_{F11}}(x_{11} + iy_{11}) & Norm_{\gamma_{F11}}(\mu_{11} + i\theta_{11})\right) & \left(Norm_{\delta_{F12}}(x_{12} + iy_{12}) & Norm_{\gamma_{F12}}(\mu_{12} + i\theta_{12})\right) \\ \left(Norm_{\delta_{F21}}(x_{21} + iy_{21}) & Norm_{\gamma_{F21}}(\mu_{21} + i\theta_{21})\right) & \left(Norm_{\delta_{F22}}(x_{22} + iy_{22}) & Norm_{\gamma_{F22}}(\mu_{22} + i\theta_{22})\right) \end{pmatrix}$$

Calculate, X_{11}, X_{12}, $X_{21}$and X_{22}, we have

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$X_{11} = \left(Norm_{\vartheta_{E11}}(\alpha_{11} + i\beta_{11}) \quad Norm_{\mu_{E11}}(\gamma_{11} + i\delta_{11})\right)$

$\cup \left(Norm_{\delta_{F11}}(x_{11} + iy_{11}) \quad Norm_{\gamma_{F11}}(\mu_{11} + i\theta_{11})\right)$

$X_{12} = \left(Norm_{\vartheta_{E12}}(\alpha_{12} + i\beta_{12}) \quad Norm_{\mu_{E12}}(\gamma_{12} + i\delta_{12})\right)$

$\cup \left(Norm_{\delta_{F12}}(x_{12} + iy_{12}) \quad Norm_{\gamma_{F12}}(\mu_{12} + i\theta_{12})\right)$

$X_{21} = (Norm_{\vartheta_{E21}}(\alpha_{21} + i\beta_{21}) \quad Norm_{\mu_{E21}}(\gamma_{21} + i\delta_{21}))$

$\cup \ (Norm_{\delta_{F21}}(x_{21} + iy_{21}) \quad Norm_{\gamma_{F21}}(\mu_{21} + i\theta_{21}))$

$X_{22} = (Norm_{\vartheta_{E22}}(\alpha_{22} + i\beta_{22}) \quad Norm_{\mu_{E22}}(\gamma_{22} + i\delta_{22}))$

$\cup \ (Norm_{\delta_{F22}}(x_{22} + iy_{22}) \quad Norm_{\gamma_{F22}}(\mu_{22} + i\theta_{22}))$

Applying formula $A_E \cup B_F = \{(x, y), \max(\mu_A(x), \mu_B(y)), \min(\vartheta_A(x), \vartheta_B(x)) : x \in E, y \in F\}$

$X_{11} = \{\max\ (Norm_{\vartheta_{E11}}(\alpha_{11} + i\beta_{11}),\ Norm_{\delta_{F11}}(x_{11} + iy_{11}),\ \min(Norm_{\mu_{E11}}(\gamma_{11} + i\delta_{11}), Norm_{\gamma_{E11}}(\mu_{11} + i\theta_{11})\}$

where $|\mu 11| = (Norm_{\vartheta_{E11}}(\alpha_{11}))^2 + (Norm_{\vartheta_{E11}}(\beta_{11}))^2 < 1$, $|\nu 11| = (Norm_{\mu_{E11}}(\gamma_{11}))2 + (Norm_{\gamma_{E11}}(\delta_{11}))\ 2 < 1$, $|\mu 11| + |\nu 11| \leq 1$.

$X_{12} = \{\max\ (Norm_{\vartheta_{E12}}(\alpha_{12} + i\beta_{12}),\ Norm_{\delta_{F12}}\ (x_{12} + iy_{12}),\ \min(Norm_{\mu_{E12}}\ (\gamma_{12} + i\delta_{12}), Norm_{\gamma_{E12}}(\mu_{12} + i\theta_{12})\}$

where $|\mu 12| = (Norm_{\vartheta_{E12}}(\alpha_{12}))^2 + (Norm_{\vartheta_{E12}}(\beta_{12}))^2 < 1$, $|\nu 11| = (Norm_{\mu_{E12}}(\gamma_{12}))2 + (Norm_{\gamma_{E12}}(\delta_{12}))\ 2 < 1$, $|\mu 12| + |\nu 12| \leq 1$.

$X_{21} = \{\max(Norm_{\vartheta_{E21}}(\alpha_{21} + i\beta_{21}),\ Norm_{\delta_{F21}}(x_{21} + iy_{21}),\ \min(Norm_{\mu_{E21}}(\gamma_{21} + i\delta_{21}), Norm_{\gamma_{E21}}(\mu_{21} + i\theta_{21})\}$

where $|\mu 21| = (Norm_{\vartheta_{E21}}(\alpha_{21}))^2 + (Norm_{\vartheta_{E21}}(\beta_{21}))^2 < 1$, $|\nu 21| = (Norm_{\mu_{E21}}(\gamma_{21}))2 + (Norm_{\gamma_{E21}}(\delta_{21}))\ 2 < 1$, $|\mu 21| + |\nu 21| \leq 1$.

$X_{22} = \{\max\ (Norm_{\vartheta_{E22}}(\alpha_{22},\ x_{22}) + i\beta_{22}),\ iNorm_{\delta_{F22}}(+iy_{22}),\ \min(Norm_{\mu_{E22}}(\gamma_{22} + i\delta_{22}), Norm_{\gamma_{E22}}(\mu_{22} + i\theta_{22})\}$

where $|\mu 22| = (Norm_{\vartheta_{E22}}(\alpha_{22}))^2 + (Norm_{\vartheta_{E22}}(\beta_{22}))^2 < 1$, $|\nu 11| = (Norm_{\mu_{E22}}(\gamma_{22}))2 + (Norm_{\gamma_{E22}}(\delta_{22}))\ 2 < 1$, $|\mu 22| + |\nu 22| \leq 1$.

$X_{11} = (Norm_{\vartheta_{E11}}(\max(\alpha_{11},\ x_{11}) + \ i\max(\beta_{11}, y_{11}))),\ \ Norm_{\mu_{E11}}(\min(\gamma_{11},\ \mu_{11}) + i\min(\delta_{11}, \theta_{11}))$

Where $|\mu 11| = (Norm_{\vartheta_{E11}}(\max(\alpha_{11},\ x_{11}))^2 + (Norm_{\vartheta_{E11}}\max(\beta_{11},\ y_{11}))^2 + (Norm_{\delta_{F11}}\ min\ (\gamma_{11}, \mu_{11}))^2$

$+(Norm_{\delta_{F11}}\ min\ (\gamma_{11},\ \mu_{11}))^2 < 1$, $|\nu 11| = (Norm_{\mu_{E11}} min(\gamma_{11},\ \mu_{11}))^2 + (Norm_{\gamma_{E11}} min(\delta_{11}, \theta_{11}))^2 < 1$, $|\mu 11| + |\nu 11| \leq 1$.

$X_{12} = (Norm_{\vartheta_{E12}}(\max(\alpha_{12},\ x_{12}) + \ i\max(\beta_{12}, y_{12})),\ \ Norm_{\mu_{E12}}(\min(\gamma_{12},\ \mu_{12}) + i\min(\delta_{12}, \theta_{12}))$

where $|\mu 12| = (Norm_{\vartheta_{E12}} max\ (\alpha_{12},\ \ x_{12}))^2 \ +(Norm_{\delta_{F12}}\ max\ (\beta_{12}, y_{12}))^2 < 1$, $|\nu 12| = (Norm_{\mu_{E12}} min(\gamma_{12},\ \mu_{12}))^2 + (Norm_{\gamma_{E12}} min(\ (\delta_{12},\ \theta_{12}))^2 < 1$, $|\mu 12| + |\nu 12| \leq 1$.

X_{21} ={($\text{Norm}_{\vartheta_{E21}} max\ (\alpha_{21}, x_{21}) + \text{iNorm}_{\delta_{F21}} max(\beta_{21}, y_{21})$, $(\text{Norm}_{\mu_{E21}} min(\gamma_{21}, \mu_{21}) + \text{iNorm}_{\gamma_{E21}} min(\delta_{21},\ \theta_{21})$}

where |μ21|=$(Norm_{\vartheta_{E21}} max\ (\alpha_{21},\ x_{21}))^2 + (Norm_{\delta_{F21}}\ max\ (\beta_{21}, y_{21}))^2 < 1$, |ν21|= $(\text{Norm}_{\mu_{E21}} min(\gamma_{21},\ \mu_{21}))^2 + (Norm_{\gamma_{E21}} min(\ (\delta_{21},\ \theta_{21}))^2 < 1$, |μ21|+|ν21| ≤ 1.

X_{22} ={max($\text{Norm}_{\vartheta_{E22}}(\alpha_{22},\ x_{22})$), $\text{iNorm}_{\delta_{F22}} max\ (\beta_{22}, y_{22})$, min $(\text{Norm}_{\mu_{E22}}(\gamma_{22,} \mu_{21})$, $\text{Norm}_{\gamma_{E22}}(\delta_{22}, \theta_{22})$}

where |μ22|=$(Norm_{\vartheta_{E22}} max\ (\alpha_{22},\ x_{22}))^2 + (Norm_{\delta_{F22}}\ max\ (\beta_{22}, y_{22}))^2 < 1$, |ν21|= $(\text{Norm}_{\mu_{E22}} min(\gamma_{22},\ \mu_{22}))^2 + (Norm_{\gamma_{E22}} min(\ (\delta_{22},\ \theta_{22}))^2 < 1$, |μ22|+|ν22| ≤ 1.

We have Norm$A_E \cup NormB_F$= $\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ is normalization of complex intuitionistic fuzzy matrix.

Example5.2.4: If A_E=$\begin{pmatrix} (0.6 + 0.2\text{i},\ 0.3 + 0.1\text{i})(0.4 + 0.1\text{i},\ 0.5 + 0.2\text{i}) \\ (0.5 + 0.3\text{i},\ 0.2 + 0.1\text{i})(0.3 + 0.2\text{i},\ 0.6 + 0.1\text{i}) \end{pmatrix}$ and

$$\text{B}_\text{F} = \begin{pmatrix} (0.4 + 0.3\text{i},\ 0.5 + 0.1\text{i})(0.3 + 0.4\text{i},\ 0.4 + 0.2\text{i}) \\ (0.6 + 0.2\text{i},\ 0.2 + 0.2\text{i})(0.5 + 0.1\text{i},\ 0.4 + 0.3\text{i}) \end{pmatrix}$$

are normalization of complex intuitionistic fuzzy matrix two intuitionistic fuzzy matrices over different universe E and F then Norm $A_E \cup Norm\ B_F$ is also normalization of complex intuitionistic fuzzy matrix.

Proof:

Given $\boldsymbol{A_E}$=$\begin{pmatrix} (0.6 + 0.2\text{i},\ 0.3 + 0.1\text{i})(0.4 + 0.1\text{i},\ 0.5 + 0.2\text{i}) \\ (0.5 + 0.3\text{i},\ 0.2 + 0.1\text{i})(0.3 + 0.2\text{i},\ 0.6 + 0.1\text{i}) \end{pmatrix}$ and

$$\boldsymbol{B_F} = \begin{pmatrix} (0.4 + 0.3\text{i},\ 0.5 + 0.1\text{i})(0.3 + 0.4\text{i},\ 0.4 + 0.2\text{i}) \\ (0.6 + 0.2\text{i},\ 0.2 + 0.2\text{i})(0.5 + 0.1\text{i},\ 0.4 + 0.3\text{i}) \end{pmatrix}$$

Norm A_E = $\begin{pmatrix} (1 + 0.67i \quad 0.125 + 0i) & (0.67 + 0.33i \quad 0.30 + 0.10i) \\ (0.83 + 1i \quad 0 + 0i) & (0.5 + 0.67i \quad 0.5 + 0i) \end{pmatrix}$

$Norm\ B_F$= $\begin{pmatrix} (0.67 + 0.75i \quad 0.375 + 0i) & (0.5 + 1i \quad 0.25 + 0.11i) \\ (1 + 0.5i \quad 0 + 0.11i) & (0.83 + 0.25i \quad 0.25 + 0.22i) \end{pmatrix}$

Norm$A_E \cup NormB_F$ = $\begin{pmatrix} (1 + 0.67i \quad 0.125 + 0i) & (0.67 + 0.33i \quad 0.30 + 0.10i) \\ (0.83 + 1i \quad 0 + 0i) & (0.5 + 0.67i \quad 0.5 + 0i) \end{pmatrix} \cup \begin{pmatrix} (0.67 + 0.75i \quad 0.375 + 0i) & (0.5 + 1i \quad 0.25 + 0.11i) \\ (1 + 0.5i \quad 0 + 0.11i) & (0.83 + 0.25i \quad 0.25 + 0.22i) \end{pmatrix}$

$X_{11} = (1 + 0.67i \quad 0.125 + 0i) \cup (0.67 + 0.75i \quad 0.375 + 0i)$

$X_{12} = (0.67 + 0.33i \quad 0.30 + 0.10i) \cup (0.5 + 1i \quad 0.25 + 0.11i)$

X_{21} = $(0.83 + 1i \quad 0 + 0i) \cup (1 + 0.5i \quad 0 + 0.11i)$

X_{22} = $(0.5 + 0.67i \quad 0.5 + 0i) \cup (0.83 + 0.25i \quad 0.25 + 0.22i)$

Applying formula $A_E \cup B_F$ = {(x, y), max (μ_A(x), μ_B(y)), min (ϑ_A(x), ϑ_B(x)): x∈E, y∈F}

X_{11} ={max$(1 + 0.67i, 0.67 + 0.75i)$, min$(0.125 + 0i, 0.375 + 0i)$}

X_{12} ={max$(0.67 + 0.33i, 0.5 + 1i)$, min$(0.30 + 0.10i, 0.25 + 0.11i)$}

X_{21} ={max$(0.83 + 1i, 1 + 0.5i)$, min$(0 + 0i, 0 + 0.11i)$}

X_{22} ={max$(0.5+0.67i, 0.83+0.25i)$, min$(0.5+0i, 0.25+0.22i)$}

X_{11} ={1+0.75i, 0.125+0i}

X_{12} ={0.67+1i, 0.25+0.10i}

X_{21} ={1+1i, 0+0i}

X_{22} ={0.83+0.67i, 0.25+0i}

We have Norm$A_E \cup NormB_F$ = $\begin{pmatrix} (1+0.75\text{i}, 0.125+0\text{i}) & (0.67+1\text{i}, 0.25+0.10\text{i}) \\ (1+1\text{i}, 0+0\text{i}) & (0.83+0.67\text{i}, 0.25+0\text{i}) \end{pmatrix}$ is also normalization of complex intuitionistic fuzzy matrix.

Theorem 5.2.5: If E and F be two universal sets. For every normalization of complex intuitionistic fuzzy matrices A_E *and* B_F are in E and F then Norm$A_E \oplus NormB_F$ is also normalization of complex intuitionistic fuzzy matrix.

Proof: If Norm A_E and $Norm\ B_F$ are two normalization intuitionistic fuzzy matrices over different univ*erses sets E and F*. If we consider two 2×2 normalization intuitionistic fuzzy matrices, Norm A_E*and* $Norm\ B_F$.

Let Norm A_E =

$$\begin{pmatrix} (Norm_{\vartheta_{E11}}(\alpha_{11}+i\beta_{11}) & Norm_{\mu_{E11}}(\gamma_{11}+i\delta_{11})) & (Norm_{\vartheta_{E12}}(\alpha_{12}+i\beta_{12}) & Norm_{\mu_{E12}}(\gamma_{12}+i\delta_{12})) \\ (Norm_{\vartheta_{E21}}(\alpha_{21}+i\beta_{21}) & Norm_{\mu_{E21}}(\gamma_{21}+i\delta_{21})) & (Norm_{\vartheta_{E22}}(\alpha_{22}+i\beta_{22}) & Norm_{\mu_{E22}}(\gamma_{22}+i\delta_{22})) \end{pmatrix}$$

and $Norm\ B_F$ =

$$\begin{pmatrix} (Norm_{\delta_{F11}}(x_{11}+iy_{11}) & Norm_{\gamma_{F11}}(\mu_{11}+i\theta_{11})) & (Norm_{\delta_{F12}}(x_{12}+iy_{12}) & Norm_{\gamma_{F12}}(\mu_{12}+i\theta_{12})) \\ (Norm_{\delta_{F21}}(x_{21}+iy_{21}) & Norm_{\gamma_{F21}}(\mu_{21}+i\theta_{21})) & (Norm_{\delta_{F22}}(x_{22}+iy_{22}) & Norm_{\gamma_{F22}}(\mu_{22}+i\theta_{22})) \end{pmatrix}$$

are two normalization intuitionistic fuzzy matrices.

Norm$A_E \oplus NormB_F$ =

$$\begin{pmatrix} (Norm_{\vartheta_{E11}}(\alpha_{11}+i\beta_{11}) & Norm_{\mu_{E11}}(\gamma_{11}+i\delta_{11})) & (Norm_{\vartheta_{E12}}(\alpha_{12}+i\beta_{12}) & Norm_{\mu_{E12}}(\gamma_{12}+i\delta_{12})) \\ (Norm_{\vartheta_{E21}}(\alpha_{21}+i\beta_{21}) & Norm_{\mu_{E21}}(\gamma_{21}+i\delta_{21})) & (Norm_{\vartheta_{E22}}(\alpha_{22}+i\beta_{22}) & Norm_{\mu_{E22}}(\gamma_{22}+i\delta_{22})) \end{pmatrix} \oplus$$
$$\begin{pmatrix} (Norm_{\delta_{F11}}(x_{11}+iy_{11}) & Norm_{\gamma_{F11}}(\mu_{11}+i\theta_{11})) & (Norm_{\delta_{F12}}(x_{12}+iy_{12}) & Norm_{\gamma_{F12}}(\mu_{12}+i\theta_{12})) \\ (Norm_{\delta_{F21}}(x_{21}+iy_{21}) & Norm_{\gamma_{F21}}(\mu_{21}+i\theta_{21})) & (Norm_{\delta_{F22}}(x_{22}+iy_{22}) & Norm_{\gamma_{F22}}(\mu_{22}+i\theta_{22})) \end{pmatrix}$$

Calculate, X_{11}, X_{12}, $X_{21}$and X_{22}, we have. Now applying both side on $A_E \oplus B_F$ = {(x, y), $(\mu_A(x) + \mu_B(y)) - \mu_A(x).\ \mu_B(y)$, $\vartheta_A(x).\vartheta_B(y)$: x∈E, y∈F}, we have

Norm$A_E \oplus NormB_F$ =

$$\begin{pmatrix} (Norm_{\vartheta_{E11}}(\alpha_{11}+i\beta_{11}) & Norm_{\mu_{E11}}(\gamma_{11}+i\delta_{11})) & (Norm_{\vartheta_{E12}}(\alpha_{12}+i\beta_{12}) & Norm_{\mu_{E12}}(\gamma_{12}+i\delta_{12})) \\ (Norm_{\vartheta_{E21}}(\alpha_{21}+i\beta_{21}) & Norm_{\mu_{E21}}(\gamma_{21}+i\delta_{21})) & (Norm_{\vartheta_{E22}}(\alpha_{22}+i\beta_{22}) & Norm_{\mu_{E22}}(\gamma_{22}+i\delta_{22})) \end{pmatrix} \oplus$$
$$\begin{pmatrix} (Norm_{\delta_{F11}}(x_{11}+iy_{11}) & Norm_{\gamma_{F11}}(\mu_{11}+i\theta_{11})) & (Norm_{\delta_{F12}}(x_{12}+iy_{12}) & Norm_{\gamma_{F12}}(\mu_{12}+i\theta_{12})) \\ (Norm_{\delta_{F21}}(x_{21}+iy_{21}) & Norm_{\gamma_{F21}}(\mu_{21}+i\theta_{21})) & (Norm_{\delta_{F22}}(x_{22}+iy_{22}) & Norm_{\gamma_{F22}}(\mu_{22}+i\theta_{22})) \end{pmatrix}$$

Calculate, X_{11}, X_{12}, $X_{21}$and X_{22}, we have

X_{11} $=\big(Norm_{\vartheta_{E11}}(\alpha_{11}+i\beta_{11}) \quad Norm_{\mu_{E11}}(\gamma_{11}+i\delta_{11})\big) + \big(Norm_{\delta_{F11}}(x_{11}+iy_{11}) \quad Norm_{\gamma_{F11}}(\mu_{11}+i\theta_{11})\big)$

Where |μ11|= $(Norm_{\vartheta_{E11}}(\max(\alpha_{11},\ x_{11}))^2 + (Norm_{\vartheta_{E11}} \max(\beta_{11},\ y_{11}))^2$ $+(Norm_{\delta_{F11}}\ min\ (\gamma_{11},\ \mu_{11}))^2$

$+(Norm_{\delta_{F11}}\ min\ (\gamma_{11},\ \mu_{11}))^2 < 1$, |ν11|= $(Norm_{\mu_{E11}} min(\gamma_{11},\ \mu_{11}))^2$ $+(Norm_{\gamma_{E11}} min(\delta_{11}, \theta_{11}))^2 < 1$, |μ11|+|ν11| ≤ 1.

$X_{12} = (Norm_{\vartheta_{E12}}(\alpha_{12} + i\beta_{12}) \quad Norm_{\mu_{E12}}(\gamma_{12} + i\delta_{12})) + (Norm_{\delta_{F12}}(x_{12} + iy_{12}) \quad Norm_{\gamma_{F12}}(\mu_{12} + i\theta_{12}))$

$X_{21} = (Norm_{\vartheta_{E21}}(\alpha_{21} + i\beta_{21}) \quad Norm_{\mu_{E21}}(\gamma_{21} + i\delta_{21})) + (Norm_{\delta_{F21}}(x_{21} + iy_{21}) \quad Norm_{\gamma_{F21}}(\mu_{21} + i\theta_{21}))$

$X_{22} = (Norm_{\vartheta_{E22}}(\alpha_{22} + i\beta_{22}) \quad Norm_{\mu_{E22}}(\gamma_{22} + i\delta_{22})) + (Norm_{\delta_{F22}}(x_{22} + iy_{22}) \quad Norm_{\gamma_{F22}}(\mu_{22} + i\theta_{22}))$

Applying formula $A_E \oplus B_F = \{(x, y), (\mu_A(x) + \mu_B(y)) - \mu_A(x).\ \mu_B(y), \vartheta_A(x).\vartheta_B(y): x \in E,\ y \in F\}$

$X_{11} = \{Norm_{\vartheta_{E11}}(\alpha_{11} + i\beta_{11}) + Norm_{\delta_{F11}}(x_{11} + iy_{11}) - Norm_{\vartheta_{E11}}(\alpha_{11} + i\beta_{11}). Norm_{\delta_{F11}}(x_{11} + iy_{11}), Norm_{\mu_{E11}}(\gamma_{11} + i\delta_{11}). Norm_{\gamma_{F11}}(\mu_{11} + i\theta_{11})\}$

$X_{12} = \{Norm_{\vartheta_{E12}}(\alpha_{12} + i\beta_{12}) + Norm_{\delta_{F12}}(x_{12} + iy_{12}) - Norm_{\vartheta_{E12}}(\alpha_{12} + i\beta_{12}). Norm_{\delta_{F12}}(x_{12} + iy_{12}), Norm_{\mu_{E12}}(\gamma_{12} + i\delta_{12}). Norm_{\gamma_{F12}}(\mu_{12} + i\theta_{12})\}$

$X_{21} = \{Norm_{\vartheta_{E21}}(\alpha_{21} + i\beta_{21}) + Norm_{\delta_{F21}}(x_{21} + iy_{21}) - Norm_{\vartheta_{E21}}(\alpha_{21} + i\beta_{21}). Norm_{\delta_{F21}}(x_{21} + y_{21}), Norm_{\mu_{E21}}(\gamma_{21} + i\delta_{21}). Norm_{\gamma_{F21}}(\mu_{21} + i\theta_{21})\}$

$X_{22} = \{Norm_{\vartheta_{E22}}(\alpha_{22} + i\beta_{22}) + Norm_{\delta_{F22}}(x_{22} + iy_{22}) - Norm_{\vartheta_{E22}}(\alpha_{22} + i\beta_{22}). Norm_{\delta_{F22}}(x_{22} + y_{22}), Norm_{\mu_{E22}}(\gamma_{22} + i\delta_{22}). Norm_{\gamma_{F22}}(\mu_{22} + i\theta_{22})\}$

$X_{11} = \{Norm_{\vartheta_{E11+}\,\delta_{F11}}((\alpha_{11} + x_{11} - \alpha_{11}.x_{11}) + i(\beta_{11} + y_{11} - \beta_{11}.y_{11})),\ Norm_{\mu_{E11+\gamma_{F11}}}(\gamma_{11}.\mu_{11} + i\delta_{11}.\theta_{11})\}$

Where |μ11|= $(Norm_{\vartheta_{E11+}\,\delta_{F11}}((\alpha_{11} + x_{11} - \alpha_{11}.x_{11}))^2 + (Norm_{\vartheta_{E11+}\,\delta_{F11}}(\beta_{11} + y_{11} - \beta_{11}.y_{11}))^2 < 1$ | ν11 |= $(Norm_{\mu_{E11+\gamma_{F11}}}(\gamma_{11}.\mu_{11})^2 + Norm_{\mu_{E11+\gamma_{F11}}}(\delta_{11}.\theta_{11})^2 < 1$, |μ11|+|ν11| ≤ 1.

$X_{12} = \{Norm_{\vartheta_{E12+}\,\delta_{F12}}((\alpha_{12} + x_{12} - \alpha_{12}.x_{12}) + i(\beta_{12} + y_{12} - \beta_{12}.y_{12})),\ Norm_{\mu_{E12+\gamma_{F12}}}(\gamma_{12}.\mu_{12} + i\delta_{12}.\theta_{12})\}$

Where |μ12|= $(Norm_{\vartheta_{E12+}\,\delta_{F12}}((\alpha_{12} + x_{12} - \alpha_{12}.x_{12}))^2 + (Norm_{\vartheta_{E12+}\,\delta_{F12}}(\beta_{12} + y_{12} - \beta_{12}.y_{12}))^2 < 1$ | ν12 |= $(Norm_{\mu_{E12+\gamma_{F12}}}(\gamma_{12}.\mu_{12})^2 + Norm_{\mu_{E12+\gamma_{F12}}}(\delta_{12}.\theta_{12})^2 < 1$, |μ12|+|ν12| ≤ 1.

$X_{21} = \{Norm_{\vartheta_{E21+}\,\delta_{F21}}((\alpha_{21} + x_{21} - \alpha_{21}.x_{21}) + i(\beta_{21} + y_{21} - \beta_{21}.y_{21})),\ Norm_{\mu_{E21+\gamma_{F21}}}(\gamma_{21}.\mu_{21} + i\delta_{21}.\theta_{21})\}$

Where |μ21|= $(Norm_{\vartheta_{E21+}\,\delta_{F21}}((\alpha_{21} + x_{21} - \alpha_{21}.x_{21}))^2 + (Norm_{\vartheta_{E21+}\,\delta_{F21}}(\beta_{21} + y_{21} - \beta_{21}.y_{21}))^2 < 1$ | ν21 |= $(Norm_{\mu_{E21+\gamma_{F21}}}(\gamma_{21}.\mu_{21})^2 + Norm_{\mu_{E21+\gamma_{F21}}}(\delta_{21}.\theta_{21})^2 < 1$, |μ21|+|ν21| ≤ 1.

$X_{22} = \{Norm_{\vartheta_{E22+}\,\delta_{F22}}((\alpha_{22} + x_{22} - \alpha_{22}.x_{22}) + i(\beta_{22} + y_{22} - \beta_{22}.y_{22})),\ Norm_{\mu_{E22+\gamma_{F22}}}(\gamma_{22}.\mu_{22} + i\delta_{22}.\theta_{22})\}$

Where |μ22|= $(Norm_{\vartheta_{E22+\delta_{F22}}}((\alpha_{22}+x_{22}-\alpha_{22}.x_{22}))^2+(Norm_{\vartheta_{E22+\delta_{F22}}}(\beta_{22}+y_{22}-\beta_{22}.y_{22}))^2<1$ | ν22 |= $(Norm_{\mu_{E22+\gamma_{F22}}}(\gamma_{22}.\mu_{22})^2+Norm_{\mu_{E22+\gamma_{F22}}}(\delta_{22}.\theta_{22})^2<1$, |μ22|+|ν22| ≤ 1.

We have Norm$A_E \oplus NormB_F$ = $\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ is also intuitionistic fuzzy matrix.

Example5.2.6: : If A_E=$\begin{pmatrix} (0.6+0.2i,\ 0.3+0.1i)(0.4+0.1i,\ 0.5+0.2i) \\ (0.5+0.3i,\ 0.2+0.1i)(0.3+0.2i,\ 0.6+0.1i) \end{pmatrix}$ and

$$B_F = \begin{pmatrix} (0.4+0.3i,\ 0.5+0.1i)(0.3+0.4i,\ 0.4+0.2i) \\ (0.6+0.2i,\ 0.2+0.2i)(0.5+0.1i,\ 0.4+0.3i) \end{pmatrix}$$

are normalization of complex intuitionistic fuzzy matrix two intuitionistic fuzzy matrices over different universe E and F then Norm$A_E \oplus NormB_F$ is also normalization of complex intuitionistic fuzzy matrix.

Proof:

Given $\boldsymbol{A_E}$=$\begin{pmatrix} (0.6+0.2i,\ 0.3+0.1i)(0.4+0.1i,\ 0.5+0.2i) \\ (0.5+0.3i,\ 0.2+0.1i)(0.3+0.2i,\ 0.6+0.1i) \end{pmatrix}$ and

$$\boldsymbol{B_F} = \begin{pmatrix} (0.4+0.3i,\ 0.5+0.1i)(0.3+0.4i,\ 0.4+0.2i) \\ (0.6+0.2i,\ 0.2+0.2i)(0.5+0.1i,\ 0.4+0.3i) \end{pmatrix}$$

Norm A_E = $\begin{pmatrix} (1+0.67i \quad 0.125+0i) & (0.67+0.33i \quad 0.30+0.10i) \\ (0.83+1i \quad 0+0i) & (0.5+0.67i \quad 0.5+0i) \end{pmatrix}$

$Norm\ B_F$= $\begin{pmatrix} (0.67+0.75i \quad 0.375+0i) & (0.5+1i \quad 0.25+0.11i) \\ (1+0.5i \quad 0+0.11i) & (0.83+0.25i \quad 0.25+0.22i) \end{pmatrix}$

Norm$A_E \oplus NormB_F$ = $\begin{pmatrix} (1+0.67i \quad 0.125+0i) & (0.67+0.33i \quad 0.30+0.10i) \\ (0.83+1i \quad 0+0i) & (0.5+0.67i \quad 0.5+0i) \end{pmatrix}$

$$\oplus \begin{pmatrix} (0.67+0.75i \quad 0.375+0i) & (0.5+1i \quad 0.25+0.11i) \\ (1+0.5i \quad 0+0.11i) & (0.83+0.25i \quad 0.25+0.22i) \end{pmatrix}$$

$X_{11} = (1+0.67i \quad 0.125+0i)\oplus(0.67+0.75i \quad 0.375+0i)$

$X_{12} = (0.67+0.33i \quad 0.30+0.10i)\oplus(0.5+1i \quad 0.25+0.11i)$

$X_{21} = (0.83+1i \quad 0+0i)\oplus(1+0.5i \quad 0+0.11i)$

$X_{22} = (0.5+0.67i \quad 0.5+0i)\oplus(0.83+0.25i \quad 0.25+0.22i)$

Applying formula $A_E \oplus B_F$ = {(x, y), (μ_A(x) + μ_B(y)) - μ_A(x). μ_B(y), ϑ_A(x).ϑ_B(y): x∈E, y∈F}

$A_E \oplus B_F = \begin{pmatrix} 1+0.9175i, 0.046+0i & 0.835+1i,\ 0.075+0.011i \\ 1+1i,\ 0+0i & 0.915+0.752i,\ 0.125+0i \end{pmatrix}$

X_{11} ={1+0.9175i, 0.046+0i}

X_{12} ={0.835+1i, 0.075+0.011i}

X_{21} ={1+1i, 0+0i}

X_{22} ={0.915+0.752i, 0.125+0i}

We have $\text{Norm}A_E \oplus NormB_F = \begin{pmatrix} (1+0.9175\text{i}, 0.046+0\text{i}) & (0.835+1\text{i}, 0.075+0.011\text{i}) \\ (1+1\text{i}, 0+0\text{i}) & (0.915+0.752\text{i}, 0.125+0\text{i}) \end{pmatrix}$ is also normalization of complex intuitionistic fuzzy matrix.

Theorem5.2.7: If E and F be two universal sets. For every normalization of complex intuitionistic fuzzy matrices A_E *and* B_F are in E and F then $\text{Norm}A_E \otimes NormB_F$ is also normalization of complex intuitionistic fuzzy matrix.

Proof: If Norm A_E and $Norm\ B_F$ are two normalization intuitionistic fuzzy matrices over different universes sets E and F. If we consider two 2×2 normalization intuitionistic fuzzy matrices, Norm A_E and $Norm\ B_F$.

Let Norm A_E =

$$\begin{pmatrix} (Norm_{\vartheta_{E11}}(\alpha_{11}+i\beta_{11}) & Norm_{\mu_{E11}}(\gamma_{11}+i\delta_{11})) & (Norm_{\vartheta_{E12}}(\alpha_{12}+i\beta_{12}) & Norm_{\mu_{E12}}(\gamma_{12}+i\delta_{12})) \\ (Norm_{\vartheta_{E21}}(\alpha_{21}+i\beta_{21}) & Norm_{\mu_{E21}}(\gamma_{21}+i\delta_{21})) & (Norm_{\vartheta_{E22}}(\alpha_{22}+i\beta_{22}) & Norm_{\mu_{E22}}(\gamma_{22}+i\delta_{22})) \end{pmatrix}$$

and $Norm\ B_F$ =

$$\begin{pmatrix} (Norm_{\delta_{F11}}(x_{11}+iy_{11}) & Norm_{\gamma_{F11}}(\mu_{11}+i\theta_{11})) & (Norm_{\delta_{F12}}(x_{12}+iy_{12}) & Norm_{\gamma_{F12}}(\mu_{12}+i\theta_{12})) \\ (Norm_{\delta_{F21}}(x_{21}+iy_{21}) & Norm_{\gamma_{F21}}(\mu_{21}+i\theta_{21})) & (Norm_{\delta_{F22}}(x_{22}+iy_{22}) & Norm_{\gamma_{F22}}(\mu_{22}+i\theta_{22})) \end{pmatrix}$$

are two normalization intuitionistic fuzzy matrices.

$\text{Norm}A_E \otimes NormB_F$ =

$$\begin{pmatrix} (Norm_{\vartheta_{E11}}(\alpha_{11}+i\beta_{11}) & Norm_{\mu_{E11}}(\gamma_{11}+i\delta_{11})) & (Norm_{\vartheta_{E12}}(\alpha_{12}+i\beta_{12}) & Norm_{\mu_{E12}}(\gamma_{12}+i\delta_{12})) \\ (Norm_{\vartheta_{E21}}(\alpha_{21}+i\beta_{21}) & Norm_{\mu_{E21}}(\gamma_{21}+i\delta_{21})) & (Norm_{\vartheta_{E22}}(\alpha_{22}+i\beta_{22}) & Norm_{\mu_{E22}}(\gamma_{22}+i\delta_{22})) \end{pmatrix} \otimes$$

$$\begin{pmatrix} (Norm_{\delta_{F11}}(x_{11}+iy_{11}) & Norm_{\gamma_{F11}}(\mu_{11}+i\theta_{11})) & (Norm_{\delta_{F12}}(x_{12}+iy_{12}) & Norm_{\gamma_{F12}}(\mu_{12}+i\theta_{12})) \\ (Norm_{\delta_{F21}}(x_{21}+iy_{21}) & Norm_{\gamma_{F21}}(\mu_{21}+i\theta_{21})) & (Norm_{\delta_{F22}}(x_{22}+iy_{22}) & Norm_{\gamma_{F22}}(\mu_{22}+i\theta_{22})) \end{pmatrix}$$

Calculate, X_{11}, X_{12}, X_{21} and X_{22}, we have

$X_{11} = (Norm_{\vartheta_{E11}}(\alpha_{11}+i\beta_{11}) \quad Norm_{\mu_{E11}}(\gamma_{11}+i\delta_{11})) \otimes (Norm_{\delta_{F11}}(x_{11}+iy_{11}) \quad Norm_{\gamma_{F11}}(\mu_{11}+i\theta_{11}))$

Where $|\mu 11| = (Norm_{\vartheta_{E11}}(\max(\alpha_{11},\ x_{11}))^2 + (Norm_{\vartheta_{E11}} \max(\beta_{11},\ y_{11}))^2 + (Norm_{\delta_{F11}}\ min\ (\gamma_{11},\ \mu_{11}))^2$

$+(Norm_{\delta_{F11}}\ min\ (\gamma_{11},\ \mu_{11}))^2 < 1$, $|\nu 11| = (\text{Norm}_{\mu_{E11}} min(\gamma_{11},\ \mu_{11}))^2 + (Norm_{\gamma_{E11}} min(\delta_{11}, \theta_{11}))^2 < 1$, $|\mu 11| + |\nu 11| \leq 1$.

$X_{12} = (Norm_{\vartheta_{E12}}(\alpha_{12}+i\beta_{12}) \quad Norm_{\mu_{E12}}(\gamma_{12}+i\delta_{12})) \otimes (Norm_{\delta_{F12}}(x_{12}+iy_{12}) \quad Norm_{\gamma_{F12}}(\mu_{12}+i\theta_{12}))$

$X_{21} = (Norm_{\vartheta_{E21}}(\alpha_{21}+i\beta_{21}) \quad Norm_{\mu_{E21}}(\gamma_{21}+i\delta_{21})) \otimes (Norm_{\delta_{F21}}(x_{21}+iy_{21}) \quad Norm_{\gamma_{F21}}(\mu_{21}+i\theta_{21}))$

$X_{22} = (Norm_{\vartheta_{E22}}(\alpha_{22}+i\beta_{22}) \quad Norm_{\mu_{E22}}(\gamma_{22}+i\delta_{22})) \otimes (Norm_{\delta_{F22}}(x_{22}+iy_{22}) \quad Norm_{\gamma_{F22}}(\mu_{22}+i\theta_{22}))$ *Applying formula* $A_E \otimes B_F = \{(x, y), \mu_A(x).\mu_B(y), \vartheta_A(x) + \vartheta_B(y) - \vartheta_A(x).\vartheta_B(y): x \in E, y \in F\}$

$X_{11} = \{Norm_{\vartheta_{E11}}(\alpha_{11}+i\beta_{11}).Norm_{\delta_{F11}}(x_{11}+iy_{11}), Norm_{\mu_{E11}}(\gamma_{11}+i\delta_{11}) + Norm_{\gamma_{F11}}(\mu_{11}+i\theta_{11}) - Norm_{\mu_{E11}}(\gamma_{11}+i\delta_{11}).Norm_{\gamma_{F11}}(\mu_{11}+i\theta_{11})\}$

X_{12} $=\{Norm_{\vartheta_{E12}}(\alpha_{12}+i\beta_{12}).Norm_{\delta_{F12}}(x_{12}+iy_{12}), Norm_{\mu_{E12}}(\gamma_{12}+i\delta_{12})+Norm_{\gamma_{F12}}(\mu_{12}+i\theta_{12})-Norm_{\mu_{E12}}(\gamma_{12}+i\delta_{12}).Norm_{\gamma_{F12}}(\mu_{12}+i\theta_{12})\}$

X_{21} $=\{Norm_{\vartheta_{E21}}(\alpha_{21}+i\beta_{21}).Norm_{\delta_{F21}}(x_{21}+y_{21}), Norm_{\mu_{E21}}(\gamma_{21}+i\delta_{21})+Norm_{\gamma_{F21}}(\mu_{21}+i\theta_{21})-Norm_{\mu_{E21}}(\gamma_{21}+i\delta_{21}).Norm_{\gamma_{F21}}(\mu_{21}+i\theta_{21})\}$

X_{22} $=\{Norm_{\vartheta_{E22}}(\alpha_{22}+i\beta_{22}).Norm_{\delta_{F22}}(x_{22}+y_{22}), Norm_{\mu_{E22}}(\gamma_{22}+i\delta_{22})+Norm_{\gamma_{F22}}(\mu_{22}+i\theta_{22})-Norm_{\mu_{E22}}(\gamma_{22}+i\delta_{22}).Norm_{\gamma_{F22}}(\mu_{22}+i\theta_{22})\}$

$X_{11}=\{Norm_{\vartheta_{E11+\delta_{F11}}}(\alpha_{11}.x_{11})+i(\beta_{11}.y_{11}),\quad Norm_{\mu_{E11+\gamma_{F11}}}(\gamma_{11}+\mu_{11}-\gamma_{11}.\mu_{11}+i(\delta_{11}+\theta_{11}-\delta_{11}.\theta_{11})\}$

Where $|\mu 11|=$ $(Norm_{\vartheta_{E11+\delta_{F11}}}(\alpha_{11}.x_{11}))^2+(Norm_{\vartheta_{E11+\delta_{F11}}}(\beta_{11}.y_{11})^2<1$ $|\nu 11|=$ $(Norm_{\mu_{E11+\gamma_{F11}}}(\gamma_{11}+\mu_{11}-\gamma_{11}.\mu_{11})^2+Norm_{\mu_{E11+\gamma_{F11}}}(\delta_{11}+\theta_{11}-\delta_{11}.\theta_{11})^2<1$, $|\mu 11|+|\nu 11|\leq 1$.

$X_{12}=\{Norm_{\vartheta_{E12+\delta_{F12}}}(\alpha_{12}.x_{12})+i(\beta_{12}.y_{12}),\quad Norm_{\mu_{E12+\gamma_{F12}}}(\gamma_{12}+\mu_{12}-\gamma_{12}.\mu_{12})+i(\delta_{12}+\theta_{12}-\delta_{12}.\theta_{12})\}$

Where $|\mu 12|=$ $(Norm_{\vartheta_{E12+\delta_{F12}}}(\alpha_{12}.x_{12})^2+(Norm_{\vartheta_{E12+\delta_{F12}}}(\beta_{12}.y_{12})^2<1$ $|\nu 12|=$ $(Norm_{\mu_{E12+\gamma_{F12}}}(\gamma_{12}+\mu_{12}-\gamma_{12}.\mu_{12})^2+Norm_{\mu_{E12+\gamma_{F12}}}(\delta_{12}+\theta_{12}-\delta_{12}.\theta_{12})^2<1$, $|\mu 12|+|\nu 12|\leq 1$.

$X_{21}=\{Norm_{\vartheta_{E21+\delta_{F21}}}(\alpha_{21}.x_{21})+i(\beta_{21}.y_{21}),\quad Norm_{\mu_{E21+\gamma_{F21}}}(\gamma_{21}+\mu_{21}-\gamma_{21}.\mu_{21}+i(\delta_{21}+\theta_{21}-\delta_{21}.\theta_{21}\}$

Where $|\mu 21|=$ $(Norm_{\vartheta_{E21+\delta_{F21}}}(\alpha_{21}.x_{21})^2+(Norm_{\vartheta_{E21+\delta_{F21}}}(\beta_{21}.y_{21})^2<1$ $|\nu 21|=$ $(Norm_{\mu_{E21+\gamma_{F21}}}(\gamma_{21}+\mu_{21}-\gamma_{21}.\mu_{21})^2+Norm_{\mu_{E21+\gamma_{F21}}}(\delta_{21}+\theta_{21}-\delta_{21}.\theta_{21})^2<1$, $|\mu 21|+|\nu 21|\leq 1$.

$X_{22}=\{Norm_{\vartheta_{E22+\delta_{F22}}}(\alpha_{22}.x_{22})+i(\beta_{22}.y_{22}),\quad Norm_{\mu_{E22+\gamma_{F22}}}(\gamma_{22}+\mu_{22}-\gamma_{22}.\mu_{22})+i(\delta_{22}+\theta_{22}-\delta_{22}.\theta_{22})\}$

Where $|\mu 22|=$ $(Norm_{\vartheta_{E22+\delta_{F22}}}(\alpha_{22}.x_{22})^2+(Norm_{\vartheta_{E22+\delta_{F22}}}(\beta_{22}.y_{22})^2<1$ $|\nu 22|=$ $(Norm_{\mu_{E22+\gamma_{F22}}}(\gamma_{22}+\mu_{22}-\gamma_{22}.\mu_{22})^2+Norm_{\mu_{E22+\gamma_{F22}}}(\delta_{22}+\theta_{22}-\delta_{22}.\theta_{22})^2<1$, $|\mu 22|+|\nu 22|\leq 1$.

We have $\text{Norm}A_E\otimes NormB_F=\begin{pmatrix}X_{11} & X_{12}\\ X_{21} & X_{22}\end{pmatrix}$ is also normalization of complex intuitionistic fuzzy matrix

Example5.2.8: If A_E=$\begin{pmatrix}(0.6+0.2\text{i},\ 0.3+0.1\text{i})(0.4+0.1\text{i},\ 0.5+0.2\text{i})\\ (0.5+0.3\text{i},\ 0.2+0.1\text{i})(0.3+0.2\text{i},\ 0.6+0.1\text{i})\end{pmatrix}$ and

$$\text{B}_\text{F}=\begin{pmatrix}(0.4+0.3\text{i},\ 0.5+0.1\text{i})(0.3+0.4\text{i},\ 0.4+0.2\text{i})\\ (0.6+0.2\text{i},\ 0.2+0.2\text{i})(0.5+0.1\text{i},\ 0.4+0.3\text{i})\end{pmatrix}$$

are normalization of complex intuitionistic fuzzy matrix two intuitionistic fuzzy matrices over different universe E and F then $\text{Norm}A_E\otimes NormB_F$ is also normalization of complex intuitionistic fuzzy matrix.

Proof:

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Given $\boldsymbol{A_E}=\begin{pmatrix}(0.6+0.2\mathrm{i},\ 0.3+0.1\mathrm{i})(0.4+0.1\mathrm{i},\ 0.5+0.2\mathrm{i})\\(0.5+0.3\mathrm{i},\ 0.2+0.1\mathrm{i})(0.3+0.2\mathrm{i},\ 0.6+0.1\mathrm{i})\end{pmatrix}$ and

$$\boldsymbol{B_F}=\begin{pmatrix}(0.4+0.3\mathrm{i},\ 0.5+0.1\mathrm{i})(0.3+0.4\mathrm{i},\ 0.4+0.2\mathrm{i})\\(0.6+0.2\mathrm{i},\ 0.2+0.2\mathrm{i})(0.5+0.1\mathrm{i},\ 0.4+0.3\mathrm{i})\end{pmatrix}$$

Norm $A_E=\begin{pmatrix}(1+0.67i \quad 0.125+0i) & (0.67+0.33i \quad 0.30+0.10i)\\ (0.83+1i \quad 0+0i) & (0.5+0.67i \quad 0.5+0i)\end{pmatrix}$

$Norm\ B_F=\begin{pmatrix}(0.67+0.75i \quad 0.375+0i) & (0.5+1i \quad 0.25+0.11i)\\ (1+0.5i \quad 0+0.11i) & (0.83+0.25i \quad 0.25+0.22i)\end{pmatrix}$

Apply the formula $A_E \otimes B_F = \{(x, y), \mu_A(x).\ \mu_B(y), \vartheta_A(x) + \vartheta_B(y) - \vartheta_A(x).\vartheta_B(y): x\in E, y\in F$

$$A_E\otimes B_F=\begin{pmatrix}(1+0.67i \quad 0.125+0i) & (0.67+0.33i \quad 0.30+0.10i)\\ (0.83+1i \quad 0+0i) & (0.5+0.67i \quad 0.5+0i)\end{pmatrix}$$

$$\otimes\begin{pmatrix}(0.67+0.75i \quad 0.375+0i) & (0.5+1i \quad 0.25+0.11i)\\ (1+0.5i \quad 0+0.11i) & (0.83+0.25i \quad 0.25+0.22i)\end{pmatrix}$$

$X_{11}=\ (1+0.67i \quad 0.125+0i)\otimes(0.67+0.75i \quad 0.375+0i)$

$X_{12}=\ (0.67+0.33i \quad 0.30+0.10i)\otimes(0.5+1i \quad 0.25+0.11i)$

$X_{21}=(0.83+1i \quad 0+0i)\otimes(1+0.5i \quad 0+0.11i)$

$X_{22}=(0.5+0.67i \quad 0.5+0i)\otimes(0.83+0.25i \quad 0.25+0.22i)$

Applying formula $A_E \otimes B_F = \{(x, y), \mu_A(x).\ \mu_B(y), \vartheta_A(x) + \vartheta_B(y) - \vartheta_A(x).\vartheta_B(y): x\in E, y\in F$

$$A_E\otimes B_F=\begin{pmatrix}0.67+0.502i, 0.453+0i & 0.335+0.33i,\ 0.65+0.199i\\ 0.83+0.5i,\ 0+0.11i & 0.415+0.1678i,\ 0.625+0.22i\end{pmatrix}$$

$X_{11}=$ {0.807+1i, 0.0.453+0i}

$X_{12}=$ {0.403+0.667i, 0.65+0.199i}

$X_{21}=$ {1+0.99i, 0+0.11i}

$X_{22}=$ {0.5+0.332i, 0.625+022i}

We have Norm$A_E\otimes NormB_F=\begin{pmatrix}(0.807+1\mathrm{i}, 0.0.453+0\mathrm{i}) & (0.403+0.667\mathrm{i}, 0.65+0.199\mathrm{i})\\ (1+0.99\mathrm{i}, 0+0.11\mathrm{i}) & (0.5+0.332\mathrm{i}, 0.625+022\mathrm{i})\end{pmatrix}$ is also normalization of complex intuitionistic fuzzy matrix.

Theorem 5.2.9: If E and F be two universal sets. For every normalization of complex intuitionistic fuzzy matrices A_E *and* B_F are in E and F then Norm$A_E@NormB_F$ is also normalization of complex intuitionistic fuzzy matrix.

Proof: If Norm A_E and $Norm\ B_F$ are two normalization intuitionistic fuzzy matrices over different univer*ses sets E and F*. If we consider two 2×2 normalization intuitionistic fuzzy matrices, Norm A_E*and* $Norm\ B_F$.

Let Norm A_E =

$$\begin{pmatrix}\left(Norm_{\vartheta_{E11}}(\alpha_{11}+i\beta_{11}) \quad Norm_{\mu_{E11}}(\gamma_{11}+i\delta_{11})\right) & \left(Norm_{\vartheta_{E12}}(\alpha_{12}+i\beta_{12}) \quad Norm_{\mu_{E12}}(\gamma_{12}+i\delta_{12})\right)\\ \left(Norm_{\vartheta_{E21}}(\alpha_{21}+i\beta_{21}) \quad Norm_{\mu_{E21}}(\gamma_{21}+i\delta_{21})\right) & \left(Norm_{\vartheta_{E22}}(\alpha_{22}+i\beta_{22}) \quad Norm_{\mu_{E22}}(\gamma_{22}+i\delta_{22})\right)\end{pmatrix}$$

and $Norm\ B_F$
$=$
$\begin{pmatrix} (Norm_{\delta_{F11}}(x_{11}+iy_{11}) & Norm_{\gamma_{F11}}(\mu_{11}+i\theta_{11})) & (Norm_{\delta_{F12}}(x_{12}+iy_{12}) & Norm_{\gamma_{F12}}(\mu_{12}+i\theta_{12})) \\ (Norm_{\delta_{F21}}(x_{21}+iy_{21}) & Norm_{\gamma_{F21}}(\mu_{21}+i\theta_{21})) & (Norm_{\delta_{F22}}(x_{22}+iy_{22}) & Norm_{\gamma_{F22}}(\mu_{22}+i\theta_{22})) \end{pmatrix}$

are two normalization intuitionistic fuzzy matrices.

Norm$A_E @ Norm B_F \quad =$
$\begin{pmatrix} (Norm_{\vartheta_{E11}}(\alpha_{11}+i\beta_{11}) & Norm_{\mu_{E11}}(\gamma_{11}+i\delta_{11})) & (Norm_{\vartheta_{E12}}(\alpha_{12}+i\beta_{12}) & Norm_{\mu_{E12}}(\gamma_{12}+i\delta_{12})) \\ (Norm_{\vartheta_{E21}}(\alpha_{21}+i\beta_{21}) & Norm_{\mu_{E21}}(\gamma_{21}+i\delta_{21})) & (Norm_{\vartheta_{E22}}(\alpha_{22}+i\beta_{22}) & Norm_{\mu_{E22}}(\gamma_{22}+i\delta_{22})) \end{pmatrix} @$
$\begin{pmatrix} (Norm_{\delta_{F11}}(x_{11}+iy_{11}) & Norm_{\gamma_{F11}}(\mu_{11}+i\theta_{11})) & (Norm_{\delta_{F12}}(x_{12}+iy_{12}) & Norm_{\gamma_{F12}}(\mu_{12}+i\theta_{12})) \\ (Norm_{\delta_{F21}}(x_{21}+iy_{21}) & Norm_{\gamma_{F21}}(\mu_{21}+i\theta_{21})) & (Norm_{\delta_{F22}}(x_{22}+iy_{22}) & Norm_{\gamma_{F22}}(\mu_{22}+i\theta_{22})) \end{pmatrix}$

Calculate, X_{11}, X_{12}, $X_{21}$and X_{22}, we have

$X_{11} = (Norm_{\vartheta_{E11}}(\alpha_{11}+i\beta_{11}) \quad Norm_{\mu_{E11}}(\gamma_{11}+i\delta_{11}))$
$@(Norm_{\delta_{F11}}(x_{11}+iy_{11}) \quad Norm_{\gamma_{F11}}(\mu_{11}+i\theta_{11}))$

Where $|\mu 11| = (Norm_{\vartheta_{E11}}(\max(\alpha_{11},\ x_{11}))^2 + (Norm_{\vartheta_{E11}} \max(\beta_{11},\ y_{11}))^2 + (Norm_{\delta_{F11}}\ min\ (\gamma_{11},\ \mu_{11}))^2$

$+(Norm_{\delta_{F11}}\ min\ (\gamma_{11},\ \mu_{11}))^2 < 1$, $|\nu 11| = (\mathrm{Norm}_{\mu_{E11}} min(\gamma_{11},\ \mu_{11}))^2 + (Norm_{\gamma_{E11}} min(\delta_{11}, \theta_{11}))^2 < 1$, $|\mu 11|+|\nu 11| \leq 1$.

$X_{12} = (Norm_{\vartheta_{E12}}(\alpha_{12}+i\beta_{12}) \quad Norm_{\mu_{E12}}(\gamma_{12}+i\delta_{12}))$
$@\ (Norm_{\delta_{F12}}(x_{12}+iy_{12}) \quad Norm_{\gamma_{F12}}(\mu_{12}+i\theta_{12}))$

$X_{21} = (Norm_{\vartheta_{E21}}(\alpha_{21}+i\beta_{21}) \quad Norm_{\mu_{E21}}(\gamma_{21}+i\delta_{21}))$
$@\ (Norm_{\delta_{F21}}(x_{21}+iy_{21}) \quad Norm_{\gamma_{F21}}(\mu_{21}+i\theta_{21}))$

$X_{22} = (Norm_{\vartheta_{E22}}(\alpha_{22}+i\beta_{22}) \quad Norm_{\mu_{E22}}(\gamma_{22}+i\delta_{22}))$
$@\ (Norm_{\delta_{F22}}(x_{22}+iy_{22}) \quad Norm_{\gamma_{F22}}(\mu_{22}+i\theta_{22}))$

Applying formula $\mathrm{A_E @ B_F} = \left\{(\mathrm{x,y}), \frac{\mu_A(x)+\mu_B(y)}{2}, \frac{\vartheta_A(x)+\vartheta_B(y)}{2} : x\in E, y\in F\right\}$

$X_{11} = \{Norm_{\vartheta_{E11@\delta_{F11}}}\left(\frac{\alpha_{11}+x_{11}+i(\beta_{11}+y_{11})}{2}\right), Norm_{\mu_{E11@\gamma_{F11}}}\left(\frac{\gamma_{11}+\mu_{11}+i(\delta_{11}+\theta_{11})}{2}\right)\}$

Where $((Norm_{\vartheta_{E11@\delta_{F11}}}\left(\frac{\alpha_{11}+x_{11}}{2}\right)$

$X_{12} = \{Norm_{\vartheta_{E11@\delta_{F11}}}\left(\frac{\alpha_{11}+x_{11}+i(\beta_{11}+y_{11})}{2}\right), Norm_{\mu_{E11@\gamma_{F11}}}\left(\frac{\gamma_{11}+\mu_{11}+i(\delta_{11}+\theta_{11})}{2}\right)\}$

$X_{21} = \{Norm_{\vartheta_{E11@\delta_{F11}}}\left(\frac{\alpha_{11}+x_{11}+i(\beta_{11}+y_{11})}{2}\right), Norm_{\mu_{E11@\gamma_{F11}}}\left(\frac{\gamma_{11}+\mu_{11}+i(\delta_{11}+\theta_{11})}{2}\right)\}$

$X_{22} = \{Norm_{\vartheta_{E11@\delta_{F11}}}\left(\frac{\alpha_{11}+x_{11}+i(\beta_{11}+y_{11})}{2}\right), Norm_{\mu_{E11@\gamma_{F11}}}\left(\frac{\gamma_{11}+\mu_{11}+i(\delta_{11}+\theta_{11})}{2}\right)\}$

$X_{11} = \{Norm_{\vartheta_{E11+\delta_{F11}}}(\alpha_{11}.x_{11}) + i(\beta_{11}.y_{11}), \quad Norm_{\mu_{E11+\gamma_{F11}}}(\gamma_{11} + \mu_{11} - \gamma_{11}.\mu_{11} + i(\delta_{11} + \theta_{11} - \delta_{11}.\theta_{11})\}$

Where $|\mu 11| = (Norm_{\vartheta_{E11+\delta_{F11}}}(\alpha_{11}.x_{11}))^2 + (Norm_{\vartheta_{E11+\delta_{F11}}}(\beta_{11}.y_{11})^2 < 1$ $|\nu 11| = (Norm_{\mu_{E11+\gamma_{F11}}}(\gamma_{11} + \mu_{11} - \gamma_{11}.\mu_{11})^2 + Norm_{\mu_{E11+\gamma_{F11}}}(\delta_{11} + \theta_{11} - \delta_{11}.\theta_{11})^2 < 1$, $|\mu 11|+|\nu 11| \leq 1$.

$X_{12} = \{Norm_{\vartheta_{E12+}\delta_{F12}}(\alpha_{12}.x_{12}) + i(\beta_{12}.y_{12}), \quad Norm_{\mu_{E12+\gamma_{F12}}}(\gamma_{12} + \mu_{12} - \gamma_{12}.\mu_{12}) + i(\delta_{12} + \theta_{12} - \delta_{12}.\theta_{12})\}$

Where |μ12|= $(Norm_{\vartheta_{E12+}\delta_{F12}}(\alpha_{12}.x_{12})^2 + (Norm_{\vartheta_{E12+}\delta_{F12}}(\beta_{12}.y_{12})^2 < 1$ | ν12 |= $(Norm_{\mu_{E12+\gamma_{F12}}}(\gamma_{12} + \mu_{12} - \gamma_{12}.\mu_{12})^2 + Norm_{\mu_{E12+\gamma_{F12}}}(\delta_{12} + \theta_{12} - \delta_{12}.\theta_{12})^2 < 1$, |μ12|+|ν12| ≤ 1.

$X_{21} = \{Norm_{\vartheta_{E21+}\delta_{F21}}(\alpha_{21}.x_{21}) + i(\beta_{21}.y_{21}), \quad Norm_{\mu_{E21+\gamma_{F21}}}(\gamma_{21} + \mu_{21} - \gamma_{21}.\mu_{21} + i(\delta_{21} + \theta_{21} - \delta_{21}.\theta_{21}\}$

Where |μ21|= $(Norm_{\vartheta_{E21+}\delta_{F21}}(\alpha_{21}.x_{21})^2 + (Norm_{\vartheta_{E21+}\delta_{F21}}(\beta_{21}.y_{21})^2 < 1$ | ν21 |= $(Norm_{\mu_{E21+\gamma_{F21}}}(\gamma_{21} + \mu_{21} - \gamma_{21}.\mu_{21})^2 + Norm_{\mu_{E21+\gamma_{F21}}}(\delta_{21} + \theta_{21} - \delta_{21}.\theta_{21})^2 < 1$, |μ21|+|ν21| ≤ 1.

$X_{22} = \{Norm_{\vartheta_{E22+}\delta_{F22}}(\alpha_{22}.x_{22}) + i(\beta_{22}.y_{22}), \quad Norm_{\mu_{E22+\gamma_{F22}}}(\gamma_{22} + \mu_{22} - \gamma_{22}.\mu_{22}) + i(\delta_{22} + \theta_{22} - \delta_{22}.\theta_{22})\}$

Where |μ22|= $(Norm_{\vartheta_{E22+}\delta_{F22}}(\alpha_{22}.x_{22})^2 + (Norm_{\vartheta_{E22+}\delta_{F22}}(\beta_{22}.y_{22})^2 < 1$ | ν22 |= $(Norm_{\mu_{E22+\gamma_{F22}}}(\gamma_{22} + \mu_{22} - \gamma_{22}.\mu_{22})^2 + Norm_{\mu_{E22+\gamma_{F22}}}(\delta_{22} + \theta_{22} - \delta_{22}.\theta_{22})^2 < 1$, |μ22|+|ν22| ≤ 1.

We have Norm$A_E \otimes NormB_F$ = $\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ is also intuitionistic fuzzy matrix

Example5.2.10: If $\boldsymbol{A_E}$=$\begin{pmatrix} (.9,.4) & (.4,.1) \\ (.5,.3) & (.8,.3) \end{pmatrix}$ and$\boldsymbol{B_F}$ = $\begin{pmatrix} (.8,.6) & (.6,.3) \\ (.7,.5) & (.9,.4) \end{pmatrix}$ are two intuitionistic fuzzy matrices over different universe E and F then Norm$\bar{A}_E @ Norm\bar{B}_F$is also fuzzy matrix set.

Proof: Given $\boldsymbol{A_E}$ =$\begin{pmatrix} (.9,.4) & (.4,.1) \\ (.5,.3) & (.8,.3) \end{pmatrix}$ and $\boldsymbol{B_F}$ = $\begin{pmatrix} (.8,.6) & (.6,.3) \\ (.7,.5) & (.9,.4) \end{pmatrix}$

$\overline{Norm}_A$ =$\begin{pmatrix} (.8,1) & (0,.3) \\ (.2,.8) & (.7,.8) \end{pmatrix}$ and $\overline{Norm}_B$=$\begin{pmatrix} (.5,1) & (0,.5) \\ (.3,.8) & (.8,.7) \end{pmatrix}$

$\overline{Norm}_A @ \overline{Norm}_B = \begin{pmatrix} (.8,1) & (0,.3) \\ (.2,.8) & (.7,.8) \end{pmatrix} @ \begin{pmatrix} (.5,1) & (0,.5) \\ (.3,.8) & (.8,.7) \end{pmatrix}$

$X_{11} = (.8,1)@(.5,1)$

$X_{12} = (0,.3)@(0,.5)$

$X_{21} = (.2,.8) @ (.3,.8)$

$X_{22} = (.7,.8)@(.8,.7)$

$Applying\ formula\ \overline{A_E}@\overline{B_F} = \{\langle x, y\rangle \frac{(\vartheta_A(x) + \vartheta_B(y))}{2}, \frac{(\mu_A(x)\,.\,\mu_B(y))}{2}: x\in E, y\in F\}$

$X_{11} = \left(\frac{.8+.5}{2}\right), \left(\frac{1\,.\,1}{2}\right)$

$X_{12} = \left(\frac{0+0}{2}\right), \left(\frac{.3\ \ .5}{2}\right)$

$X_{21} = \left(\frac{.2+.3}{2}\right), \left(\frac{.8\ \ .8}{2}\right)$

$X_{22} = \left(\frac{.7+.8}{2}\right), \left(\frac{.8\ \ .7}{2}\right)$

X_{11} ={.7, .5}

X_{12} ={0, .07}

X_{21} ={.3, .3}

X_{22} ={.8, .3}

We have Norm$\bar{A}_E$@ $Norm\bar{B}_F$ = $\begin{pmatrix} (.7,\ .5) & (0,.07) \\ (.3,\ .3) & (.8\ ,.3) \end{pmatrix}$ is also intuitionistic fuzzy matrix

Theorem 5.2.6: If E and F be two universal sets. For every normalization of an intuitionistic fuzzy matrices $\overline{A_E}$ *and* $\overline{B_F}$ are in E and F then Norm$\bar{A}_E$ \$ $Norm\bar{B}_F$is also normalization of an intuitionistic fuzzy matrices.

Proof: If $Norm_A$ and $Norm_B$ are two normalization intuitionistic fuzzy matrices over different universes sets E and F. If we consider two 2×2 matrix

$\overline{Norm_A}$ $= \begin{pmatrix} (Norm_{\vartheta_{E11}}(x) & Norm_{\mu_{E11}}(x)) & (Norm_{\vartheta_{E12}}(x) & Norm_{\mu_{E12}}(x)) \\ (Norm_{\vartheta_{E21}}(x) & Norm_{\mu_{E21}}(x)) & (Norm_{\vartheta_{E22}}(x) & Norm_{\mu_{E22}}(x)) \end{pmatrix}$ and $\overline{Norm_B}$=$\begin{pmatrix} (Norm_{\delta_{F11}}(y) & Norm_{\gamma_{F11}}(y)) & (Norm_{\delta_{F12}}(y) & Norm_{\gamma_{F12}}(y)) \\ (Norm_{\delta_{F21}}(y) & Norm_{\gamma_{F21}}(y)) & (Norm_{\delta_{F22}}(y) & Norm_{\gamma_{F22}}(y)) \end{pmatrix}$

$$\overline{Norm_A}\ \$\ \overline{Norm_B} = \begin{pmatrix} (Norm_{\vartheta_{E11}}(x) & Norm_{\mu_{E11}}(x)) & (Norm_{\vartheta_{E12}}(x) & Norm_{\mu_{E12}}(x)) \\ (Norm_{\vartheta_{E21}}(x) & Norm_{\mu_{E21}}(x)) & (Norm_{\vartheta_{E22}}(x) & Norm_{\mu_{E22}}(x)) \end{pmatrix}\ \$$$

$$\begin{pmatrix} (Norm_{\delta_{F11}}(y) & Norm_{\gamma_{F11}}(y)) & (Norm_{\delta_{F12}}(y) & Norm_{\gamma_{F12}}(y)) \\ (Norm_{\delta_{F21}}(y) & Norm_{\gamma_{F21}}(y)) & (Norm_{\delta_{F22}}(y) & Norm_{\gamma_{F22}}(y)) \end{pmatrix}$$

X_{11} =$(Norm_{\vartheta_{E11}}(x)\quad Norm_{\mu_{E11}}(x))$ \$ $(Norm_{\delta_{F11}}(y)\quad Norm_{\gamma_{F11}}(y))$

X_{12} =$(Norm_{\vartheta_{E12}}(x)\quad Norm_{\mu_{E12}}(x))$ \$ $(Norm_{\delta_{F12}}(y)\quad Norm_{\gamma_{F12}}(y))$

X_{21} =$(Norm_{\vartheta_{E21}}(x)\quad Norm_{\mu_{E21}}(x))$ \$ $(Norm_{\delta_{F21}}(y)\quad Norm_{\gamma_{F21}}(y))$

X_{22} =$(Norm_{\vartheta_{E22}}(x)\quad Norm_{\mu_{E22}}(x))$ \$ $(Norm_{\delta_{F22}}(y)\quad Norm_{\gamma_{F22}}(y))$

Applying formula $\overline{A_E}$ \$ $\overline{B_F}$ = $\{\langle x,y\rangle, \sqrt{\vartheta_A(x)\,.\,\vartheta_B(y)}, \sqrt{\mu_A(x)\,.\,\mu_B(y)}\ ,: x \in E,\ y \in F\ \}$

$$X_{11} = \left(\sqrt{Norm_{\vartheta_{E11}}(x).Norm_{\delta_{F11}}(y)}\ , \sqrt{Norm_{\mu_{E11}}(x).Norm_{\gamma_{F11}}(y)}\right)$$

$$X_{12} = \left(\sqrt{Norm_{\vartheta_{E12}}(x).Norm_{\delta_{F12}}(y)}\ , \sqrt{Norm_{\mu_{E12}}(x).Norm_{\gamma_{F12}}(y)}\right)$$

$$X_{21} = \left(\sqrt{Norm_{\vartheta_{21}}(x).Norm_{\delta_{F21}}(y)}\ , \sqrt{Norm_{\mu_{E21}}(x).Norm_{\gamma_{F21}}(y)}\right)$$

$$X_{22} = \left(\sqrt{Norm_{\vartheta_{E22}}(x).Norm_{\delta_{F22}}(y)}\ , \sqrt{Norm_{\mu_{E22}}(x).Norm_{\gamma_{F22}}(y)}\right)$$

We have Norm$\bar{A}_E$\$ $Norm\bar{B}_F$ = $\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ is also intuitionistic fuzzy matrix

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Example: If $\boldsymbol{A_E}=\begin{pmatrix} (.9,.4) & (.4,.1) \\ (.5,.3) & (.8,.3) \end{pmatrix}$ and $\boldsymbol{B_F}=\begin{pmatrix} (.8,.6) & (.6,.3) \\ (.7,.5) & (.9,.4) \end{pmatrix}$ are two intuitionistic fuzzy matrices over different universe E and F then Norm$\bar{A}_E \$ Norm\bar{B}_F$ is also fuzzy matrix set.

Proof: Given $\boldsymbol{A_E}=\begin{pmatrix} (.9,.4) & (.4,.1) \\ (.5,.3) & (.8,.3) \end{pmatrix}$ and $\boldsymbol{B_F}=\begin{pmatrix} (.8,.6) & (.6,.3) \\ (.7,.5) & (.9,.4) \end{pmatrix}$

$\overline{Norm_A}=\begin{pmatrix} (.8,1) & (0,.3) \\ (.2,.8) & (.7,.8) \end{pmatrix}$ and $\overline{Norm_B}=\begin{pmatrix} (.5,1) & (0,.5) \\ (.3,.8) & (.8,.7) \end{pmatrix}$

$\overline{Norm_A} \$ \overline{Norm_B}=\begin{pmatrix} (.8,1) & (0,.3) \\ (.2,.8) & (.7,.8) \end{pmatrix} \$ \begin{pmatrix} (.5,1) & (0,.5) \\ (.3,.8) & (.8,.7) \end{pmatrix}$

$X_{11}= (.8,1)\$(.5,1)$

$X_{12}= (0,.3)\$(0,.5)$

$X_{21}= (.2,.8)\$(.3,.8)$

$X_{22}= (.7,.8)\$(.8,.7)$

Applying formula $\overline{A_E} \$ \overline{B_F}=\{\langle x,y\rangle, \sqrt{\vartheta_A(x).\vartheta_B(y)}, \sqrt{\mu_A(x).\mu_B(y)}, : x\in E, y\in F\}$

$X_{11}=\left(\sqrt{.8\ .5}, \sqrt{1.\ 1}\right)$

$X_{12}=\left(\sqrt{0.\ 0}, \sqrt{.3\ \ .5}\right)$

$X_{21}=\left(\sqrt{.2\ .3}, \sqrt{.8\ .8}\right)$

$X_{22}=\left(\sqrt{.7\ .8}, \sqrt{.8\ .7}\right)$

$X_{11}=\{.6,\ .1\}$

$X_{12}=\{0\ ,.4\}$

$X_{21}=\{.2,\ .8\}$

$X_{22}=\{.7\ ,.7\}$

We have Norm$\bar{A}_E \$ Norm\bar{B}_F=\begin{pmatrix} (.6,\ .1) & (0\ ,.4) \\ (.2,\ .8) & (.7\ ,.7) \end{pmatrix}$ is also intuitionistic fuzzy matrix.

Theorem 5.2.7: If E and F be two universal sets. For every normalization of an intuitionistic fuzzy matrices $\overline{A_E}$ *and* $\overline{B_F}$ are in E and F then Norm$\bar{A}_E \# Norm\bar{B}_F$ is also normalization of an intuitionistic fuzzy matrices.

Proof: If $Norm_A$ and $Norm_B$ are two normalization intuitionistic fuzzy matrices over different univer*ses sets E and F. If we consider two* 2×2 *matrix*

$\overline{Norm_A}=\begin{pmatrix} (Norm_{\vartheta_{E11}}(x) & Norm_{\mu_{E11}}(x)) & (Norm_{\vartheta_{E12}}(x) & Norm_{\mu_{E12}}(x)) \\ (Norm_{\vartheta_{E21}}(x) & Norm_{\mu_{E21}}(x)) & (Norm_{\vartheta_{E22}}(x) & Norm_{\mu_{E22}}(x)) \end{pmatrix}$ and

$\overline{Norm_B}=\begin{pmatrix} (Norm_{\delta_{F11}}(y) & Norm_{\gamma_{F11}}(y)) & (Norm_{\delta_{F12}}(y) & Norm_{\gamma_{F12}}(y)) \\ (Norm_{\delta_{F21}}(y) & Norm_{\gamma_{F21}}(y)) & (Norm_{\delta_{F22}}(y) & Norm_{\gamma_{F22}}(y)) \end{pmatrix}$

$\overline{Norm_A} \# \overline{Norm_B}=\begin{pmatrix} (Norm_{\vartheta_{E11}}(x) & Norm_{\mu_{E11}}(x)) & (Norm_{\vartheta_{E12}}(x) & Norm_{\mu_{E12}}(x)) \\ (Norm_{\vartheta_{E21}}(x) & Norm_{\mu_{E21}}(x)) & (Norm_{\vartheta_{E22}}(x) & Norm_{\mu_{E22}}(x)) \end{pmatrix} \#$

$$\begin{pmatrix} (Norm_{\delta_{F11}}(y) & Norm_{\gamma_{F11}}(y)) & (Norm_{\delta_{F12}}(y) & Norm_{\gamma_{F12}}(y)) \\ (Norm_{\delta_{F21}}(y) & Norm_{\gamma_{F21}}(y)) & (Norm_{\delta_{F22}}(y) & Norm_{\gamma_{F22}}(y)) \end{pmatrix}$$

X_{11} =$(Norm_{\vartheta_{E11}}(x) \quad Norm_{\mu_{E11}}(x))$ # $(Norm_{\delta_{F11}}(y) \quad Norm_{\gamma_{F11}}(y))$

X_{12} =$(Norm_{\vartheta_{E12}}(x) \quad Norm_{\mu_{E12}}(x))$ # $(Norm_{\delta_{F12}}(y) \quad Norm_{\gamma_{F12}}(y))$

X_{21} =$(Norm_{\vartheta_{E21}}(x) \quad Norm_{\mu_{E21}}(x))$ # $(Norm_{\delta_{F21}}(y) \quad Norm_{\gamma_{F21}}(y))$

X_{22} =$(Norm_{\vartheta_{E22}}(x) \quad Norm_{\mu_{E22}}(x))$ # $(Norm_{\delta_{F22}}(y) \quad Norm_{\gamma_{F22}}(y))$

$Applying\ formula\ \overline{A_E}$ # $\overline{B_F}$ = $\{\langle x, y\rangle, \frac{2\vartheta_A(x).\vartheta_B(y)}{\vartheta_A(x)+\vartheta_B(y)}, \frac{2\mu_A(x).\mu_B(y)}{\mu_A(x)+\mu_B(y)} : x\in E, y\in F\}$

$$X_{11} = \left(\frac{2Norm_{\vartheta_{E11}}(x).Norm_{\delta_{F11}}(y)}{Norm_{\vartheta_{E11}}(x) + Norm_{\delta_{F11}}(y)}, \frac{2Norm_{\mu_{E11}}(x).Norm_{\gamma_{F11}}(y)}{Norm_{\mu_{E11}}(x) + Norm_{\gamma_{F11}}(y)}\right)$$

$$X_{12} = \left(\frac{2Norm_{\vartheta_{E12}}(x).Norm_{\delta_{F12}}(y)}{Norm_{\vartheta_{E12}}(x) + Norm_{\delta_{F12}}(y)}, \frac{2Norm_{\mu_{E12}}(x).Norm_{\gamma_{F12}}(y)}{Norm_{\mu_{E12}}(x) + Norm_{\gamma_{F12}}(y)}\right)$$

$$X_{21} = \left(\frac{2Norm_{\vartheta_{E21}}(x).Norm_{\delta_{F21}}(y)}{Norm_{\vartheta_{E21}}(x) + Norm_{\delta_{F21}}(y)}, \frac{2Norm_{\mu_{E21}}(x).Norm_{\gamma_{F21}}(y)}{Norm_{\mu_{E21}}(x) + Norm_{\gamma_{F21}}(y)}\right)$$

$$X_{22} = \left(\frac{2Norm_{\vartheta_{E22}}(x).Norm_{\delta_{F22}}(y)}{Norm_{\vartheta_{E22}}(x) + Norm_{\delta_{F22}}(y)}, \frac{2Norm_{\mu_{E22}}(x).Norm_{\gamma_{F22}}(y)}{Norm_{\mu_{E22}}(x) + Norm_{\gamma_{F22}}(y)}\right)$$

We have Norm$\bar{A}_E$# $Norm\bar{B}_F$ = $\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ is also intuitionistic fuzzy matrix

Example: If $\boldsymbol{A_E}$=$\begin{pmatrix} (.9,.4) & (.4,.1) \\ (.5,.3) & (.8,.3) \end{pmatrix}$ and$\boldsymbol{B_F} = \begin{pmatrix} (.8,.6) & (.6,.3) \\ (.7,.5) & (.9,.4) \end{pmatrix}$ are two intuitionistic fuzzy matrices over different universe E and F then Norm$\bar{A}_E$\$$Norm\bar{B}_F$is also fuzzy matrix set.

Proof: Given $\boldsymbol{A_E}$ =$\begin{pmatrix} (.9,.4) & (.4,.1) \\ (.5,.3) & (.8,.3) \end{pmatrix}$ and $\boldsymbol{B_F} = \begin{pmatrix} (.8,.6) & (.6,.3) \\ (.7,.5) & (.9,.4) \end{pmatrix}$

$\overline{Norm}_A$ =$\begin{pmatrix} (.8,1) & (0,.3) \\ (.2,.8) & (.7,.8) \end{pmatrix}$ and $\overline{Norm}_B$=$\begin{pmatrix} (.5,1) & (0,.5) \\ (.3,.8) & (.8,.7) \end{pmatrix}$

$$\overline{Norm}_A \# \overline{Norm}_B = \begin{pmatrix} (.8,1) & (0,.3) \\ (.2,.8) & (.7,.8) \end{pmatrix} \# \begin{pmatrix} (.5,1) & (0,.5) \\ (.3,.8) & (.8,.7) \end{pmatrix}$$

$X_{11} = (.8, 1)\#(.5, 1)$

$X_{12} = (0,.3)\ \#(0,.5)$

$X_{21} = (.2,.8)\#\ (.3,.8)$

$X_{22} = (.7,.8)\#(.8,.7)$

$Applying\ formula\overline{A_E}$ # $\overline{B_F}$ = $\{\langle x, y\rangle, \frac{2\vartheta_A(x).\vartheta_B(y)}{\vartheta_A(x)+\vartheta_B(y)}, \frac{2\mu_A(x).\mu_B(y)}{\mu_A(x)+\mu_B(y)} : x\in E, y\in F\}$

$$X_{11} = \left(\frac{2(.8).(.5)}{(.8) + (.5)}, \frac{2(1).(1)}{1 + 1}\right)$$

$$X_{12} = \left(\frac{2(0).(0)}{(0) + (0)}, \frac{2(.3).(5)}{(.3) + (.5)}\right)$$

$X_{21} = \left(\frac{2(.2).(.3)}{(.2)+(.3)}, \frac{2(.8).(.8)}{(.8)+(.8)}\right)$

$X_{22} = \left(\frac{2(.7).(.8)}{(.7)+(.8)}, \frac{2(.8).(7)}{(.8)+(.7)}\right)$

X_{11} ={.6 , 1}

X_{12} ={0 , .4}

X_{21} ={.2, .8}

X_{22} ={.7 ,.7}

We have Norm$\bar{A}_E\$\, Norm\bar{B}_F$ = $\begin{pmatrix} (.6, & .1) & (0\,, .4) \\ (.2, & .8) & (.7\,, .7) \end{pmatrix}$ is also intuitionistic fuzzy matrix.

Theorem5.2.8: If E and F be two universal sets. For every normalization of an intuitionistic fuzzy matrices $\overline{A_E}$ *and* $\overline{B_F}$ are in E and F then Norm$\bar{A}_E * Norm\bar{B}_F$is also normalization of an intuitionistic fuzzy matrices.

Proof: If $Norm_A$ and $Norm_B$ are two normalization intuitionistic fuzzy matrices over different universes *sets E and F.If we consider two* 2×2 *matrix*

$\overline{Norm_A} = \begin{pmatrix} (Norm_{\vartheta_{E11}}(x) & Norm_{\mu_{E11}}(x)) & (Norm_{\vartheta_{E12}}(x) & Norm_{\mu_{E12}}(x)) \\ (Norm_{\vartheta_{E21}}(x) & Norm_{\mu_{E21}}(x)) & (Norm_{\vartheta_{E22}}(x) & Norm_{\mu_{E22}}(x)) \end{pmatrix}$ and

$\overline{Norm_B} = \begin{pmatrix} (Norm_{\delta_{F11}}(y) & Norm_{\gamma_{F11}}(y)) & (Norm_{\delta_{F12}}(y) & Norm_{\gamma_{F12}}(y)) \\ (Norm_{\delta_{F21}}(y) & Norm_{\gamma_{F21}}(y)) & (Norm_{\delta_{F22}}(y) & Norm_{\gamma_{F22}}(y)) \end{pmatrix}$

$$\overline{Norm_A} * \overline{Norm_B} = \begin{pmatrix} (Norm_{\vartheta_{E11}}(x) & Norm_{\mu_{E11}}(x)) & (Norm_{\vartheta_{E12}}(x) & Norm_{\mu_{E12}}(x)) \\ (Norm_{\vartheta_{E21}}(x) & Norm_{\mu_{E21}}(x)) & (Norm_{\vartheta_{E22}}(x) & Norm_{\mu_{E22}}(x)) \end{pmatrix} * \begin{pmatrix} (Norm_{\delta_{F11}}(y) & Norm_{\gamma_{F11}}(y)) & (Norm_{\delta_{F12}}(y) & Norm_{\gamma_{F12}}(y)) \\ (Norm_{\delta_{F21}}(y) & Norm_{\gamma_{F21}}(y)) & (Norm_{\delta_{F22}}(y) & Norm_{\gamma_{F22}}(y)) \end{pmatrix}$$

$X_{11} = (Norm_{\vartheta_{E11}}(x) \quad Norm_{\mu_{E11}}(x)) * (Norm_{\delta_{F11}}(y) \quad Norm_{\gamma_{F11}}(y))$

$X_{12} = (Norm_{\vartheta_{E12}}(x) \quad Norm_{\mu_{E12}}(x)) * (Norm_{\delta_{F12}}(y) \quad Norm_{\gamma_{F12}}(y))$

$X_{21} = (Norm_{\vartheta_{E21}}(x) \quad Norm_{\mu_{E21}}(x)) * (Norm_{\delta_{F21}}(y) \quad Norm_{\gamma_{F21}}(y))$

$X_{22} = (Norm_{\vartheta_{E22}}(x) \quad Norm_{\mu_{E22}}(x)) * (Norm_{\delta_{F22}}(y) \quad Norm_{\gamma_{F22}}(y))$

Applying formula $\overline{A_E} * \overline{B_F} = \{\langle x, y\rangle, \frac{\vartheta_A(x)+\vartheta_B(y)}{2(\vartheta_A(x).\vartheta_B(y)+1)}, \frac{\mu_A(x)+\mu_B(y)}{2(\mu_A(x).\mu_B(y)+1)} : x\in E, y\in F\}$

$$X_{11} = \left(\frac{Norm_{\vartheta_{E11}}(x) + Norm_{\delta_{F11}}(y)}{2(Norm_{\vartheta_{E11}}(x).Norm_{\delta_{F11}}(y)+1)}, \frac{Norm_{\mu_{E11}}(x) + Norm_{\gamma_{F11}}(y)}{2(Norm_{\mu_{E11}}(x).\ Norm_{\gamma_{F11}}(y)+1)}\right)$$

$$X_{12} = \left(\frac{Norm_{\vartheta_{E12}}(x) + Norm_{\delta_{F12}}(y)}{2(Norm_{\vartheta_{E12}}(x).Norm_{\delta_{F12}}(y)+1)}, \frac{Norm_{\mu_{E12}}(x) + Norm_{\gamma_{F12}}(y)}{2(Norm_{\mu_{E12}}(x).\ Norm_{\gamma_{F12}}(y)+1)}\right)$$

$$X_{21} = \left(\frac{Norm_{\vartheta_{E21}}(x) + Norm_{\delta_{F21}}(y)}{2(Norm_{\vartheta_{E21}}(x).Norm_{\delta_{F21}}(y) + 1)}, \frac{Norm_{\mu_{E21}}(x) + Norm_{\gamma_{F21}}(y)}{2(Norm_{\mu_{E21}}(x).\ Norm_{\gamma_{F21}}(y) + 1)}\right)$$

$$X_{22} = \left(\frac{Norm_{\vartheta_{E22}}(x) + Norm_{\delta_{F22}}(y)}{2(Norm_{\vartheta_{E11}}(x).Norm_{\delta_{F11}}(y) + 1)}, \frac{Norm_{\mu_{E22}}(x) + Norm_{\gamma_{F22}}(y)}{2(Norm_{\mu_{E22}}(x).\ Norm_{\gamma_{F22}}(y) + 1)}\right)$$

We have Norm$\bar{A}_E *\ Norm\bar{B}_F = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ is also intuitionistic fuzzy matrix

Example: If $\boldsymbol{A_E}$=$\begin{pmatrix} (.9,.4) & (.4,.1) \\ (.5,.3) & (.8,.3) \end{pmatrix}$ and$\boldsymbol{B_F} = \begin{pmatrix} (.8,.6) & (.6,.3) \\ (.7,.5) & (.9,.4) \end{pmatrix}$ are two intuitionistic fuzzy matrices over different universe E and F then Norm$\bar{A}_E * Norm\bar{B}_F$is also fuzzy matrix set.

Proof: Given $\boldsymbol{A_E}$ =$\begin{pmatrix} (.9,.4) & (.4,.1) \\ (.5,.3) & (.8,.3) \end{pmatrix}$ and $\boldsymbol{B_F} = \begin{pmatrix} (.8,.6) & (.6,.3) \\ (.7,.5) & (.9,.4) \end{pmatrix}$

$\overline{Norm}_A$ =$\begin{pmatrix} (.8,1) & (0,.3) \\ (.2,.8) & (.7,.8) \end{pmatrix}$ and $\overline{Norm}_B$=$\begin{pmatrix} (.5,1) & (0,.5) \\ (.3,.8) & (.8,.7) \end{pmatrix}$

$\overline{Norm}_A * \overline{Norm}_B = \begin{pmatrix} (.8,1) & (0,.3) \\ (.2,.8) & (.7,.8) \end{pmatrix} * \begin{pmatrix} (.5,1) & (0,.5) \\ (.3,.8) & (.8,.7) \end{pmatrix}$

$X_{11} = (.8,1) * (.5,1)$

$X_{12} = (0,.3) * (0,.5)$

$X_{21} = (.2,.8) * (.3,.8)$

$X_{22} = (.7,.8) * (.8,.7)$

$Applying\ formula\ \overline{A_E} * \overline{B_F} = \{\langle x, y\rangle, \frac{\vartheta_A(x)+\vartheta_B(y)}{2(\vartheta_A(x).\vartheta_B(y)+1)}, \frac{\mu_A(x)+\mu_B(y)}{2(\mu_A(x).\mu_B(y)+1)} : x\in E, y\in F\}$

$$X_{11} = \left(\frac{(.8) + (.5)}{2((.8).(.5) + 1)}, \frac{(1) + (1)}{2(1.1 + 1)}\right)$$

$$X_{12} = \left(\frac{(0) + (0)}{2((0).(0) + 1)}, \frac{(.3) + (.5)}{2(.3\ .5 + 1)}\right)$$

$$X_{21} = \left(\frac{(.2) + (.3)}{2((.2).(.3) + 1)}, \frac{(.8) + (.8)}{2(.8\ .8 + 1)}\right)$$

$$X_{22} = \left(\frac{(.7) + (.8)}{2((.7).(.8) + 1)}, \frac{(.8) + (.7)}{2(.8\ .7 + 1)}\right)$$

X_{11} ={.5 , .5}

X_{12} ={0 , .3}

X_{21} ={.2, .5}

X_{22} ={.5 ,.5}

We have Norm$\bar{A}_E * Norm\bar{B}_F = \begin{pmatrix} (.5,\ .5) & (0\ ,.3) \\ (.2,\ .5) & (.5\ ,.5) \end{pmatrix}$ is also intuitionistic fuzzy matrix.

Theorem 5.2.9: If E and F be two universal sets. For every normalization of an intuitionistic fuzzy matrices $\overline{A_E}$ *and* $\overline{B_F}$ are in E and F then Norm$\bar{A}_E\Delta$ $Norm\bar{B}_F$ is also normalization of an intuitionistic fuzzy matrices.

Proof: If $Norm_A$ and $Norm_B$ are two normalization intuitionistic fuzzy matrices over different universes *sets E and F. If we consider two* 2×2 *matrix*

$$\overline{Norm_A} = \begin{pmatrix} (Norm_{\vartheta_{E11}}(x) & Norm_{\mu_{E11}}(x)) & (Norm_{\vartheta_{E12}}(x) & Norm_{\mu_{E12}}(x)) \\ (Norm_{\vartheta_{E21}}(x) & Norm_{\mu_{E21}}(x)) & (Norm_{\vartheta_{E22}}(x) & Norm_{\mu_{E22}}(x)) \end{pmatrix} \text{ and}$$

$$\overline{Norm_B} = \begin{pmatrix} (Norm_{\delta_{F11}}(y) & Norm_{\gamma_{F11}}(y)) & (Norm_{\delta_{F12}}(y) & Norm_{\gamma_{F12}}(y)) \\ (Norm_{\delta_{F21}}(y) & Norm_{\gamma_{F21}}(y)) & (Norm_{\delta_{F22}}(y) & Norm_{\gamma_{F22}}(y)) \end{pmatrix}$$

$$\overline{Norm_A}\,\Delta\,\overline{Norm_B} = \begin{pmatrix} (Norm_{\vartheta_{E11}}(x) & Norm_{\mu_{E11}}(x)) & (Norm_{\vartheta_{E12}}(x) & Norm_{\mu_{E12}}(x)) \\ (Norm_{\vartheta_{E21}}(x) & Norm_{\mu_{E21}}(x)) & (Norm_{\vartheta_{E22}}(x) & Norm_{\mu_{E22}}(x)) \end{pmatrix} \Delta$$

$$\begin{pmatrix} (Norm_{\delta_{F11}}(y) & Norm_{\gamma_{F11}}(y)) & (Norm_{\delta_{F12}}(y) & Norm_{\gamma_{F12}}(y)) \\ (Norm_{\delta_{F21}}(y) & Norm_{\gamma_{F21}}(y)) & (Norm_{\delta_{F22}}(y) & Norm_{\gamma_{F22}}(y)) \end{pmatrix}$$

$$X_{11} = (Norm_{\vartheta_{E11}}(x) \quad Norm_{\mu_{E11}}(x))\,\Delta\,(Norm_{\delta_{F11}}(y) \quad Norm_{\gamma_{F11}}(y))$$

$$X_{12} = (Norm_{\vartheta_{E12}}(x) \quad Norm_{\mu_{E12}}(x))\,\Delta\,(Norm_{\delta_{F12}}(y) \quad Norm_{\gamma_{F12}}(y))$$

$$X_{21} = (Norm_{\vartheta_{E21}}(x) \quad Norm_{\mu_{E21}}(x))\,\Delta\,(Norm_{\delta_{F21}}(y) \quad Norm_{\gamma_{F21}}(y))$$

$$X_{22} = (Norm_{\vartheta_{E22}}(x) \quad Norm_{\mu_{E22}}(x))\,\Delta\,(Norm_{\delta_{F22}}(y) \quad Norm_{\gamma_{F22}}(y))$$

Applying formula $\overline{A_E}\ \Delta\,\overline{B_F} = \{\langle x, y\rangle, \frac{\vartheta_A(x)+\vartheta_B(y)}{\mu_A(x)+\mu_B(y)+\vartheta_A(x)+\vartheta_B(y)}, \frac{\mu_A(x)+\mu_B(y)}{\mu_A(x)+\mu_B(y)+\vartheta_A(x)+\vartheta_B(y)} : x\in E, y\in F\}$

$$X_{11} = \frac{Norm_{\vartheta_{E11}}(x)+Norm_{\delta_{F11}}(y)}{Norm_{\mu_{E11}}(x)+Norm_{\gamma_{F11}}(y)+Norm_{\vartheta_{E11}}(x)+Norm_{\delta_{F11}}(y)}, \frac{Norm_{\mu_{E11}}(x)+Norm_{\gamma_{F11}}(y)}{Norm_{\mu_{E11}}(x)+Norm_{\gamma_{F11}}(y)+Norm_{\vartheta_{E11}}(x)+Norm_{\delta_{F11}}(y)}$$

$$X_{12} = \frac{Norm_{\vartheta_{E12}}(x)+Norm_{\delta_{F12}}(y)}{Norm_{\mu_{E12}}(x)+Norm_{\gamma_{F12}}(y)+Norm_{\vartheta_{E12}}(x)+Norm_{\delta_{F12}}(y)}, \frac{Norm_{\mu_{E12}}(x)+Norm_{\gamma_{F12}}(y)}{Norm_{\mu_{E12}}(x)+Norm_{\gamma_{F12}}(y)+Norm_{\vartheta_{E12}}(x)+Norm_{\delta_{F12}}(y)}$$

$$X_{21} = \frac{Norm_{\vartheta_{21}}(x)+Norm_{\delta_{F21}}(y)}{Norm_{\mu_{E21}}(x)+Norm_{\gamma_{F21}}(y)+Norm_{\vartheta_{E21}}(x)+Norm_{\delta_{F21}}(y)}, \frac{Norm_{\mu_{E21}}(x)+Norm_{\gamma_{F21}}(y)}{Norm_{\mu_{E21}}(x)+Norm_{\gamma_{F21}}(y)+Norm_{\vartheta_{E21}}(x)+Norm_{\delta_{F21}}(y)}$$

$$X_{22} = \frac{Norm_{\vartheta_{E22}}(x)+Norm_{\delta_{F22}}(y)}{Norm_{\mu_{E22}}(x)+Norm_{\gamma_{F22}}(y)+Norm_{\vartheta_{E22}}(x)+Norm_{\delta_{F22}}(y)}, \frac{Norm_{\mu_{E22}}(x)+Norm_{\gamma_{F22}}(y)}{Norm_{\mu_{E22}}(x)+Norm_{\gamma_{F22}}(y)+Norm_{\vartheta_{E22}}(x)+Norm_{\delta_{F22}}(y)}$$

We have Norm$\bar{A}_E\Delta$ $Norm\bar{B}_F = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ is also intuitionistic fuzzy matrix.

Example: If $\boldsymbol{A_E}=\begin{pmatrix}(.9,.4) & (.4,.1)\\(.5,.3) & (.8,.3)\end{pmatrix}$ and $\boldsymbol{B_F}=\begin{pmatrix}(.8,.6) & (.6,.3)\\(.7,.5) & (.9,.4)\end{pmatrix}$ are two intuitionistic fuzzy matrices over different universe E and F then Norm$\bar{A}_E\Delta Norm\bar{B}_F$ is also fuzzy matrix set.

Proof: Given $\boldsymbol{A_E}=\begin{pmatrix}(.9,.4) & (.4,.1)\\(.5,.3) & (.8,.3)\end{pmatrix}$ and $\boldsymbol{B_F}=\begin{pmatrix}(.8,.6) & (.6,.3)\\(.7,.5) & (.9,.4)\end{pmatrix}$

$\overline{Norm_A}=\begin{pmatrix}(.8,1) & (0,.3)\\(.2,.8) & (.7,.8)\end{pmatrix}$ and $\overline{Norm_B}=\begin{pmatrix}(.5,1) & (0,.5)\\(.3,.8) & (.8,.7)\end{pmatrix}$

$\overline{Norm_A}\Delta\overline{Norm_B}=\begin{pmatrix}(.8,1) & (0,.3)\\(.2,.8) & (.7,.8)\end{pmatrix}\Delta\begin{pmatrix}(.5,1) & (0,.5)\\(.3,.8) & (.8,.7)\end{pmatrix}$

$X_{11}=(.8,1)\Delta(.5,1)$

$X_{12}=(0,.3)\Delta(0,.5)$

$X_{21}=(.2,.8)\Delta(.3,.8)$

$X_{22}=(.7,.8)\Delta(.8,.7)$

$Applying\ formula \quad \overline{A_E}\ \Delta\ \overline{B_F}=\{\langle x,y\rangle,\frac{\vartheta_A(x)+\vartheta_B(y)}{\mu_A(x)+\mu_B(y)+\vartheta_A(x)+\vartheta_B(y)},\frac{\mu_A(x)+\mu_B(y)}{\mu_A(x)+\mu_B(y)+\vartheta_A(x)+\vartheta_B(y)}:x\in E,y\in F\}$

$X_{11}=\frac{(.8)+(.5)}{(1)+(1)+(.8)+(.5)},\frac{(1)+(1)}{(1)+(1)+(.8)+(.5)}$

$X_{12}=\frac{(0)+(0)}{(.3)+(.5)+(0)+(0)},\frac{(.3)+(.5)}{(.3)+(.5)+(0)+(0)}$

$X_{21}=\frac{(.2)+(.3)}{(.8)+(.8)+(.2)+(.3)},\frac{(.8)+(.8)}{(.8)+(.8)+(.2)+(.3)}$

$X_{22}=\frac{(.7)+(.8)}{(.8)+(.7)+(.7)+(.8)},\frac{(.8)+(.7)}{(.8)+(.7)+(.7)+(.8)}$

$X_{11}=\{.4, .6\}$

$X_{12}=\{0, 1\}$

$X_{21}=\{.2, .8\}$

$X_{22}=\{.5, .5\}$

We have Norm$\bar{A}_E\Delta\ Norm\bar{B}_F=\begin{pmatrix}(.4, .6) & (0,.1)\\(.2, .8) & (.5,.5)\end{pmatrix}$ is also intuitionistic fuzzy matrix.

Conclusion:

Normalization of intuitionistic fuzzy matrices ensures that the values are scaled within a specific range, facilitating their use in decision-making algorithms or other computational models. The key is to handle the three components—membership, non-membership, and indeterminacy—carefully to preserve their relationships while standardizing the values across the matrix.

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