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Fixed Point Theorem for $\mathscr{H}-\mathscr{F}-$ Contractive Mapping in Complete Fuzzy Metric Spaces

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Abstract: This study introduces a new class of contractive mappings, namely $\mathscr{H}$–$\mathscr{F}$–contractive mappings, in the setting of complete fuzzy metric spaces. We provide the sufficient conditions under which such mappings admit unique fixed points. Our approach builds on classical results by employing a novel functional inequality that involves a strictly increasing function $F \in \mathscr{F}$ and auxiliary function $H \in \mathscr{H}$. An illustrative example demonstrates the applicability of the main results. This work enriches the framework of the fixed point theory in fuzzy metric spaces and opens pathways for further exploration.

Keywords: Fixed point, fuzzy metric space, $\mathscr{H}$–$\mathscr{F}$–contractive mapping, fuzzy analysis, contraction principle.

1. Introduction

The theory of fuzzy metric spaces, initiated by Kramosil and Michalek [1] and further developed by George and Veeramani [2], has received considerable attention for addressing the uncertainties in mathematical models. Several classical fixed point results, such as Banach, Kannan, and Chatterjea contractions, have been generalized to fuzzy metric settings [3, 4, 5, 6, 7]. In this context, new contraction types, such as $\mathscr{F}$-contractions and simulation functions have broadened the scope of fixed point analysis.

Motivated by these developments, we propose the concept of $\mathscr{H}$–$\mathscr{F}$–contractive mappings. These mappings incorporate an auxiliary function $\mathscr{H}$ and transformation function $\mathscr{F}$, which allows us to derive a fixed point theorem that generalizes several known results. Our main contribution lies in establishing a unique fixed point under the new contractive condition.

2. Preliminaries

Here, we recall the key definitions required for our analysis. Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0$ denotes the set of all non-negative integers, $\mathbb{R}^+ = (0, \infty)$ and $\mathbb{I} = [0, 1]$.

Definition 1. [8] A binary operation $* : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ is called a continuous t-norm, if the following conditions holds:

(i) $*$ is associative and commutative.

(ii) For any $a \in \mathbb{I}$, $a * 1 = a$.

(iii) For $a, b, c, d \in \mathbb{I}$ with $a \leq b$ and $c \leq d$, $a * c \leq b * d$.

(iv) $*$ is continuous.

For example, $a *_1 b = min\{a, b\}$, $a *_2 b = ab$ and $a *_3 b = max\{a + b - 1, 0\}$ are commonly used continuous t-norms.

Definition 2. [2] The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space (referred to as GV - Fuzzy Metric Space), if X is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy set defined on $X^2 \times \mathbb{R}^+$ satisfying the following conditions;

(GV-1) $M(x, y, 0) > 0$ for all $x, y \in X$.

(GV-2) $M(x, y, t) = 1$ for all $t \in \mathbb{R}^+$ iff $x = y$.

(GV-3) $M(x, y, t) = M(y, x, t)$ for any $t \in \mathbb{R}^+$.

(GV-4) $M(x, z, t+s) \geq M(x, y, t)*M(y, z, s)$ for all $x, y, z \in X$ and $t, s \in \mathbb{R}^+$.

(GV-5) $M(x, y, .) : \mathbb{R}^+ \to \mathbb{I}$ is continuous.

Remarks:

1. [2] For any $r \in (0, 1)$, whenever $M(x, y, t) > 1 - r$ for $x, y \in X$ and $t \in \mathbb{R}^+$, there exists $t_0 \in (0, t)$ such that $M(x, y, t_0) > 1 - r$.
2. For any $r_1 > r_2$, there exists r_3 such that $r_1 * r_3 > r_2$, and for any r_4, we can find an r_5 such that $r_5 * r_5 > r_4$, where $r_1, r_2, r_3, r_4, r_5 \in (0, 1)$.
3. [3, 2] It is well known and easy to verify that for every $x, y \in X$, $M(x, y, .)$ is a non-decreasing continuous function of $\mathbb{R}^+$.

Definition 3. Let $(X, M, *)$ be a fuzzy metric space.

(i) A sequence $\{x_n\}$ converges to $x \in X$, if every $\delta \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \delta$ for all $n \geq n_0$. This implies that $\lim_{n\to\infty} M(x_n, x, t) = 1$.

(ii) A sequence $\{x_n\}$ is said to be a Cauchy, if for every $\epsilon > 0$ and $t \in \mathbb{R}^+$, there exists $n_0 \in \mathbb{N}$ such that $M(x_m, x_n, t) > 1 - \epsilon$ for all $m > n \geq n_0$. Moreover, $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$, for every $p \in \mathbb{N}$ and $t \in \mathrm{R}^+$.

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is called the complete fuzzy metric space.

3. Main Result

In the following, $\mathscr{F}$ denotes the set of all strictly increasing functions $F:(0,1)\to\mathbb{R}$ satisfying conditions $\lim_{p\to 0^+} F(p) = -\infty$ and $\lim_{p\to 1^-} F(p) = \infty$. In addition, $\mathscr{H}$ denotes the set of mappings $H:X\times X\to\mathbb{I}$ satisfies the conditions, for any sequence $\{x_n\} \subset X$, $\lim_{n\to\infty} H(x_n, x_{n+1}) = 0$ and $H(x\ y) = 0$, if $x = y$.

Definition 4. For any $F \in \mathscr{F}$, a mapping $T : X \to X$ is said to be $\mathscr{H}$– $\mathscr{F}$– contractive if there exists a function $H \in \mathscr{H}$ such that

$$F(M(Tx, Ty, t)) \geq H(x, y) + F(M(x, y, t)) \tag{1}$$

for all $x, y \in X$ with $x \neq y$.

Theorem 1. Let $(X, M, *)$ be a complete fuzzy metric space. If $T : X \to X$ is an $\mathscr{H}$– $\mathscr{F}$– contractive mapping, then T has a unique fixed point in X.

Proof: Let $x_0 \in X$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}_0$. Assume that $T : X \to X$ is an $\mathscr{H}$– $\mathscr{F}$– contractive mapping. For some $n \in \mathbb{N}_0$, if $x_{n+1} = Tx_n = x_n$, then x_n is a fixed point. Assume that for any $n \in \mathbb{N}_0$, $Tx_n = x_{n+1} \neq x_n$. From Equation (1), for every $n \in \mathbb{N}_0$ and $t > 0$, we have

$$F(M(Tx_n, Tn_{n+1}, t)) \geq H(x_n, x_{n+1}) + F(M(x_n, x_{n+1}, t)). \tag{2}$$

Because $H(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$,

$$F(M(Tx_n, Tn_{n+1}, t)) \geq H(x_n, x_{n+1}) + F(M(x_n, x_{n+1}, t)) > F(M(x_n, x_{n+1}, t)). \tag{3}$$

We know that F is a strictly increasing function, therefore,

$$M(Tx_n, Tn_{n+1}, t) = M(x_{n+1}, x_{n+2}, t) > M(x_n, x_{n+1}, t).$$

Thus, for every $t > 0$, $\{M(x_n, x_{n+1}, t)\}$ is an increasing sequence bounded from above in $\mathbb{I}$. Hence, for every $t > 0$, $\{M(x_n, x_{n+1}, t)\}$ converges in $\mathbb{I}$. That is, for every $t > 0$, there exists $\delta(t) \in (0, 1)$ and $N \in \mathbb{N}_0$, such that $M(x_n, x_{n+1}, t) > 1 - \delta(t)$ for all $n \geq N$. Moreover, for every $t > 0$, suppose that there exists $\alpha(t) \in \mathbb{I}$ such that as $n \to \infty$ the sequence $\{M(x_n, x_{n+1}, t)\}$ approaches to its limit $\alpha(t)$ from its left side.

$$\Rightarrow \lim_{n\to\infty} M(x_n, x_{n+1}, t) = \alpha(t)^-.$$

And, this imply

$$\lim_{n\to\infty} F\left(M(x_n, x_{n+1}, t)\right) = F\left(\alpha(t)^-\right).$$

By taking limit both sides of equation (3), we get

$$F\left(\alpha(t)^-\right) \geq \lim_{n\to\infty} H(x_n, x_{n+1}) + F\left(\alpha(t)^-\right) > F\left(\alpha(t)^-\right).$$

This is possible when

$$\lim_{n\to\infty} H(x_n, x_{n+1}) = 0 \tag{4}$$

and $F\left(\alpha(t)^-\right) = 0,$ which contradict with $F\left(\alpha(t)^-\right) > 0.$ Therefore, we have

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1^-. \tag{5}$$

Furthermore, we need to prove that $\{x_n\}$ is a Cauchy sequence. For $m, n \in \mathbb{N}_0$ and equation (2) implies that

$$\begin{aligned}
F\left(M(x_{n+m}, x_{n+m+1}, t)\right) &= F\left(M(Tx_{n+m-1}, Tx_{n+m}, t)\right) \\
&\geq H(x_{n+m-1}, x_{n+m}) + F\left(M(x_{n+m-1}, x_{n+m}, t)\right) \\
&\geq H(x_{n+m-1}, x_{n+m}) + H(x_{n+m-2}, x_{n+m-1}) + F\left(M(x_{n+m-2}, x_{n+m-2}, t)\right) \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&\geq \sum_{k=0}^{m} H(x_{n+k-1}, x_{n+k}) + F\left(M(x_n, x_{n+1}, t)\right)
\end{aligned}$$

By taking the limits of both sides and together from equations (3) and (5), we obtain

$$\lim_{n\to\infty} F\left(M(x_{n+m}, x_{n+m+1}, t)\right) \geq \lim_{n\to\infty} \sum_{k=0}^{m} H(x_{n+k-1}, x_{n+k}) + F\left(1^-\right) > F\left(1^-\right).$$

From Equation (4), it is clear that for every $k \in \mathbb{N}_0$, $\lim_{n\to\infty} H(x_{n+k-1}, x_{n+k}) = 0.$ Hence

$\lim_{n\to\infty} F\left(M(x_{n+m}, x_{n+m+1}, t)\right) = F\left(1^-\right)$, and hence $\lim_{n\to\infty} M(x_{n+m}, x_{n+m+1}, t) = 1^-$.

Therefore $\{x_n\}$ is a Cauchy sequence. Since $(X, M, *)$ is complete, therefore there exists $z \in X$ such that

$$\lim_{n\to\infty} x_n = z. \tag{6}$$

It is easy to verify that z is a fixed point of T. Since T is continuous and equation (7), implies that

$$T(z) = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z. \tag{7}$$

Finally, to prove the uniqueness of the fixed point z of T, assume that there exists another fixed point u of T in X. That is $T(u) = u$, where $z \neq u$. Now, by using equation (1) and (3), we get

$$F(M(Tz, Tu, t)) \geq H(z, u) + F(M(z, u, t)) > F(M(z, u, t))$$

$$\Rightarrow \quad F(M(z, u, t)) \geq H(z, u) + F(M(z, u, t)) > F(M(z, u, t)).$$

As a consequence, we have $F(M(z, u, t)) > F(M(z, u, t))$, this imply, $M(z, u, t) > M(z, u, t)$. This is a contradiction, hence $z = u$. That is T has a unique fixed point in X.

Example 1. Let $X = \mathrm{R}$, $a * b = min\{a, b\}$ for all $a, b \in \mathrm{I}$ and $M(x, y, t) = \dfrac{t}{t + |x - y|}$ for all $x, y \in X$ and $t > 0$. Subsequently, $(X, M, *)$ is a complete fuzzy metric space. Now, let $F \in \mathscr{F}$ be defined as $F(p) = \dfrac{2p-1}{p(1-p)}$ for all $p \in (0, 1)$, and let $H \in \mathscr{H}$ defined as $H(x, y) = |x - y|$. Now, define $T{:}X \to X$, $T(x) = \dfrac{x}{4}$ for all $x \in X$.

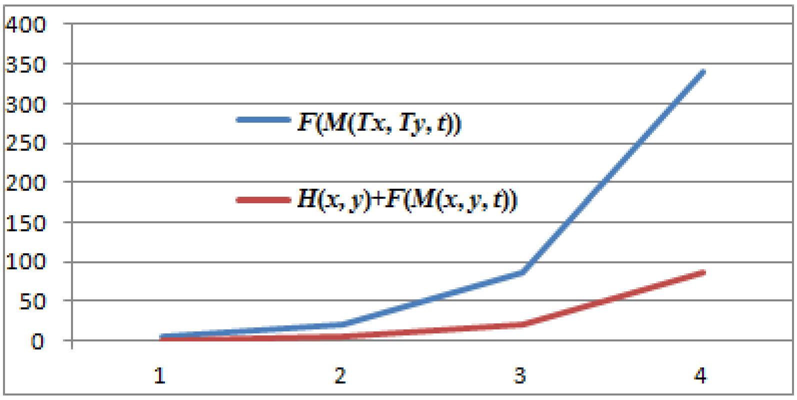

Figure 1: $\mathscr{H}$– $\mathscr{F}$– contractive

From the graph given in the Figure 1, it is clear that, for any $x, y \in X$ and $t > 0$, T is a $\mathscr{H}$– $\mathscr{F}$– contractive. Thus, all the conditions of Theorem 1 are satisfied, therefore T has a unique fixed point

$z = 0$. To verify the above calculations, the reader is referred to the computational data for $t = 1$ presented to the table shown in the below Table 2.

n	x_n	$x_{n+1} = T(x_n)$	$M(x_n, x_{n+1}, t)$	$H(x_n, x_{n+1})$	$F(M(Tx_n, Tx_{n+1}, t))$	LHS
1	1.000000000000	0.250000000000	0.842105263158	0.750000000000	21	6
2	0.250000000000	0.062500000000	0.955223880597	0.187500000000	85	21
3	0.062500000000	0.015625000000	0.988416988417	0.046875000000	341	85
4	0.015625000000	0.003906250000	0.997078870497	0.011718750000	1365	341
5	0.003906250000	0.000976562500	0.999268114174	0.002929687500	5461	1365
6	0.000976562500	0.000244140625	0.999816928053	0.000732421875	21845	5461
7	0.000244140625	0.000061035156	0.999954225728	0.000183105469	87381	21845
8	0.000061035156	0.000015258789	0.999988556039	0.000045776367	349525	87381
9	0.000015258789	0.000003814697	0.999997138985	0.000011444092	1398101	349525
10	0.000003814697	0.000000953674	0.999999284745	0.000002861023	5592405	1398101
11	0.000000953674	0.000000238419	0.999999821186	0.000000715256	22369621	5592405
12	0.000000238419	0.000000059605	0.999999955297	0.000000178814	89478485	22369621
13	0.000000059605	0.000000014901	0.999999988824	0.000000044703	357913940	89478485
14	0.000000014901	0.000000003725	0.999999997206	0.000000011176	1431655764	357913940
15	0.000000003725	0.000000000931	0.999999999302	0.000000002794	5726623060	1431655764
16	0.000000000931	0.000000000233	0.999999999825	0.000000000698	22906492244	5726623060
17	0.000000000233	0.000000000058	0.999999999956	0.000000000175	91625968980	22906492244
18	0.000000000058	0.000000000015	0.999999999989	0.000000000044	366503875924	91625968980
19	0.000000000015	0.000000000004	0.999999999997	0.000000000011	1466015503700	366503875924
20	0.000000000004	0.000000000001	0.999999999999	0.000000000003	5864062014804	1466015503700
21	0.000000000001	0.000000000000	1.000000000000	0.000000000001	23456248059220	5864062014804
...	...	...	...	...	...	...
$n \to \infty$	$x_n \to 0$	$T(0) = 0$	$M(x_n, x_{n+1}, t) \to 1^-$	$H(x_n, x_{n+1}) \to 0$	$F(M(Tx_n, Tx_{n+1}, t)) \to \infty$	$LHS \to \infty$

Table 1: Computation of Fixed Point for $t = 1$, where $LHS = H(x_n, x_{n+1}) + F(M(x_n, x_{n+1}, t))$.

However, varying the value of t influences the number of iterations required to approach a fixed point. As illustrated in Figure 2, increasing the value of t reduces the number of iterations required to approximate the fixed point $x = 0$.

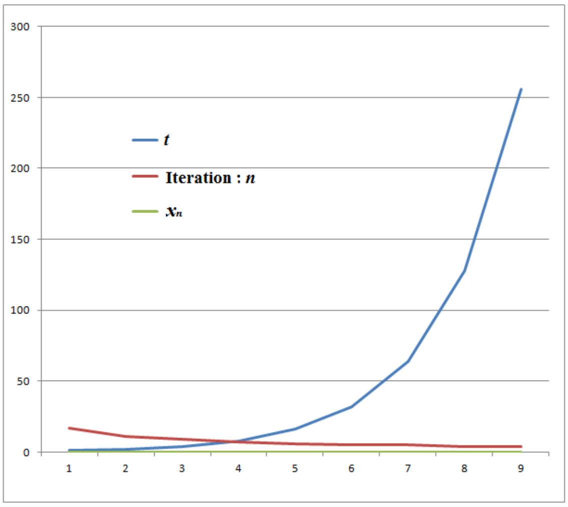

Figure 2: Effect of t on the Number of Iterations n.

Example 2 Let $X = C[0, 1]$ be a fuzzy metric space with $a * b = ab$ for all $a, b \in \mathbb{I}$ and fuzzy metric

$$M(f,g,t) = \exp\left(\frac{-\|f-g\|_{\infty}}{t}\right)$$

for all $f, g \in X$ and $t > 0$. It is clear that $(X, M, *)$ is a fuzzy metric space, to verify its completeness, let us consider the sequence of functions $\left\{ f_n(x) = x + \frac{1}{n}\sin(\pi x) \right\} \subset X$ and a function $f(x) = x$ in X.

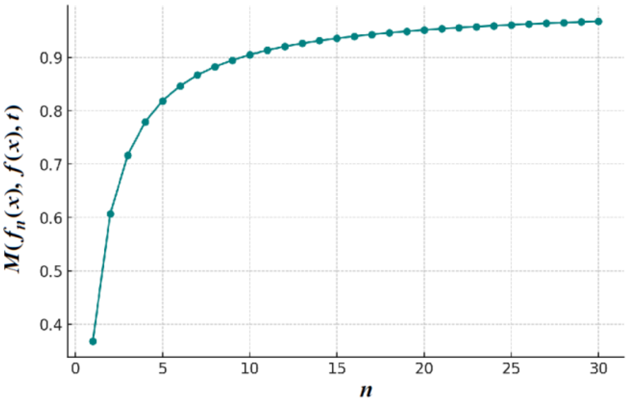

Figure 3: Convergence of $M(f_n(x), f(x), t)$ as $n \to \infty$ for $t = 1$.

From the graph shown in Figure 3, it is clear that $(X, M, *)$ is a complete fuzzy metric space.

Now, for any $g \in X$ define $H(f,g) = \frac{1}{2}\|f-g\|_{\infty}$ and for any $q \in (0, 1)$ define $F(p) = \ln\left(\frac{p}{1-p}\right)$. It is easy to see that, for any $x \in \mathbb{I}$, $H(f, f) = 0$ and a sequence of functions $\{f_n\} \subset X$, $\lim_{n\to\infty} H(f_n, f_{n+1}) = 0$. Also, we have $\lim_{p\to 0^+} F(p) = -\infty$, and $\lim_{p\to 1^-} F(p) = \infty$. Now, define $T : X \times X \to X$,

$$T(f)(x) = \int_0^1 xyf(y)dy \tag{8}$$

for all $f, g \in X$ and $x, y \in \mathbb{I}$. Let $g(x) = \sin x$ in X, then we have,

$$T(f)(x) = \int_0^1 xyf(y)dy = \int_0^1 xy^2 dy = \frac{x}{3},$$

$$T(g)(x) = \int_0^1 xyg(y)dy = \int_0^1 xy\sin y\, dy = x(\sin 1 - \cos 1).$$

And $\|f-g\|_{\infty} = \sup_{x\in I} |T(f)(x) - T(g)(x)| = \left|\frac{1}{3} - \sin 1 + \cos 1\right| \approx 0.0322.$ For $t = 1$, we have

$$M(f,g,t) = \exp\left(-\|f-g\|_{\infty}\right) \approx 0.8534,$$

$$M(T(f),T(g),t) = \exp\left(-\|T(f)-T(g)\|_{\infty}\right) \approx 0.9683,$$

$$H\left(f(x),g(x)\right) = \frac{1}{2}\|f-g\|_{\infty} = 0.07925,$$

$$H\left(T(f)(x),T(g)(x)\right) = \frac{1}{2}\|T(f)-T(g)\|_{\infty} = 0.0161.$$

Now, from contraction (1), we obtain

$$F\,(M\,(T\,(f\,)(x),\ T\,(g)(x),\ t)) \approx 3.42 > 1.84 \approx H(f\,(x),\ g(x)) + F\,(M\,(f\,(x),\ g(x),\ t)).$$

Hence, it can be concluded that for any $f, g \in X$, T is a $\mathscr{H}$– $\mathscr{F}$– contractive, and hence from Theorem 1, T has a unique fixed point in X.

Let us consider $h^* \in X$ is a fixed point of T, that is,

$$T\,(h^*)(x) = h^*(x) \qquad \forall\, x \in \mathbb{I}. \tag{9}$$

to find h^* take any function h from X. By the definition of T in the equation (8), we construct a sequence of functions as follows;

$$T(h)(x) = \int_0^1 xyh(y)dy = x\left(\int_0^1 yh(y)dy\right) = cx = h_1(x)$$

$$T(h_1)(x) = \int_0^1 xyh_1(y)dy = x\left(\int_0^1 yh_1(y)dy\right) = \frac{cx}{3} = h_2(x)$$

$$T(h_2)(x) = \int_0^1 xyh_2(y)dy = x\left(\int_0^1 yh_2(y)dy\right) = \frac{cx}{3^2} = h_3(x)$$

$$T(h_3)(x) = \int_0^1 xyh_3(y)dy = x\left(\int_0^1 yh_3(y)dy\right) = \frac{cx}{3^3} = h_4(x)$$

$$\vdots$$

$$T(h_n)(x) = \int_0^1 xyh_n(y)dy = x\left(\int_0^1 yh_n(y)dy\right) = \frac{cx}{3^n} = h_{n+1}(x)$$

where $c = \int_0^1 yh(y)dy$. This implies that, $\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} \frac{cx}{3^n - 1} = 0,$ and

$$\lim_{n\to\infty} T(h_n)(x) = \lim_{n\to\infty} h_{n+1}(x)$$

$$T\left(\lim_{n\to\infty} h_n(x)\right) = \lim_{n\to\infty} \frac{cx}{3^n} \;\Rightarrow\; T(0) = 0.$$

Hence from the equation (9), the zero function $h^*(x) = 0$ for all $x \in \mathbb{I}$, is a unique fixed point of T.

Remark: In the above Example 2, consider the evaluation $x = 0$. we observe that $T(h)(0) = \int_0^1 0\,yh(y)dy = 0,$ which implies that, any function $h \in X$ satisfying $h(0) = 0$ also fulfills the fixed point condition at the specific point, that is $T\,(h)(0) = h(0)$. For instance, the non-trivial function $h(x) = \sin x \in X$, satisfies this condition since $h(0) = 0$, and consequently $T\,(h)(0) = 0 = h(0)$.

However, the point-wise existence of the fixed point condition does not ensure that T admits a global fixed point on X. This highlights the limitations of point-wise analysis and motivates the use of an iterative contraction framework, where the sequence $\{T^n(f)\}$ converges uniformly to a unique fixed point in X, providing a more rigorous foundation for the existence and uniqueness results.

Definition 5. Let $(X, M, *)$ be a complete fuzzy metric space. For any $F \in \mathscr{F}$, a self mapping $T{:}X \rightarrow X$ is said to be iterated $\mathscr{H}$– $\mathscr{F}$–contractive if there exists a function $H \in \mathscr{H}$ and $N \in \mathbb{N}$ such that

$$F\,(M\,(T^n x,\, T^n y,\, t)) \geq H(x, y) + F\,(M\,(x, y, t)) \tag{10}$$

for all $x, y \in X$ and $n \geq N$.

Corollary 1. Let $(X, M, *)$ be a complete fuzzy metric space. If $T : X \rightarrow X$ is an iterated $\mathscr{H}$– $\mathscr{F}$–contractive mapping, then T has a unique fixed point in X.

4. Conclusion

We introduced the concept of $\mathscr{H}$–$\mathscr{F}$–contractive mappings and proved the fixed point theorem in complete fuzzy metric spaces. The result generalizes the existing work and provides a versatile framework for further developments in fuzzy fixed point theory. Future work may involve exploring multivalued mappings, partial fuzzy metrics, and applications of control theory.

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