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Bicomplex Sequence Spaces: Duality via Idempotent Decomposition

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Abstract: This paper investigates the duals of some bicomplex sequence spaces corresponding to bicomplex functions that are holomorphic in the bicomplex space $\mathbb{C}_2$, or entire bicomplex sequence spaces. We investigate these spaces through their idempotent decompositions and examine the β-dual, γ-dual, and δ-dual. Precise definitions and analyses of these duals are presented. Our results demonstrate that the duals of the original sequence spaces are strictly contained within the duals of their corresponding idempotent subclasses. These findings are also discussed in the context of algebra homomorphisms between the original sequence space ℵ and its idempotent subclasses $1_{\aleph}$ and $2_{\aleph}$.

Keywords: bicomplex numbers, entire bicomplex sequence spaces, idempotent sequence spaces, Köthe – Toeplitz duals

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1. Introduction

1.1 DUALS OF SEQUENCE SPACES

There are two primary types of duals associated with a sequence space: the **algebraic dual** and the **topological dual**. The **algebraic dual** of a linear space V is the set of all linear functionals from V to a scalar field K, and is denoted by $L(V, K) = V^{\#}$. On the other hand, the **topological dual** consists of all *continuous* linear functionals on V and is denoted by V*.

The only sequence space with a well-behaved algebraic dual consisting of sequences is ϕ, whose dual is ω. Therefore, in duality theory, it is more effective to study sequence spaces with linear topologies, though finding their topological duals is often challenging. To address this, Köthe and Toeplitz [3] introduced the α-dual and β-dual, which facilitate a more practical dual system. Later, Garling [1] proposed the more general γ-dual, and for symmetric sequence spaces, the δ-dual was introduced by Garling [2] and Ruckle [5].

1.2 BICOMPLEX SPACE $\mathbb{C}_2$

Bicomplex Numbers were defined by Corrado Segre (1860 – 1924) in 1892. Infinite set of algebras and the concept of multicomplex numbers was given in [6]. The set of bicomplex numbers is given by

$\mathbb{C}_2 = \{\mu_1 + i_1\mu_2 + i_2\mu_3 + i_1 i_2\mu_4 : \mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{C}_0\}$, where

${i_1}^2 = {i_2}^2 = -1$, $i_1 i_2 = i_2 i_1$.

The binary operations of addition and scalar multiplication on $\mathbb{C}_2$ are defined coordinate-wise, and multiplication is defined component-wise (i.e., term by term). With these operations, $\mathbb{C}_2$ forms a commutative algebra with identity. There are several notable differences between the algebraic structures of $\mathbb{C}_2$ and $\mathbb{C}_1$, as outlined by Price in [4]. Although bicomplex numbers, like quaternions, form a four-dimensional algebra, they differ in that bicomplex numbers are commutative, whereas quaternions are not.

Idempotent Elements – Apart from 0 and 1, the structure contains two distinct nontrivial idempotent elements given by $e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$. The addition of these two idempotent elements is 1 and their product is zero. There are two principal ideals generated by these idempotent elements. Intersection of these ideals is zero and their union is the set of all singular elements of $\mathbb{C}_2$. Two bicomplex numbers are zero divisors precisely when one is a complex multiple of one idempotent element and the other is a complex multiple of the other idempotent element. The detailed study of $\mathbb{C}_2$ is provided in [13].

2. Objectives

The objective of this paper is to investigate the β-, γ-, and δ-duals of certain classes of bicomplex sequences, and to explore their relationships with the duals of corresponding idempotent subclasses, supported by illustrative examples and counterexamples

3. Methods

Idempotent technique has been used to investigate the duals of bicomplex sequence spaces and their subclasses.

4. Results

BICOMPLEX KÖTHE – TOEPLITZ DUALS

If ω′ is the family of all bicomplex sequences $\xi = (\xi_k)$ with $\xi_k \in \mathbb{C}_2$, $k \geq 1$, where $\mathbb{C}_2$ is the space of all bicomplex numbers. If ψ be a bicomplex sequence space, then we denote α –, β –, γ –, and δ – duals of ψ by $\psi_\alpha, \psi_\beta, \psi_\gamma$ and ψ_δ respectively. These duals have been defined in [10]. Let us see these definitions for our ready reference.

$$\psi_\alpha = \left\{\xi : \xi \in \omega', \sum_{i\geq 1} \|\xi_i \eta_i\| < \infty, \forall\, \eta_i \in \lambda\right\}$$

$$\psi_\beta = \{\xi : \xi \in \omega', \|\textstyle\sum_{i\geq 1} \xi_i \eta_i\| < \infty, \forall\, \eta_i \in \lambda\}$$

$$\psi_\gamma = \left\{\xi : \xi \in \omega', \sup_n \left\|\sum_{i=1}^{n} \xi_i \eta_i\right\| < \infty, \forall\, \eta_i \in \lambda\right\}$$

$$\lambda_\delta = \left\{\xi : \xi \in \omega', \textstyle\sum_{i\geq 1} \left\|\xi_i \eta_{\rho(i)}\right\| < \infty, \forall \eta_i \in \lambda\, and \rho \in \pi\right\}$$

where π is the set of all permutations of $\mathbb{N}$.

4.1 BICOMPLEX SEQUENC SPACES

Now, let us first describe the classes of sequences whose duals we aim to study:

(i) $\aleph = \left\{ f : f = \{\mathcal{X}_j\} = \left\{ 1_{x_j}.e_1 + 2_{x_j}.e_2 \right\} : \sup_{j\geq 1} j^j \left| 1_{x_j} \right|_{\mathbb{C}_1} < \infty, \sup_{j\geq 1} j^j \left| 2_{x_j} \right|_{\mathbb{C}_1} < \infty \right\}$

(ii) $1_\aleph = \left\{ f : f = \left\{ 1_{x_j}.e_1 \right\} : \sup_{j\geq 1} j^j \left| 1_{x_j} \right|_{\mathbb{C}_1} < \infty \right\}$

(iii) $2_\aleph = \left\{ f : f = \left\{ 2_{x_j}.e_1 \right\} : \sup_{j\geq 1} j^j \left| 2_{x_j} \right|_{\mathbb{C}_1} < \infty \right\}$

The class $\aleph$ given in (i) has been studied in [8] by Srivastava & Srivastava and the subspaces given in (ii) and (iii) have been studied in [9] by Wagh. The subclass in (ii) have been studied with a functional analytic viewpoint in [11] by Wagh. These subclasses are the subspaces of our space $\aleph$ in the sense that they are formed by idempotent sequences of $\aleph$.

In the above spaces, the notation $|.|_{\mathbb{C}_1}$ represents the complex norm.

For any bicomplex sequence $\{\mathcal{X}_j\} = \left\{ 1_{x_j}.e_1 + 2_{x_j}.e_2 \right\}$, the following two conditions

(iv) and (v) are equivalent:

(iv) $\sup_{j\geq 1} j^j \left| 1_{x_j} \right|_{\mathbb{C}_1} < \infty$ and $\sup_{j\geq 1} j^j \left| 2_{x_j} \right|_{\mathbb{C}_1} < \infty$

(v) $\sup_{j\geq 1} j^j \left\| \mathcal{X}_j \right\|_{\mathbb{C}_2} < \infty$

Where $\| \quad \|_{\mathbb{C}_2}$ represents the bicomplex norm, given by

$\|\varsigma\|_{\mathbb{C}_2} = \left\{ \frac{\left|1_\varsigma\right|_{\mathbb{C}_1}^{\ 2} + \left|2_\varsigma\right|_{\mathbb{C}_1}^{\ 2}}{2} \right\}^{1/2}$, where

(vi) $\varsigma = u_1 + i_2 u_2 = (u_1 - i_1 u_2)e_1 + (u_1 + i_1 u_2)e_2 = 1_\varsigma e_1 + 2_\varsigma e_2 \in \mathbb{C}_2, 1_\varsigma, 2_\varsigma \in \mathbb{C}_1$

1_ς and 2_ς are first idempotent component and second idempotent component of ς respectively. The

idempotent representation given in (vi) is unique and was given by Srivastava in [7].

Thus, the class in (i) has an equivalent representation given by

$$\aleph = \left\{ f : f = \{\mathcal{X}_j\} : \sup_{j \geq 1} j^{j} \|\mathcal{X}_j\|_{\mathbb{C}_2} < \infty \right\}$$

$(\xi_j) \in \aleph$ can be written as $\xi_j = z_{1j} + i_2 z_{2j} = \beta_{1j} e_1 + \beta_{2j} e_2$, where,

$\beta_{1j} = z_{1j} - i_1 z_{2j}$,

$\beta_{2j} = z_{1j} + i_1 z_{2j}$

(β_{1j}) and (β_{2j}) are complex sequences i.e., $\beta_{1j}, \beta_{2j} \in \mathbb{C}_1(i_1)$.

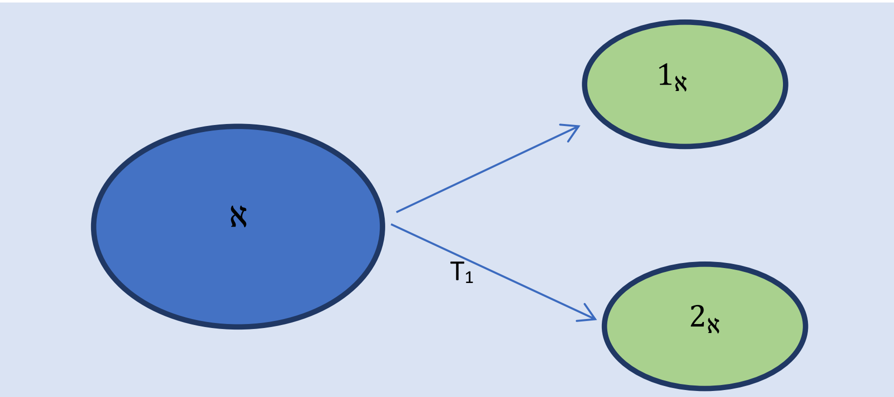
4.2 ALGEBRA HOMOMORPHISM BETWEEN ℵ AND ITS SUBCLASSES

Algebra homomorphism (denoted by T_1 and T_2) between ℵ and its subclasses $1_\aleph$ and $2_\aleph$ have been investigated in [10] given by:

$T_1 : \aleph \to 1_\aleph$ as $T_1(f) = T_1(\{\xi_j\}) = \left\{1_{\xi_j} e_1\right\} \in 1_\aleph$

and $T_2 : \aleph \to 2_\aleph$ as $T_2(f) = T_2(\{\xi_j\}) = \left\{2_{\xi_j} e_2\right\} \in 2_\aleph$, $\{\xi_k\} \in B$

Its pictorial representation is given below



Thus we can say that

$(\eta_j) \in \aleph_\alpha$ if and only if $T_1(\eta_j) \in (1_\aleph)_\alpha, T_2(\eta_j) \in (2_\aleph)_\alpha$

Or in words we can also say that a sequence belongs to α - dual of the class ℵ if and only if its T_1 –

image belongs to α - dual of the first idempotent component of ℵ or its first subclass and the T_2 –

image belongs to the α - dual of its second idempotent component or its second subclass. This is to

be noted that the α – dual of the class ℵ has been studied in [12]. In this paper, we are going to

analyze other duals of these spaces.

β – dual of the class ℵ and its subclasses $1_{\aleph}$ and $2_{\aleph}$ are given by

$$\aleph_\beta = \left\{ (\eta_j) \in \omega' : \left\| \sum_{j\geq 1} \eta_j \xi_j \right\| < \infty, \forall (\xi_j) \in \aleph \right\}$$

Or

$$\left\{ (\eta_j) \in \omega' : \left| \sum_{j\geq 1} 1\eta_j{}^1\xi_j \right| < \infty, \left| \sum_{j\geq 1} 2\eta_j{}^2\xi_j \right| < \infty, \forall (\xi_j) \in \aleph \right\}$$

$$(1_\aleph)_\beta = \left\{ (\eta_j) \in \omega' : \left| \sum_{j\geq 1} 1\eta_j{}^1\xi_j \right| < \infty, \forall (\xi_j) \in \aleph \right\}$$

$$(2_\aleph)_\beta = \left\{ (\eta_j) \in \omega' : \left| \sum_{j\geq 1} 2\eta_j{}^2\xi_j \right| < \infty, \forall (\xi_j) \in \aleph \right\}$$

Theorem 4.1: (i) $\aleph_\beta \subset (1_\aleph)_\beta$.

(ii) $\aleph_\beta \subset (2_\aleph)_\beta$

Proof: (i) We know $1_\aleph \subseteq \aleph$ and

$\aleph_\beta \subseteq (1_\aleph)_\beta$ (since dual of a set is contained in the dual of its subset)

To establish that the inclusion is proper, we construct a sequence contained in the β – dual of $1_\aleph$ which does not belong to β – dual of ℵ.

Consider the sequence,

$(\eta_j) = \left(\frac{1}{j^3}e_1 + j^{3+j}e_2\right).$

First we show that $(\eta_j) \in (1_\aleph)_\beta$.

Let $\left(1_{\xi_j}e_1\right) \in 1_\aleph$

$$\Rightarrow \sup_j j^j \left|1_{\xi_j}\right| < \infty$$

$\Rightarrow j^j \left|1_{\xi_j}\right| < M, \forall j \geq 1$ for some M.

$\Rightarrow \left|1_{\xi_j}\right| < \frac{M}{j^j}, \forall j \geq 1$ (I)

$$j^j > j^3, \forall j \geq 3 \Rightarrow \frac{1}{j^j} < \frac{1}{j^3}, \forall j \geq 3$$

and $\sum \frac{1}{j^3}$ is convergent $\Rightarrow \sum \frac{1}{j^j}$ is convergent, by comparison test.

$\sum \frac{M}{j^j}$ is a convergent series. (II)

$\therefore$ from (I) and (II) $\sum \left|1_{\xi_j}\right|$is convergent.

And $\left|\sum 1_{\xi_j}\right| \leq \sum \left|1_{\xi_j}\right| < \infty$

In $(\eta_j) = \left(\frac{1}{j^3}e_1 + j^{3+j}e_2\right)$,

$\left|\sum 1_{\eta_j}\right| \leq \sum \left|1_{\eta_j}\right| = \sum \frac{1}{j^3} < \infty$ and$\left|\sum 1_{\xi_j}\right| < \infty$,

The term-by-term product of two convergent series is itself convergent,

Therefore $\left|\sum \frac{1}{j^3}^1 \xi_j\right| < \infty, \forall \left(1_{\xi_j}e_1\right) \in 1_\aleph$.

Thus$(\eta_j) \in (1_\aleph)_\beta$. (a)

To show that$(\eta_j) \notin \aleph_\beta$, we must show that for some element$(\xi_j) \in \aleph$, $\sum \xi_j \eta_j$ is not convergent.

Consider the sequence$(\xi_j) = j^{-j}\left(\frac{1}{j^2}e_1 + \frac{1}{j^3}e_2\right) = \left(j^{-j-2}e_1 + j^{-j-3}e_2\right)$.

Note first that,

$$\sup_j j^j \|\xi_j\| = \sup_j j^j \left(\frac{\left|1_{\xi_j}\right|^2 + \left|2_{\xi_j}\right|^2}{2}\right)^{1/2}$$

$$= \sup_j j^j \left(\frac{|j^{-2j-4}| + |j^{-2j-6}|}{2}\right)^{1/2}$$

$$= \sup_j \left\{\frac{1}{2}\left(\frac{1}{j^4} + \frac{1}{j^6}\right)\right\}^{1/2} < \infty$$

So that $(\xi_j) \in \aleph$.

Now, $\left\|\sum_{j\geq 1} \xi_j \eta_j\right\| \leq \sum_{j\geq 1} \|\xi_j \eta_j\| = \sum_{j\geq 1} \left\|\left(\frac{1}{j^{2+j}} e_1 + \frac{1}{j^{3+j}} e_2\right) . \left(\frac{1}{j^2} e_1 + j^{2+j} e_2\right)\right\|$

$$= \sum_{j\geq 1} \left\|\left(\frac{1}{j^{4+j}} e_1 + \frac{1}{j} e_2\right)\right\|$$

$$j^{4+j} > j \Rightarrow \frac{1}{j^{4+j}} < \frac{1}{j^4}$$

and $\sum_{j\geq 1} \left|\frac{1}{j^4}\right|$ is convergent therefore by comparison test $\sum_{j\geq 1} \left|\frac{1}{j^{4+j}}\right|$ is also convergent, but $\sum_{j\geq 1} \left|\frac{1}{j^4}\right|$ is not convergent.

Therefore, $(\eta_j) \notin \aleph_\beta$ (b)

Hence from (a) and (b) we get

$(\eta_j) \in (1_\aleph)_\beta$ but $(\eta_j) \notin \aleph_\beta$.

Note 1: Beta dual of the class $\aleph$ is properly contained in the beta dual of its T_1 image.

Next, gamma dual of the class $\aleph$ and its subclasses are given by

$$\aleph_\gamma = \left\{(\eta_j) \in \omega' : \sup_n \left\|\sum_{j=1}^n \eta_j \xi_j\right\| < \infty, \forall (\xi_j) \in \aleph\right\} \qquad \text{(III)}$$

Or

$$\left\{(\eta_j) \in \omega' : \sup_n \left|\sum_{j=1}^n 1_{\eta_j} 1_{\xi_j}\right| < \infty, \sup_n \left|\sum_{j=1}^n 1_{\eta_j}{}^2 \xi_j\right| < \infty, \forall (\xi_j) \in \aleph\right\}.$$

$(1_{\aleph})_\gamma = \left\{(\eta_j) \in \omega' : \sup_n \left|\sum_{j=1}^{n} 1_{\eta_j} 1_{\xi_j}\right| < \infty, \forall(\xi_j) \in \aleph\right\}$ (IV)

$(2_{\aleph})_\gamma = \left\{(\eta_j) \in \omega' : \sup_n \left|\sum_{j=1}^{n} 2_{\eta_j} 2_{\xi_j}\right| < \infty, \forall(\xi_j) \in \aleph\right\}$ (V)

Theorem 4.2: A sequence (η_k) belongs to γ - dual of the class $\aleph$ if and only if its first idempotent sequence belongs to γ - dual of the class $1_\aleph$ and the second idempotent sequence belongs to γ - dual of the class $2_\aleph$ that is

$(\eta_j) \in B_\gamma \Leftrightarrow \left(1_{\eta_j} e_1\right) \in (1_\aleph)_\gamma \; and \left(2_{\eta_j} e_2\right) \in (2_\aleph)_\gamma.$

Proof: Let $(\eta_j) \in \aleph_\gamma$ be any sequence.

By (III),

$$(\eta_j) \in B_\gamma \Leftrightarrow \sup_n \left\|\sum_{j=1}^{n} \eta_j \xi_j\right\| < \infty, \forall(\xi_j) \in \aleph$$

$$\Leftrightarrow \left\{\sup_n \left|\sum_{j=1}^{n} 1\eta_j{}^1\xi_j\right|\right\} e_1 + \left\{\sup_n \left|\sum_{j=1}^{n} 2\eta_j{}^2\xi_j\right|\right\} e_2 < \infty, \forall(\xi_j) \in \aleph$$

$$\Leftrightarrow \sup_n \left|\sum_{j=1}^{n} 1j^1\xi_j\right| < \infty \text{ and } \sup_n \left|\sum_{j=1}^{n} 2\eta_j{}^2\xi_j\right| < \infty, \forall(\xi_j) \in \aleph$$

Hence, $(\eta_j) \in \aleph_\gamma \Leftrightarrow (1\eta_j e_1) \in (1_\aleph)_\gamma$ and $(2\eta_j e_2) \in (2_\aleph)_\gamma$, by (III) and (IV).

(c)

Corollary 1:

Thus from (b) and (c) we can say that

$(\eta_j) \in \aleph_\gamma$ if and only if $T_1(\eta_j) \in (1_{\aleph)}, T_2(\eta_j) \in (2_\aleph)_\gamma$

Or in words we can also say that a sequence belongs to γ - dual of the class $\aleph$ if and only if its T_1 –

image belongs to γ - dual of the first idempotent component of $\aleph$ or its first subclass and the T_2 – image

belongs to the γ - dual of its second idempotent component or its second subclass.

Theorem 4.3: (i)$\aleph_\gamma \subset (1_\aleph)_\gamma$.

(ii) $\aleph_\gamma \subset (2_\aleph)_\gamma$

Proof: Similar example may be considered.

Note 2: Gamma dual of the class ℵ is properly contained in the gamma dual of its idempotent parts/ T_1\]

,and T_2 – images.

δ – dual of the class ℵ and its subclasses

$$\aleph_\delta = \{(\eta_j) \in \omega' : \textstyle\sum_{j\geq 1} \|\eta_j \xi_{\rho(j)}\| < \infty, \forall(\xi_j) \in \aleph \quad, \rho \in \pi\} \qquad \text{(VI)}$$

$$= \left\{(\eta_j) \in \omega' : \sum_{j\geq 1} \left|1_{\eta_j} 1_{\xi_{\rho(j)}}\right| < \infty, \ \sum_{j\geq 1} \left|2_{\eta_j} 2_{\xi_{\rho(j)}}\right| < \infty, \ \forall(\xi_j) \in \aleph, \rho \in \pi\right\}$$

$$(1_\aleph)_\delta = \left\{(\eta_j) \in \omega' : \textstyle\sum_{j\geq 1} \left|1_{\eta_j}{}^1 \xi_{\rho(j)}\right| < \infty, \forall(\xi_j) \in \aleph \; and \; \rho \in \pi\right\} \qquad \text{(VII)}$$

$$(2_\aleph)_\delta = \left\{(\eta_j) \in \omega' : \textstyle\sum_{j\geq 1} \left|2_{\eta_j}{}^2 \xi_{\rho(j)}\right| < \infty, \forall(\xi_j) \in \aleph \; and \; \rho \in \pi\right\} \qquad \text{(VIII)}$$

Theorem 4.4: A sequence (η_j) belongs to δ - dual of the class ℵ if and only if its first idempotent sequence belongs to δ - dual of the class $1_\aleph$ and the second idempotent sequence belongs to δ - dual of the class $2_\aleph$, that is

$$(\eta_j) \in B_\delta \Leftrightarrow (1\eta_j e_1) \in (1_\aleph)_\delta \text{ and } (2\eta_j e_2) \in (2_\aleph)_\delta$$

Proof: Let $(\eta_k) \in B_\delta$be any sequence.

From (VI),

$$(\eta_j) \in \aleph_\delta \Leftrightarrow \sum_{j\geq 1} \|\eta_j \xi_{\rho(j)}\| < \infty, \forall(\xi_j) \in \aleph \text{ and } \rho \in \pi$$

$$\Leftrightarrow \left\{\{\textstyle\sum_{j\geq 1} |1\eta_j{}^1 \xi_{\rho(j)}|\} e_1 + \{\sum_{j\geq 1} |2\eta_j{}^2 \xi_{\rho(j)}|\} e_2\right\} < \infty, \forall(\xi_j) \in \aleph$$

$$\Leftrightarrow \textstyle\sum_{j\geq 1} |1\eta_j{}^1 \xi_{\rho(j)}| < \infty \text{ and } \sum_{j\geq 1} |2\eta_j{}^2 \xi_{\rho(j)}| < \infty, \forall(\xi_j) \in \aleph$$

Hence, $(\eta_j) \in \aleph_\delta \Leftrightarrow (1\eta_j e_1) \in (1_\aleph)_\delta$ and $(2\eta_j e_2) \in (2_\aleph)_\delta$, by (VII) and (VIII).

(d)

Corollary 2:

From (d) and the definitions of T_1 and T_2 we can say that

$(\eta_j) \in \aleph_\delta$ if and only if $T_1(\eta_j) \in (1_\aleph), T_2(\eta_j) \in (2_\aleph)$

Or in words we can also say that a sequence belongs to δ - dual of the Class ℵ if and only if its T_1 –

image belongs to δ - dual of the first idempotent component of ℵ or its first subclass and the T_2 – image

belongs to the δ - dual of its second idempotent component or its second subclass.

Theorem 4.5: (i)$\aleph_\delta \subset (1_\aleph)_\delta$.

(ii) $\aleph_\delta \subset (2_\aleph)_\delta$

Proof: Can be shown easily with the help of similar example.

Note 3: Delta dual of the class ℵ is properly contained in the delta dual of its idempotent parts/T_1 and T_2 images.

Constructing other counter – examples

We can also construct other counter – examples. For instance, for part (i) of the theorems 4.1, 4.3 and 4.5,

we can take $(\xi_j) = j^{-j}(\frac{1}{j^3}e_1 + \frac{1}{j^4}e_2) = \frac{1}{j^{3+j}}e_1 + \frac{1}{j^{4+j}}e_2$, $(\eta_j) = \left(\frac{1}{j^2}e_1 + j^{j+3}e_2\right)$

For part (ii), take $(\eta_j) = \left(j^{j+2}e_1 + \frac{1}{j^2}e_2\right)$ or $\left(j^{j+2}e_1 + \frac{1}{j^3}e_2\right)$

Generalized counter – example,

for part (i) of the theorems 4.1, 4.3 and 4.5, take

$$(\xi_j) = j^{-j}(\frac{1}{j^n}e_1 + \frac{1}{j^{n+1}}e_2) = \frac{1}{j^{n+j}}e_1 + \frac{1}{j^{n+1+j}}e_2, n \geq 2$$

$$(\eta_j) = \left(\frac{1}{j^{n-1}}e_1 + j^{j+n}e_2\right)$$

For part (ii), take $(\xi_j) = j^{-j}(\frac{1}{j^n}e_1 + \frac{1}{j^{n+1}}e_2) = \frac{1}{j^{n+j}}e_1 + \frac{1}{j^{n+1+j}}e_2, n \geq 2$

$(\eta_j) = \left(j^{j+n-1}e_1 + \frac{1}{j^{n-2}}e_2\right)$.

Conclusion: we have studied beta, gamma and delta duals of the idempotent subspaces of an entire bicomplex sequence space and shown that these duals of idempotent sequences are properly contained in the dual of our original space. We have also given the generalized example for this proper containment.

Conflict of interest: there is no conflict of interest.

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