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More Results on Complementary Tree Domination Number of Semi Total Point Graph

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Abstract:

A set $D \subseteq V$ of a graph $G = (V, E)$ is a complementary tree dominating set if the induced subgraph $< V(G) - D >$ is a tree. The complementary tree domination number $\gamma_{ctd}(G)$ is the minimum cardinality of a complementary tree dominating set (ctd-set) of G. The semi total point graph $T_2(G)$ is the graph G whose vertex set is $V(G) \cup E(G)$. Where two vertices are adjacent if and only if (i) they are adjacent vertices of G or (ii) one is a vertex and the other is an edge of G incident with it. In this paper complementary tree domination number of semi total point graph of graphs, its bounds and relation between $\gamma_{ctd}(G)$ and $\gamma_{ctd}(T_2(G))$ are obtained.

Keywords: Dominating set, Complementary tree dominating set, Semi total point graph.

1. Introduction

A Graph $G(V, E)$ discussed in this paper be a simple, finite, undirected, connected graph with p vertices and q edges. A set of vertices in a graph G is independent, if no two vertices are adjacent. The largest number of vertices in such a set is called the independence number and is denoted by $\beta_0(G)$. The corona $G_1 \circ G_2$ of two graphs G_1 and G_2 are defined as the graph G obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. The Corona $G_1 \circ G_2$ has $p_1(1 + p_2)$ vertices and $q_1 + p_1 q_2 + p_1 p_2$ edges. The graph $C_n^{(t)}$ is the one point union of *t* cycles of length *n*. A graph G is unicyclic if it contains exactly one cycle. A broom graph $B_{n,m}$ is a graph of n vertices which have a path P_m and $n - m$ pendant vertices, all of these vertices are adjacent to either the origin *u* or the terminus *v* of the path. Any undefined term in this paper may be found in Harary [2].

The concept of domination in graphs was introduced by Ore [5]. A set $D \subseteq V$ is said to be a dominating set of G, if every vertex in $V - D$ is adjacent to some vertex in D. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. The complementary tree domination number of a graph was introduced byS. Muthammai, M. Bhanumathi and P. Vidhya [4] have established some results on complementary tree domination number of graphs. A set $D \subseteq V(G)$ is said to be complementary tree dominating set (ctd-set) if the

induced subgraph $\langle V(G) - D \rangle$ is a tree. The minimum cardinality of a ctd-set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$. E. Sampath Kumar and S.B. Chikkodimath[3] introduced the concept of semi total point graphs of a graph. Also B. Basavanagoud, S.M. Hosamani and S.H. Malghan[1] have obtained some results on domination number of semi total point graph. Given a graph G, the semi total point graph $T_2(G)$ of G is the graph whose point set is $V(G) \cup E(G)$ where two points are adjacent if and only if (i) they are adjacent points of G or (ii) one is a point of G and the other is a line of G, incident with it.
For notation convenience, an edge $(u_1, u_2) \in E(G)$ then its corresponding edge vertex is denoted by u'_{12} in $T_2(G)$.
In this paper complementary tree domination number of semi total point graph of graphs, its bounds and relation between $\gamma_{ctd}(G)$ and $\gamma_{ctd}(T_2(G))$ are obtained.

2. Prior Results

Observation 2.1. [6]

(i) For the path P_p, $\gamma_{ctd}\left(T_2(P_p)\right) = p - 1$, where $(p \geq 2)$.

(ii) For the cycle C_p, $\gamma_{ctd}\left(T_2(C_p)\right) = p - 1$, where $(p \geq 3)$.

(iii) For the star graph $K_{1,p-1}$, $\gamma_{ctd}\left(T_2(K_{1,p-1})\right) = p - 1$ or $q, p \geq 2$.

(iv) For a complete graph K_p, $p \geq 4$ then $\gamma_{ctd}\left(T_2(K_p)\right) = \frac{p^2-5p+12}{2}$.

(v) For a wheel graph W_p, $p \geq 4\gamma_{ctd}\left(T_2(W_p)\right) = p$ where $W_p = C_{p-1} + K_1$ for $(p \geq 4)$.

(vi) For a corona graph $P_p \circ K_1$, $p \geq 2$ then $\gamma_{ctd}\left(T_2(P_p \circ K_1)\right) = 2p - 1$.

(vii) For a corona graph $C_p \circ K_1$, $p \geq 3$ then $\gamma_{ctd}\left(T_2(C_p \circ K_1)\right) = 2p$.

(viii) $\gamma_{ctd}\left(T_2(K_1 + P_n)\right) = p, p \geq 3$.

Proposition 2.2. [6]

Let $G(p, q)$ be a connected graph with $\delta(G) \geq 2$ then atleast one vertex of G is the member of ctd-set of $T_2(G)$.

Proposition 2.3. [6]

For any connected graph $G(p, q)$ with $p \geq 2, \left\lceil \frac{p}{\Delta(G)+1} \right\rceil \leq \gamma_{ctd}\left(T_2(G)\right) \leq p + q - 2$.

Theorem 2.4. [6]

$\gamma_{ctd}\left(T_2(G)\right) = 1$ if and only if $G \cong K_2$.

3. Characterisation of Complementary Tree Dominating Set in Semi Total Point Graph T2(G)

In the following, a necessary and sufficient condition for a ctd-set of a graph *G* to be a ctd-set of its semi total point graph $T_2(G)$ is found.

Theorem 3.1.

A ctd-set *D* of a connected graph $G = (V, E)$ is also a ctd-set of $T_2(G)$ if and only if

(i) $\langle D \rangle$ has an isolated vertices.

(ii) For each $v \in D, N(v) \cap (V - D) \neq \varphi$ and

(iii) $\langle V - D\rangle \cong K_1$ and if $v \in V - D$ then $N(v) \cap D \neq \varphi$.

Proof.

Let *D* be a ctd-set of both *G* and $T_2(G)$.

(i) Let $v \in D$ be not an isolated vertex in $\langle D \rangle$ then its edge vertex v' in $T_2(G)$ is isolated in $\langle V(\mathrm{T_2(G)}) - \mathrm{D} \rangle$ which contradicts the ctd-set of $T_2(G)$. Therefore, $\langle D \rangle$ has an isolated vertices.

(ii) Let there exists a vertex $v \in D$ such that $N(v) \cap (V - D) = \varphi$. Then, its edge vertex v' is isolated in $\langle V(T_2(G)) - D \rangle$.

(iii) If K_2 is an induced subgraph of $\langle V - D \rangle$. Then $\langle V(T_2(G)) - D \rangle$ contains a cycle. Therefore, $\langle V - D\rangle \cong K_1$. Let $v \in K_1$. Since *G* is connected so that remaining vertices are adjacent to *v*. There exists a edge vertices $v_i' \in (v, v_i)$ such that $\langle v, v_i, v_i'\rangle \cong C_3$ in $T_2(G)$.Hence $N(v) \cap D \neq \varphi$.

Conversely, if (i) is true, *D* is dominating set of $T_2(G)$. If (ii) holds, then $\langle V(T_2(G)) - D \rangle$ is connected and if (iii) holds, then $\langle V(T_2(G)) - D \rangle$ is acyclic. Therefore, $\langle V(T_2(G)) - D \rangle$ is a tree. Hence, *D* is also a ctd-set of $T_2(G)$. □

In the following, exact values of complementary tree domination number of semi total point graph of some classes of graphs are given.

Proposition 3.2.

Let $C_p^{(t)}$, $t \geq 2$ be the one point union of *t* cycles of length $p (p \geq 3)$ then$\gamma_{ctd}\left(T_2\left(C_p^{(t)}\right)\right) = (p-1)t, p \geq 3$.

Proof.

Let $G = C_p^{(t)}$ and *u* be the point of union of *t* cycles of length *p*. Let the vertex set of k^{th} cycle in $C_p^{(t)}$ be $V_k = \{u, u_{k1}, u_{k2}, \dots, u_{k,p-1}\} k = 1,2,\dots,t (t \geq 2)$ $V_k\left(T_2\left(C_p^{(t)}\right)\right) = \{u, u_{k1}, u_{k2}, \dots, u_{k,p-1}\} \cup \{u'_{k1}, u'_{k2} \dots u'_{kp}\}$ where $u'_{k1}, u'_{k2}, \dots, u'_{kp}$ are the corresponding edge vertices of $(u, u_{k1}), (u_{k1}, u_{k2}), \dots, (u, u_{k,p-1})$. Let $D_k = \{u_{k1}, u'_{k3}, \dots, u'_{kp}\}$, $k = 1, 2, \dots, t$ $D = \bigcup_{k=1}^{t} D_k \subseteq V(T_2(G))$. Then $\langle V(T_2(G)) - D \rangle$ is a tree. Hence *D* is a minimum ctd-set of $T_2(G)$. Therefore, $|D| = \gamma_{ctd}(T_2(G)) = (p-1)t$. □

Proposition 3.3.

Let G be a unicyclic graph by attaching a path of length $(n \geq 1)$ to the $t(\leq p)$ consecutive vertices of $C_p(p \geq 3)$. Then $\gamma_{ctd}\big(T_2(G)\big) = nt + p - 1$.

Proof.

In the cycle $C_p(p \geq 3)$ say $v_1, v_2, \dots, v_p$. Consider a path of length P_{p-1} in C_p say $v_1, v_2, \dots, v_{p-1}$ and attach a path $P'_n(n \geq 1)$ say $v_i, u_2, \dots, u_n$, $i = 1, 2, \dots, t$ to the $t(\leq p)$ consecutive vertices of C_p. In $T_2(G)$, the set of all edge vertices of path P'_n of *t* consecutive vertices of C_p, edge vertices of a path P_{p-1} and a vertex v_p forms a minimum ctd-set of $T_2(G)$. Therefore, $\gamma_{ctd}\big(T_2(G)\big) = nt + p - 1$.

□

Corollary 3.4.

G be a unicyclic graph by attaching one pendant vertex to exactly one vertex of C_p then $\gamma_{ctd}\big(T_2(G)\big) = p$.

Corollary 3.5.

G be a unicyclic graph by attaching one pendant vertex to $p - 1$ vertices of C_p. Then $\gamma_{ctd}\big(T_2(G)\big) = 2p - 2$.

Corollary 3.6.

G be a unicyclic graph by attaching a path of length *n* to exactly one vertex of C_p then$\gamma_{ctd}\big(T_2(G)\big) = p + n + 1$.

4. Bounds and Some Exact Values for the Complementary Tree Domination Number of Semi Total Point Graph of Graphs

Theorem 4.1.

$\gamma_{ctd}\big(T_2(G)\big) = 2$ if and only if $G \cong K_{1,2}$ or C_3.

Proof.

Let *D* be a γ_{ctd}-set of $T_2(G)$ such that $|D| = 2$.
Let $D = \{u_1, u_2\}$ where $u_1, u_2 \in V\big(T_2(G)\big)$.

Case 1.

u_1 and u_2 are vertices in *G*. Then, *D* is also a γ_{ctd}-set of *G*. By Theorem 3.1, it can be seen that $G \cong K_{1,2}$.

Case 2.

$u_1, u_2 \in V\big(T_2(G)\big) - V(G)$. Let $u_1 = u'_1$ and $u_2 = u'_2$ where (u, u_1) and $(u, u_2) \in E(G)$. Since u'_1 and u'_2 are edge vertices in $T_2(G)$ and $\langle V(T_2(G)) - D\rangle$ is connected and acyclic. Hence it can be seen that $G \cong K_{1,2}$.

Case 3.

Let $u_1 \in V(G)$ and $u_2 = u'_2 \in V\big(T_2(G)\big) - V(G)$

Sub case 3.1. $u_2 = u_2'$.

That is $D = \{u_1, u_2'\}$ is a γ_{ctd}-set of $T_2(G)$. Hence $G \cong C_3$.

Sub case 3.2. $u_2 \neq u_2'$.

Let $u_2 = v'$ for some $(u_1, v) \in E(G)$ and $v' \neq u_2'$. Then $D = \{u, v'\}$ is a γ_{ctd}-set of $T_2(G)$. $u_2' \neq v'$ implies that $\langle u_2', v, x \rangle$ form a cycle in $\langle V(T_2(G)) - D \rangle$ forms either cycle or disconnected where $x, v \in V(G)$.

Conversely, if $G \cong K_{1,2}$ or C_3. Then $\gamma_{ctd}(T_2(G)) = 2$. □

Theorem 4.2.

$\gamma_{ctd}(T_2(G)) = p + q - 2$ if and only if $G \cong K_2$.

Proof.

Assume $\gamma_{ctd}(T_2(G)) = p + q - 2$. Let D be a minimum ctd-set of $T_2(G)$ having $p + q - 2$ vertices. Since $\langle V(T_2(G)) - D \rangle \cong K_2$. Let $V(T_2(G)) - D = \{u, v\}$ either

(i) $u, v \in V(G)$ or

(ii) $u \in V(G)$ and $v = u' \in V(T_2(G)) - V(G)$ where $u' \in (u, v)$.

Case 1.

Let $u' \in D$ where u' be the edge vertex of (u, v) in $T_2(G)$. Therefore, no vertex of $V(G)$ is an element of D hence $u' \in D$. Therefore $G \cong K_2$.

Case 2.

Let $u \in V(G)$, $u' \in V(T_2(G)) - V(G)$. Since $\langle V(T_2(G)) - D \rangle \cong K_2$. u' is adjacent to u in $\langle V(T_2(G)) - D \rangle$. That is v is adjacent to a vertex u in G. Therefore, $G \cong K_2$.

Conversely, if $G \cong K_2$ then $\gamma_{ctd}(T_2(G)) = p + q - 2$. □

Remark 4.3.

If $p \geq 3$ then $\gamma_{ctd}(T_2(G)) \leq p + q - 3$. Equality holds if $G \cong P_3$.

Theorem 4.4.

Let $G(p, q)$ be a complete graph with $4 \leq p \leq 8$ then $\gamma_{ctd}(T_2(G)) \leq \left\lceil \frac{p+q}{2} \right\rceil$.

Proof.

We prove induction on p. Let $p = 4$, $e = (u_1, u_2) \in E(G)$ where $u_1, u_2 \in V(G)$. Let $D = \{u_1, u_2, u_{12}', u_{34}'\} \subseteq V(T_2(G))$ is a ctd-set of $T_2(G)$ where $u_{12}', u_{34}' \in V(T_2(G)) - V(G)$ and $D' = V(T_2(G)) - D = \{u_3, u_4, u_{13}', u_{23}', u_{24}', u_{41}'\}$ since $p \geq 4$ and $\delta(G) \geq 2$ each vertex in $\langle V(T_2(G)) - D' \rangle$ is adjacent to atleast one vertex in D and $\langle V(T_2(G)) - D \rangle \cong S_{m,m}, m, m \geq 2$ in $T_2(G)$. Therefore, $|D| \leq \left\lceil \frac{p+q}{2} \right\rceil$. Hence $\gamma_{ctd}(T_2(G)) \leq \left\lceil \frac{p+q}{2} \right\rceil$. Equality holds if $G \cong K_8$. □

Theorem 4.5.

Let *G* be a connected graph such that $\delta(G) \geq 2$ then $\gamma_{ctd}\big(T_2(G)\big) \leq q - \Delta(G) + 1$.

Proof.

Let *v* be a vertex of maximum degree in *G*. Let $S = \{v_i' : (v, v_i) \in E(G), i = 1, \dots, \Delta(G)\}$. Any set $D \subseteq V\big(T_2(G)\big)$ such that $V\big(T_2(G)\big) - D = \big\{\bigcup_{i=1}^{p-1} v_i\big\} \cup S$. Since $\delta(G) \geq 2$, $deg(v_i) \geq 2, i = 1,2,\dots,p-1$ and hence v_i is adjacent to the vertices of *G* other than *v*, Then v_i' is adjacent to the vertex v_i, $i = 1,2,\dots,p-1$ where v_i is a vertex in $V\big(T_2(G)\big) - D$. Also, *v* is adjacent to atleast one vertex of *G* and hence in *S*. Therefore, *D* is a dominating set of $V\big(T_2(G)\big)$. Moreover $\big\langle V\big(T_2(G)\big) - D\big\rangle \cong T \circ K_1$ and hence *D* is a ctd-set of $T_2(G)$. Therefore,

$$\gamma_{ctd}\big(T_2(G)\big) \leq \big|V\big(T_2(G)\big) - (V(G) - v) - S\big|$$
$$= p + q - p + 1 - \Delta(G)$$
$$\leq q - \Delta(G) + 1$$

Equality holds if $G \cong K_3, K_4 \& K_1 + P_{n-1}$. □

Theorem 4.6.

Let $T_2\big(G_1(p_1, q_1)\big)$ and $T_2\big(G_2(p_2, q_2)\big)$ be two connected graphs of order atleast two. Let *T* be an induced sub-graph of $T(G_1)$ having maximum number of vertices such that *T* is a tree. If β_0 is the independence number of $T_2(G_2)$ and vertices corresponding to the edge joining from copies of $T(G_2)$ to $T(G_1)$ then

$$\gamma_{ctd}\big(T_2(G_1 \circ G_2)\big) \leq 2p_1p_2 + p_1(1 + q_2) + q_1 - (t - 2)\beta_0 - t.$$

Proof.

Let *T* be an induced sub-graph of $T(G_1)$ having maximum number of vertices such that *T* is a tree and $|T| = t$.

Let *S* be a maximum independent set of $T(G_2)$ and vertices corresponding to the edge joining to the vertices of each copies of $T(G_2)$ to $T(G_1)$ such that $|S| = \beta_0$ and D' be the set of vertices in *S* in copies of $T(G_2)$ which are adjacent to the vertices of *T* then $|D'| = (t - 2)\beta_0$ if $\delta(G_1) \geq 2$.

Let $D = (V(T_2(G_1 \circ G_2))) - (V(T) \cup D')$ then $V\big(T_2(G_1 \circ G_2)\big) - D = V(T) \cup D'$ and $(t - 2)$ vertices of $V(T)$ are adjacent to $(p_2 - \beta_0)$ vertices in a copy of $T_2(G_2)$. Also each vertex in D' is adjacent to atleast one $(p_2 - \beta_0)$ vertices in a copy of $T_2(G_2)$.

Therefore, *D* is a dominating set of $T_2(G_1 \circ G_2)$ and
$\langle V(T_2(G_1 \circ G_2)) - D \rangle$ is a tree.

$$\gamma_{ctd}\big(T_2(G_1 \circ G_2)\big) \leq |D|$$
$$\leq \big|V\big(T_2(G_1 \circ G_2)\big) - (V(T) \cup D')\big|$$
$$= 2p_1p_2 + p_1(1 + q_2) + q_1 - t(1 + \beta_0) - 2\beta_0.$$ □

5. Relation Between γ_{ctd}(G) and γ_{ctd}(T_2(G))

Observation 5.1.

For any connected graph $G(p, q)$, $p \geq 2$ then $\gamma_{ctd}(G) \leq \gamma_{ctd}\big(T_2(G)\big)$. Equality holds if $G \cong K_{1,p-1}$.

Theorem 5.2.

For any connected graph $G(p,q), p \geq 2$ with $\delta(G) = 1$ then $\gamma_{ctd}(T_2(G)) \leq p - 1 + \gamma_{ctd}(G)$.

Proof.

Let D be a minimum ctd-set of G and hence $|D| = \gamma_{ctd}(G)$. Therefore, $\langle V(G) - D \rangle$ is a tree. Now, the set $D' = D \cup \left(V(T_2(G)) - V(G)\right)$ is a minimum ctd-set of $T_2(G)$. Hence, $\gamma_{ctd}(T_2(G)) \leq |D'| = \gamma_{ctd}(G) + p - 1$. □

Theorem 5.3.

Given two integers a and b with $2 \leq a \leq b$ there exists a graph with $a + b + 1$ vertices such that $\gamma(T_2(G)) = a + 1$ and $\gamma_{ctd}(T_2(G)) = a + b$. Also $\gamma_{ctd}(T_2(G)) \leq \gamma(T_2(G)) + b$.

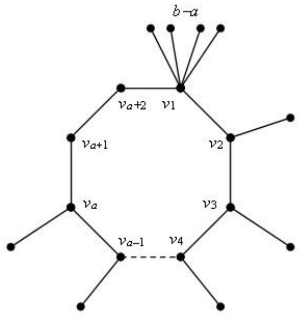

Figure 1.

Proof.

In the cycle $C_{a+2} (a \geq 2)$ by $\{v_1, v_2, \dots, v_{a+2}\}$ of length $a + 2$. Consider a path of length a. In this path attach $(b - a)$ pendant edges at exactly one vertex and attach one pendant edge at each of the remaining $(a - 1)$ vertices. Let the graph thus obtained be denoted by G and G has $a + b + 1$ vertices.

In $T_2(G)$, edges of cycle C_{a+2} and edges of pendant vertex of G and the vertices of G are the vertex set of $T_2(G)$. Therefore, $|V(T_2(G))| = 2a + 2b + 2$.

The set $\{v_1, v_2, \dots, v_{a+1}\}$ forms a minimum dominating set of $T_2(G)$ and the set consisting of edge vertex of path P_{a+1} a vertex C_{a+2} and all the edge vertex of pendant vertices of G forms a minimum ctd-set of $T_2(G)$. Therefore, $\gamma_{ctd}(T_2(G)) = a + b$.
If $a = b$ then, the equality holds. □

Theorem 5.4.

If $\gamma_{ctd}(G) = 1$ then $\gamma_{ctd}(T_2(G)) = p - 1$ where $p \geq 2$ is the number of vertices in G.

Proof.

Assume $\gamma_{ctd}(G) = 1$, then $G \cong K_1 + T$ where T is a tree with atleast two vertices. Let $V(K_1) = v$ and $V(T) = \{v_1, v_2, \dots, v_{p-1}\}$ then $V(G) = \{v, v_1, v_2, \dots v_{p-1}\}$ and $V(T_2(G)) = V(G) \cup v_i' \cup v_{i,i+1}'$ where $v_i' \in (v, v_i)$ and $v_{i,i+1}' \in (v_i, v_{i+1})$ where $i = 1, 2, \dots, p - 1$. Let $D = \{v_{i,i+1}'/i = 1,2, \dots, p - 2\} \cup \{v\}$. Then $D \subseteq V(T_2(G))$ and $\langle V(T_2(G)) - D \rangle$ is a tree. Therefore D is a ctd-set of $T_2(G)$ and hence, $\gamma_{ctd}(T_2(G)) = p - 1$. □

Theorem 5.5.

Let G be a connected graph, if $\gamma_{ctd}(G) = 2$ then

$$\gamma_{ctd}(T_2(G)) = \begin{cases} p & if\ G \cong G_1 \& G_2 \\ p-1 & if\ G \cong G_3 \end{cases}.$$

(i) G_1 is the graph obtained from $K_1 + T$ with one pendant edge attached at the vertex of K_1, where T is any tree with $p-2$ vertices.

(ii) G_2 is the graph obtained from a tree T where ($|T| = p-2$) by joining each of the vertices of the tree to the vertices of K_2 such that $\deg_G(v) \geq 2$ for all $v \in V(K_2)$.

(iii) G_3 is the graph obtained from a tree by joining each of the vertices of the tree to the vertices of $2K_1$ such that $\deg_G(v) \geq 1$ for all $v \in V(2K_1)$ and $|V(G)| = p$.

Proof.

Assume $\gamma_{ctd}(G) = 2$.

Let G be a connected graph with $p \geq 4$ and $S_1 = \{u_1, u_2\}$ is a minimum ctd-set of G then $\langle V(G) - S_1 \rangle$ is a tree T. Hence $|T| = |V(G) - S_1| = p - 2$.

Now construct $T_2(G)$, the vertices of complementary dominating set of G (tree T) and its edge vertex forms a cycle C_3.

Let$S_2 = \{v'_{ij} / i \neq j, i = 1,2, \dots, p-3, j = 1,2, \dots, p-2\} \subseteq V(T_2(G))$where$(v_i, v_j) \in E(V(G) - S_1)$.

Case 1.

u_1 and u_2 are connected then $G \cong G_1$ or G_2. Let $D = S_1 \cup S_2 \cup \{u'_{12}\}$ is a minimum ctd-set of $T_2(G_1)$ or $T_2(G_2)$. Hence $|D| = |S_1| + |S_2| + 1 = p$.Therefore, $\gamma_{ctd}(T_2(G)) = p$ if $G \cong G_1$ or G_2.

Case 2.

u_1 and u_2 are not connected. Then $G \cong G_3$. Let $D = S_1 \cup S_2$ is a minimum ctd-set of $T_2(G_3)$. Hence $|D| = |S_1| + |S_2| = p - 1$. Therefore, $\gamma_{ctd}(T_2(G)) = p - 1$ if $G \cong G_3$. □

Theorem 5.6.

Let G be a connected graph with p vertices ($p \geq 3$), $V(T_2(G)) = V(G) \cup V'(G)$ then, $V'(G) = V(T_2(G)) - V(G)$ is a ctd-set of $T_2(G)$ if and only if G is a tree.

Proof.

Assume $V'(G)$ is a ctd-set of $T_2(G)$. Then, each vertex in $V(T_2(G)) - V(G)$ is adjacent to atleast one vertex in $V'(G)$ and $\langle V(T_2(G)) - V'(G) \rangle$ is a tree. That is $\langle V(G) \rangle$ is a tree.

Conversely, Assume G is a tree.Let$D = V'(G)$ that is D contains all the edge vertices of G. Since G is connected each vertex v in $V(T_2(G)) - D = V(G)$ forms a cycle C_3 in $T_2(G)$ and $V(T_2(G)) - D = V(G)$ is a tree. Hence, D is a ctd-set of G. □

Theorem 5.7.

For any connected (p, q) graph G,

$$\gamma_{ctd}(T_2(G)) + \Delta(G) = 2p - 2 \text{ or } p + q - 1 \text{ if and only if } G \cong K_{1,p-1}\ (p \geq 4).$$

Proof.

When $G \cong K_{1,p-1}$, $\gamma_{ctd}\big(T_2(G)\big) + \Delta(G) = p + q - 1$.

Conversely, $\gamma_{ctd}\big(T_2(G)\big) + \Delta(G) = 2p - 2$ is possible if $\gamma_{ctd}\big(T_2(G)\big) = p - 1$ and $\Delta(G) = p - 1$ is possible only if G is a star on p vertices. □

Theorem 5.8.

For any connected (p, q) graph G, $\gamma_{ctd}\big(T_2(G)\big) + \Delta(G) = 2p - n$ $(p \geq 4)$ where $n = diam(G)$, $n \geq 2$ if $G \cong$ broom graph.

Proof.

For the graphs given in the theorem $\Delta(G) = p - n$, $\gamma_{ctd}\big(T_2(G)\big) = p$ then $\gamma_{ctd}\big(T_2(G)\big) + \Delta(G) = 2p - n$.

Conversely, $\gamma_{ctd}\big(T_2(G)\big) + \Delta(G) = 2p - n$ only possible if

(i) $\gamma_{ctd}\big(T_2(G)\big) = p - 1$ and $\Delta(G) = p - (n - 1)$ in this case $\gamma_{ctd}\big(T_2(G)\big) = p - 1$ if and only if G is a tree on p vertices. But for a star $\Delta(G) = p - 1$ and $diam(G) = 2$.

(ii) If G is a broom graph on p vertices with path of length 2, $\Delta(G) = p - 2$ and $diam(G) = 3$.

$$\gamma_{ctd}\big(T_2(G)\big) + \Delta(G) = p - 1 + p - 2 = 2p - 3$$

Therefore G is a broom graph of length $n - 1$.i.e., $n = diam(G)(n \geq 2)$. □

Theorem 5.9.

Let t_1 and t_2 be two trees with order $p_1 \geq 2$ and $p_2 \geq 2$ respectively. Then

$$\gamma_{ctd}\big(T_2(t_1 \circ t_2)\big) \leq 2\gamma_{ctd}(t_1 \circ t_2) + p_1(1 + p_2) - 1.$$

Proof.

We have $\gamma_{ctd}(G) \leq p_1(p_2 - 1)$[4]. Let D be the minimum ctd-set of $t_1 \circ t_2$. Hence $|D| \leq p_1(p_2 - 1)$.

Let D' be the number of edge vertices of $T_2(t_1 \circ t_2)$. Then $D \cup D' \subseteq V\big(T_2(G)\big)$ is a minimum ctd-set of $T_2(G)$. Therefore,

$$\begin{aligned}\gamma_{ctd}\big(T_2(G)\big) &\leq |D \cup D'| \\ &\leq 2p_1(p_2 - 1) + p_1(1 + p_2) - 1 \\ &\leq 2\gamma_{ctd}(G) + p_1(1 + p_2) - 1.\end{aligned}$$

□

Theorem 5.10.

Let G be a (p, q), $p \geq 5$, graph such that both G and $\overline{G}$ are connected then

(i) $8 \leq \gamma_{ctd}\big(T_2(G)\big) + \gamma_{ctd}\left(T_2(\overline{G})\right) \leq 2(p + q - 4)$

(ii) $4 \leq \gamma_{ctd}\big(T_2(G)\big).\gamma_{ctd}\left(T_2(\overline{G})\right) \leq (p + q - 4)^2$

Proof.

By Theorem 3.4 $\gamma_{ctd}(T_2(G)) = p + q - 2$ if and only if G is a graph K_2. But in this case $\overline{G}$ is disconnected. $\gamma_{ctd}(T_2(G)) = p + q - 3$ if and only if G is the graph $P_3 \& C_4$. But in this case $\overline{G}$ is disconnected. Therefore, $\gamma_{ctd}(T_2(G)) \leq p + q - 4$. Hence, $\gamma_{ctd}(T_2(G)) + \gamma_{ctd}\left(T_2(\overline{G})\right) \leq 2(p + q - 4)$.

For lower bound $\gamma_{ctd}(T_2(G)) = 4$ if and only if $G \cong P_4$ and C_5. In this case $\overline{G}$ is connected. Hence $\gamma_{ctd}(G) + \gamma_{ctd}(\overline{G}) \geq 4$. (ii) follows similarly. □

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