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Some Common Fixed Point Theorems for Weakly Compatible Mappings in Rectangular S-Metric Spaces

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Abstract:
This paper explores the existence of common fixed points for weakly compatible mappings in the framework of rectangular S-metric spaces, a generalization of traditional S-metric spaces. By introducing suitable contractive conditions, we establish new common fixed point theorems that unify and extend various known results in the field of fixed point theory. These theorems provide broader applicability and deeper insights into the behaviour of such mappings under generalized metric conditions. The developed results not only contribute to the theoretical understanding but also enhance the scope of fixed point analysis in more generalized settings. To illustrate the validity and applicability of the main results, we provide several relevant examples. These examples serve to clarify the theoretical findings and demonstrate how the established theorems can be effectively applied in practice. Overall, the study offers a meaningful advancement in the general theory of fixed points within the context of rectangular S-metric spaces.

Keywords: S-metric space; rectangular S-metric space; Common fixed point.

1. Introduction

Fixed-point theory finds applications in numerous disciplines. Within this context, coincidence points, which generalize fixed points, hold significant importance. Jungck [5] was the first to introduce the concept of commuting mappings and to define common fixed points in metric spaces. Although commuting mappings possess strong structural properties, their stringent conditions restrict their applicability. To overcome this limitation, Sessa [15] proposed the concept of weakly commuting mappings, offering a more flexible alternative. Expanding on this notion, Jungck [6] introduced compatible mappings, which further generalized weak commutatively. He showed that every weakly commuting pair is compatible, although the converse is not necessarily true. Later, Jungck [7, 8] introduced the concept of weak compatibility, defining a pair of self-maps as weakly compatible if they commute at their coincidence points.

Building upon this foundational work, several researchers have investigated coincidence points for various types of mappings in metric spaces, where distances are traditionally measured between pairs of points. While effective in many settings, this pairwise approach can be limiting in more complex scenarios. To address such limitations, Sedghi et al. [13] introduced the concept of S-metric spaces, where distances are defined over triplets of points, thereby broadening the scope of fixed-point theory. Abbas and Jungck [2] established results on coincidence and common fixed points under contractive conditions in cone metric spaces, while Abbas and Rhoades [3] further generalized fixed-point theorems by eliminating the requirement of commutatively. Saluja [12] extended fixed-point results for weak contractions within complete S-metric spaces.

In this paper, we investigate coincidence and common fixed points for weakly compatible mappings satisfying weak contraction conditions in complete S-metric spaces. The theoretical results are illustrated with relevant examples.

2. Methodology

We begin by introducing the necessary definitions and properties of rectangular S-metric spaces and weakly compatible mappings. Using suitable contractive conditions, we formulate and prove new common fixed point theorems within this generalized framework. The results are supported by illustrative examples that verify the applicability of the proposed theorems. This approach extends existing results in fixed point theory to a broader class of metric spaces.

3. Preliminaries

In this section, we present the basic definitions and foundational concepts necessary for the development of our main results.

Definition 3.1 [14] Let X be a non empty set and $S: X^3 \to \mathbb{R}^+$ be a function satisfying the properties given below;

• $S(x, y, z) = 0$ if and only if $x = y = z$

• $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $a, x, y, z \in X$ (termed as rectangle inequality).

Then, the pair (X, S) is named as a S-metric space.

Definition 3.2 [1] Let X be a non void set and $S: X^3 \to \mathbb{R}^+$ be a function satisfying the conditions listed below:

• $S(x, y, z) = 0$ if and only if $x = y = z$

• $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z \in X$ and all distinct points $a \in X - \{x, y, z\}$. Then, the pair (X, S) is referred as a rectangular S-metric space.

Definition 3.3 [14] Let (X, S) be an S-metric space and $A \subset X$.

• A sequence $\{x_n\}$ in X is converge to x if $S(x_n, x_n, x) \to 0 \; as \; n \to \infty$. In other words, for every $\epsilon > 0$ there exists $n_0 \in N$ such that for all $n \geq n_0, S(x_n, x_n, x) < \epsilon$. In this case, we write $lim_{n\to\infty} x_n = x$ and we say that x is the limit of $\{x_n\}$ in X.

• A sequence $\{x_n\}$ in X is reffered as Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in N$ such that for each $n, m \geq n_0, S(x_n, x_n, x_m) < \epsilon$.

• The S-metric space (X, S) is said to be complete if every Cauchy sequence in X converges to a limit in X.

Definition 3.4 [12] Let (X,S) be a S-metric space. A mapping $T: X \to X$ is said to be a weak contraction on X if there exists a function $\psi: [0,\infty) \to [0,\infty)$ with $\psi(t) = 0$ if and only if $t = 0$ that satisfies

$$S(Tx,Tx,Ty) \leq S(x,x,y) - \delta\psi(S(x,x,y))$$

for all $x, y \in X$, where $0 \leq \delta < 1$.

Example 3.5 Let $X = R$ and $S: X \times X \times X \to R^{+}$ be a function such that

$$S(x,x,y) = \begin{cases} 9x^2 + y^2 & \text{if} x \neq y, \\ 0 & \text{if} x = y, \end{cases} \tag{3.1}$$

for all $x, y \in X$. Then (X,S) becomes a S-metric space.

Let $f: X \to X$ is defined as $f(x) = \frac{x}{9}$ and $\psi(t) = 80t$ for all $t \geq 0$, where $\psi: [0,\infty) \to [0,\infty)$ is non decreasing continuous function. Then

$$\begin{aligned} S(fx,fx,fy) &= S\left(\frac{x}{9},\frac{x}{9},\frac{y}{9}\right) \\ &= 9\left(\frac{x}{9}\right)^2 + \left(\frac{y}{9}\right)^2 \\ &= 9\frac{x^2}{81} + \frac{y^2}{81} \\ &= 9x^2 + y^2 - \frac{80}{81}(9x^2 + y^2) \\ &= S(x,x,y) - \frac{1}{81}\psi(S(x,x,y)). \end{aligned}$$

Thus, we found that f is a weak contraction on X. Since, it fulfills the requirements of Definition (3.4) for $\delta = \frac{1}{81}$.

Proposition 3.6 [2] If weakly compatible functions f and g have exactly only one point of coincidence $w = fx = gx$ on X, then w is the unique common fixed point of f and g.

Lemma 3.7 [14] A condition $S(x,x,y) = S(y,y,x)$ holds $\forall x, y \in X$ in S-metric space (X,S).

Lemma 3.8 [14] Let (X,S) be an S-metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X converging to x and y respectively, that is, $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $S(x_n,x_n,y_n) \to S(x,x,y)$ as $n \to \infty$.

Lemma 3.9 [14] Let (X,S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then the limit x is unique.

Lemma 3.10 [14] Let (X,S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then $\{x_n\}$ is a Cauchy sequence.

4. Main results

In this section, we present new common fixed point theorems for weakly compatible mappings defined on complete rectangular S-metric spaces. These results are obtained by imposing specific contractive conditions that ensure the existence and uniqueness of common fixed points.

Theorem 4.1 Let f and g be a self mapping on complete S-metric space (X, S). Assume that f and g satisfies the following restrictions,

$$\psi(S(fx, fx, fy), S(fy, fy, fx)) \leq q\psi(S(gx, gx, gy), S(gy, gy, gx)),$$

where $0 < q < 1$ and $\psi: [0, \infty)^2 \to [0, \infty)^2$ is a continuous function on $[0, \infty)^2$ with $\psi(a, b) = 0$ if and only if $a = 0 = b$. Also,

- $f(X) \subseteq g(X)$,
- If $g(X)$ is complete.

Then f and g have a unique coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Let x_0 be any point in X. Since $fx_0 \in f(X)$ and $f(X) \subseteq g(X)$, there exists a point x_1 in X such that $fx_0 = gx_1$. Clearly $x_1 \in X$ and again, by inclusion $fx_1 \in f(X)$ there exists x_2 in X such that $fx_1 = gx_2$. Proceeding in this way, we generate a sequence $\{x_n\}$ in X, such that each $x_{n+1} \in X$ satisfies $fx_n = gx_{n+1}$, $\forall n$.

Now, consider the following expression:

$$\begin{aligned}
&\psi\big(S(gx_{n+1}, gx_{n+1}, gx_{n+2}), S(gx_{n+2}, gx_{n+2}, gx_{n+1})\big) \\
&\quad = \psi(S(fx_n, fx_n, fx_{n+1}), S(fx_{n+1}, fx_{n+1}, fx_n)) \\
&\quad \leq q\psi(S(gx_n, gx_n, gx_{n+1}), S(gx_{n+1}, gx_{n+1}, gx_n)) \\
&\quad \leq \ q^2\psi(S(gx_{n-1}, gx_{n-1}, gx_n), S(gx_n, gx_n, gx_{n-1})).
\end{aligned}$$

By continuing this process inductively, we obtain:

$$\psi(S(gx_{n+1}, gx_{n+1}, gx_{n+2}), S(gx_{n+2}, gx_{n+2}, gx_{n+1})) \leq \psi(S(gx_0, gx_0, gx_1), S(gx_1, gx_1, gx_0)).$$

Since $0 < q < 1$ taking the limit as $n \to \infty$, gives

$$\psi(S(gx_{n+1}, gx_{n+1}, gx_{n+2}), S(gx_{n+2}, gx_{n+2}, gx_{n+1})) \to 0.$$

Due to continuity of ψ, it follows that

$$0 = \lim_{n\to\infty} \psi(S(gx_{n+1}, gx_{n+1}, gx_{n+2}), S(gx_{n+2}, gx_{n+2}, gx_{n+1}))$$

$$0 = \psi(\lim_{n\to\infty} S(gx_{n+1}, gx_{n+1}, gx_{n+2}), \lim_{n\to\infty} S(gx_{n+2}, gx_{n+2}, gx_{n+1})).$$

Thus, by using the property of ψ, we conclude

$$\lim_{n\to\infty} S(gx_{n+1}, gx_{n+1}, gx_{n+2}) = \lim_{n\to\infty} S(gx_{n+2}, gx_{n+2}, gx_{n+1}) = 0.$$

Now, we will prove that the sequence $\{gx_n\}$ is a Cauchy sequence. Assume, for contradiction, that $\{gx\_n\}$ is not a Cauchy sequence. Then there is $\epsilon > 0$ and subsequences $\{gx_{n(k)}\}$ and $\{gx_{m(k)}\}$ such that, $\forall\ k \in N$. Then, we have

$$n(k) > m(k) > k, S(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) \geq \epsilon \text{ and } S(gx_{n(k)}, gx_{n(k)}, gx_{m(k)-1}) < \epsilon.$$

Hence,

$$\epsilon \leq S(gx_{n(k)}, gx_{n(k)}, gx_{m(k)})$$
$$= S(gx_{m(k)}, gx_{m(k)}, gx_{n(k)})$$
$$\leq S(gx_{m(k)}, gx_{m(k)}, gx_{m(k)-1}) + S(gx_{m(k)}, gx_{m(k)}, gx_{m(k)-1})$$
$$+S(gx_{n(k)}, gx_{n(k)}, gx_{m(k)-1})$$
$$< 0 + 0 + \epsilon.$$

Which is leads to a contradiction. So,

$$S(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) = \epsilon.$$

Additionally,

$$S(gx_{n(k)-1}, gx_{n(k)-1}, gx_{m(k)-1}) \leq S(gx_{n(k)-1}, gx_{n(k)-1}, gx_{n(k)}) + S(gx_{n(k)-1}, gx_{n(k)-1}, gx_{n(k)})$$
$$+S(gx_{m(k)-1}, gx_{m(k)-1}, gx_{n(k)})$$
$$= 2S(gx_{n(k)-1}, gx_{n(k)-1}, gx_{n(k)}) + S(gx_{n(k)}, gx_{n(k)}, gx_{m(k)-1})$$
$$< 2(0) + \epsilon = \epsilon.$$

Thus, $S(gx_{n(k)-1}, gx_{n(k)-1}, gx_{m(k)-1}) < \epsilon$.

Consequently, $S(gx_{m(k)-1}, gx_{m(k)-1}, gx_{n(k)-1}) = S(gx_{n(k)-1}, gx_{n(k)-1}, gx_{m(k)-1}) < \epsilon$.

Now,

$$\psi(S(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}), S(gx_{m(k)}, gx_{m(k)}, gx_{n(k)}))$$
$$= \psi(S(fx_{n(k)-1}, fx_{n(k)-1}, fx_{m(k)-1}), S(fx_{m(k)-1}, fx_{m(k)-1}, fx_{n(k)-1}))$$
$$\leq q\psi(S(gx_{n(k)-1}, gx_{n(k)-1}, gx_{m(k)-1}), S(gx_{m(k)-1}, gx_{m(k)-1}, gx_{n(k)-1}))$$
$$< q\psi(\epsilon, \epsilon),$$

which gives $\psi(\epsilon, \epsilon) < q\psi(\epsilon, \epsilon)$.

Since $0 < q < 1$, this inequality is only possible if $\epsilon = 0\psi(\epsilon, \epsilon) = 0$.

By the property of ψ we conclude

$$S(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) = 0.$$

This leads to a contradiction. Thus we obtain, $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, there exist a point q in $g(X)$ such that $gx_n \to q$, as $n \to \infty$. Thus, there exists a point p in X with $gp = q$. Hence,

$$\psi(S(gx_{n+1}, gx_{n+1}, fp), S(fp, fp, gx_{n+1})) = \psi(S(fx_n, fx_n, fp), S(fp, fp, fx_n))$$
$$\leq q\psi(S(gx_n, gx_n, gp), S(gp, gp, gx_n)),$$

taking the limit as $n \to \infty$, we get

$$\psi(S(q,q,fp),S(fp,fp,q)) \leq q\psi(S(q,q,q),S(q,q,q)) = q\psi(0,0).$$

So,

$$\psi(S(q,q,fp),S(fp,fp,q)) = 0fp = q.$$

Thus, $fp = gp = q$ showing that q is the point of coincidence of f and g.

To show uniqueness, suppose another point $w \in X$ such that $fw = gw = w$. Now

$$\begin{aligned}\psi(S(q,q,w),S(w,w,q)) &= \psi(S(fp,fp,fw),S(fw,fw,fp)) \\ &\leq \; q\psi(S(gp,gp,gw),S(gw,gw,gp)) \\ &= \; q\psi(S(q,q,w),S(w,w,q)).\end{aligned}$$

Again, since $0 < q < 1$, this yields;

$$\psi(S(q,q,w),S(w,w,q)) = 0q = w.$$

Therefore f and g have a unique point of coincidence. Since, f and g are weakly compatible, by Proposition (3.6), it follows that they have a unique common fixed point in X.

Theorem 4.2 Let (X,S) be a complete rectangular S-metric space and $f,g: X \to X$ be two mappings such that for all $x,y \in X$,

$$S(fx,fx,fy) \leq \alpha S(gx,gx,gy) + \beta[S(gx,gx,fx) + S(gy,gy,fy)],$$

where $\alpha, \beta > 0$ and $\alpha + 2\beta < 1$.

Assume the following conditions hold;

- $f(X) \subseteq g(X)$,
- If $g(X)$ is complete.

Then f and g have a unique coincidence point in X (i.e. there exists unique $z \in X$ such that $fz = gz$). Moreover, if f and g are weakly compatible (i.e., they commute at their coincidence point), then f and g have a unique common fixed point in X (that is, there exists a unique $p \in X$ such that $fp = gp = p$).

Proof. Let x_0 be any point in X. Since $fx_0 \in f(X)$ and $f(X) \subseteq g(X)$, there exists a point x_1 in X such that $fx_0 = gx_1$. As $x_1 \in X$, we have $fx_1 \in f(X)$, so there exists x_2 in X such that $fx_1 = gx_2$. Continuing in this way, we construct a sequence $\{x_n\}$ in X such that, $fx_n = gx_{n+1}$ for all n, where $x_{n+1} \in X$

Now, consider

$$\begin{aligned}&S(gx_{n+1},gx_{n+1},gx_n) = S(fx_n,fx_n,fx_{n-1}) \\ &\leq \alpha S(gx_n,gx_n,gx_{n-1}) + \beta[S(gx_n,gx_n,fx_n) + S(gx_{n-1},gx_{n-1},fx_{n-1})] \\ &= \alpha S(gx_n,gx_n,gx_{n-1}) + \beta[S(gx_n,gx_n,gx_{n+1}) + S(gx_{n-1},gx_{n-1},gx_n)]\end{aligned}$$

$$= \alpha S(gx_n, gx_n, gx_{n-1}) + \beta[S(gx_{n+1}, gx_{n+1}, gx_n) + S(gx_n, gx_n, gx_{n-1})]$$

$$= (\alpha + \beta)S(gx_n, gx_n, gx_{n-1}) + \beta S(gx_{n+1}, gx_{n+1}, gx_n)$$

Rewriting;

$$(1 - \beta)S(gx_{n+1}, gx_{n+1}, gx_n) \leq (\alpha + \beta)S(gx_n, gx_n, gx_{n-1})$$

This yields;

$$S(gx_{n+1}, gx_{n+1}, gx_n) \leq \frac{\alpha+\beta}{1-\beta} S(gx_n, gx_n, gx_{n-1})$$

$$= \ hS(gx_n, gx_n, gx_{n-1})$$

$$\leq \ h^n S(gx_1, gx_1, gx_0),$$

where, $h = \frac{\alpha+\beta}{1-\beta} < 1$ by assumption.

By recursion:

$$S(gx_{n+1}, gx_{n+1}, gx_n) \leq \ h^n S(gx_1, gx_1, gx_0),$$

Let $z = S(gx_1, gx_1, gx_0)$. Then, for $m < n$, we have

$$S(gx_n, gx_n, gx_m) \leq \ S(gx_n, gx_n, gx_{n-1}) + S(gx_n, gx_n, gx_{n-1}) + S(gx_m, gx_m, gx_{n-1})$$

$$\leq \ h^{n-1}S(gx_1, gx_1, gx_0) + h^{n-1}S(gx_1, gx_1, gx_0) + S(gx_m, gx_m, gx_{n-1})$$

$$= \ 2h^{n-1}S(gx_1, gx_1, gx_0) + S(gx_m, gx_m, gx_{n-1})$$

$$\leq \ 2h^{n-1}z + [S(gx_m, gx_m, gx_{n-2}) + S(gx_m, gx_m, gx_{n-2})$$

$$+ S(gx_{n-1}, gx_{n-1}, gx_{n-2})]$$

$$= \ 2h^{n-1}z + [2S(gx_m, gx_m, gx_{n-2}) + S(gx_{n-1}, gx_{n-1}, gx_{n-2})]$$

$$\leq \ 2h^{n-1}z + [2S(gx_m, gx_m, gx_{n-2}) + h^{n-2}S(gx_1, gx_1, gx_0)]$$

$$= \ 2h^{n-1}z + h^{n-2}z + 2S(gx_m, gx_m, gx_{n-2})$$

$$\leq \ 2h^{n-1}z + h^{n-2}z + 2[2S(gx_m, gx_m, gx_{n-3}) + S(gx_{n-2}, gx_{n-2}, gx_{n-3})]$$

$$\leq \ 2h^{n-1}z + h^{n-2}z + 4S(gx_m, gx_m, gx_{n-3}) + 2h^{n-3}z$$

$$\leq \ 2h^{n-1}z + h^{n-2}z + 2h^{n-3}z + 4[2S(gx_m, gx_m, gx_{n-4})$$

$$+S(gx_{n-3}, gx_{n-3}, gx_{n-4})]$$

$$\leq \ 2h^{n-1}z + h^{n-2}z + 2h^{n-3}z + 8S(gx_m, gx_m, gx_{n-4}) + 4h^{n-4}z$$

$$= \ 2h^{n-1}z + h^{n-2}z + 2h^{n-3}z + 4h^{n-4}z + \cdots$$

$$= \ 2h^{n-1}z + h^{n-2}z\left(\ 1 + \frac{2}{h} + \frac{4}{h^2} + \frac{8}{h^3} + \cdots \right)$$

$$= \ 2h^{n-1}z + h^{n-2}z\left(\ 1 + \left(\frac{2}{h}\right) \ + \left(\frac{2}{h}\right)^2 + \left(\frac{2}{h}\right)^3 + \cdots \right)$$

$$= \ 2h^{n-1}z + h^{n-2}z\left(\ 1 - \frac{2}{h}\right)^{-1},$$

Since $0 < h < 1$, by taking limit as $n \to \infty$, we obtain,

$$\lim_{n\to\infty} S(gx_n, gx_n, gx_m) = 0.$$

Hence, we obtain that the sequence $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Given that $g(X)$ is complete, there exist a point q in $g(X)$ such that $gx_n \to q$, as $n \to \infty$. Since, $q \in g(X)$ there exists a point p in X with $gp = q$. Now,

$$S(fx_n, fx_n, fp) \leq \alpha S(gx_n, gx_n, gp) + \beta[S(gx_n, gx_n, fx_n) + S(gp, gp, fp)]$$

$$= \alpha S(gx_n, gx_n, gp) + \beta[S(gx_n, gx_n, gx_{n+1}) + S(gp, gp, fp)].$$

Taking the limit as $n \to \infty$ and using continuity and convergence

$$S(q, q, fp) \leq \alpha S(q, q, q) + \beta[S(q, q, q) + S(q, q, fp)]$$

Which implies

$$S(q, q, fp) \leq \beta S(q, q, fp).$$

Since, $0 < \beta < 1$, the only solution is $S(q, q, fp) = 0, fp = q$. But also, $gp = q$ so $fp = gp = q$ showing that, q is the point of coincidence of f and g. To prove uniqueness, suppose $w \in X$ is another coincidence point, i.e., $fw = gw = w$. Then

$$\begin{aligned} S(q, q, w) &= S(fp, fp, fw) \\ &\leq \alpha S(gp, gp, gw) + \beta[S(gp, gp, fp) + S(gw, gw, fw)] \\ &= \alpha S(q, q, w) + \beta[S(q, q, q) + S(w, w, w)] \\ &= \alpha S(q, q, w). \end{aligned}$$

Since, $0 < \alpha < 1$, we can conclude that $S(q, q, w) = 0 q = w$. Hence, the coincidence point is unique. Finally since, f and g are weakly compatible, by Proposition (3.6), they have a unique common fixed point in X.

Theorem 4.3 Let (X, S) be a complete rectangular S-metric space, and let $f, g: X \to X$ be two self-maps satisfying the inequality

$$S(fx, fx, gy) \leq \alpha S(x, x, y) + \beta[S(x, x, fx) + S(y, y, gy)] + \gamma[S(x, x, gy) + S(y, y, fx)]$$

for all $x, y \in X$, where $\alpha, \beta, \gamma > 0$ and $\alpha + 2\beta + 3\gamma < 1$.

Then f and g have a unique common fixed point in X. Moreover, every fixed point of f is also a fixed point of g, and conversely.

Proof. Let x_0 be an arbitrary starting point of X, and define the sequence $\{x_n\}$ by

$$x_{2n+1} = fx_{2n}, x_{2n+2} = gx_{2n+1}, n = 0,1,2, \cdots$$

Now,

$$S(x_{2n+1},x_{2n+1},x_{2n+2})=S(fx_{2n},fx_{2n},gx_{2n+1})$$
$$\le \alpha S(x_{2n},x_{2n},x_{2n+1})+\beta[S(x_{2n},x_{2n},fx_{2n})+S(x_{2n+1},x_{2n+1},gx_{2n+1})]$$
$$+\gamma[S(x_{2n},x_{2n},gx_{2n+1})+S(x_{2n+1},x_{2n+1},fx_{2n})]$$
$$=\alpha S(x_{2n},x_{2n},x_{2n+1})+\beta[S(x_{2n},x_{2n},x_{2n+1})+S(x_{2n+1},x_{2n+1},x_{2n+2})]$$
$$+\gamma[S(x_{2n},x_{2n},x_{2n+2})+S(x_{2n+1},x_{2n+1},x_{2n+1})]$$
$$=\alpha S(x_{2n},x_{2n},x_{2n+1})+\beta[S(x_{2n},x_{2n},x_{2n+1})+S(x_{2n+1},x_{2n+1},x_{2n+2})]$$
$$+\gamma[S(x_{2n},x_{2n},x_{2n+1})+S(x_{2n},x_{2n},x_{2n+1})+S(x_{2n+2},x_{2n+2},x_{2n+1})]$$
$$=\alpha S(x_{2n},x_{2n},x_{2n+1})+\beta[S(x_{2n},x_{2n},x_{2n+1})+S(x_{2n+1},x_{2n+1},x_{2n+2})]$$
$$+\gamma[2S(x_{2n},x_{2n},x_{2n+1})+S(x_{2n+1},x_{2n+1},x_{2n+2})]$$
$$=(\alpha+\beta+2\gamma)S(x_{2n},x_{2n},x_{2n+1})+(\beta+\gamma)S(x_{2n+1},x_{2n+1},x_{2n+2})$$

Rearranging gives

$$(1-(\beta+\gamma))S(x_{2n+1},x_{2n+1},x_{2n+2})\le(\alpha+\beta+2\gamma)S(x_{2n},x_{2n},x_{2n+1})$$
$$\le\frac{(\alpha+\beta+2\gamma)}{(1-(\beta+\gamma))}S(x_{2n},x_{2n},x_{2n+1})$$

By taking $\delta=\frac{(\alpha+\beta+2\gamma)}{(1-(\beta+\gamma))}<1$, We get

$$S(x_{2n+1},x_{2n+1},x_{2n+2})\le\delta S(x_{2n},x_{2n},x_{2n+1})$$

Applying recursively,

$$S(x_{2n+1},x_{2n+1},x_{2n+2})\le\delta^{2n+1}S(x_0,x_0,x_1),$$

For $m,n\in\mathbb{N}$ with $m<n$ and some $N\in\mathbb{N}$, we estimate

$$S(x_n,x_n,x_m)\le\ S(x_n,x_n,x_{n+1})+S(x_n,x_n,x_{n+1})+S(x_m,x_m,x_{n+1})$$
$$=2S(x_n,x_n,x_{n+1})+S(x_m,x_m,x_{n+1})$$

By recursively applying the triangle-like inequality, we proceed as,

$$S(x_n,x_n,x_m)\le2\delta^nS(x_0,x_0,x_1)+[S(x_m,x_m,x_{n+2})+S(x_m,x_m,x_{n+2})+S(x_{n+1},x_{n+1},x_{n+2})]$$

By taking $S(gx_0,gx_0,gx_1)=z$, this yields

$$S(x_n,x_n,x_m)\le2\delta^nz+[2S(x_m,x_m,x_{n+2})+S(x_{n+1},x_{n+1},x_{n+2})]$$
$$\le2\delta^nz+[2S(x_m,x_m,x_{n+2})+\delta^{n+1}z]$$
$$\le2\delta^nz+\delta^{n+1}z+2[S(x_m,x_m,x_{n+3})+S(x_m,x_m,x_{n+3})+S(x_{n+2},x_{n+2},x_{n+3})]$$
$$\le2\delta^nz+\delta^{n+1}z+2[2S(x_m,x_m,x_{n+3})+\delta^{n+2}z]$$

$$\leq 2\delta^n z + \delta^{n+1} z + 2\delta^{n+2} z + 4[S(x_m, x_m, x_{n+4}) + S(x_m, x_m, x_{n+4})$$

$$\leq 2\delta^n z + \delta^{n+1} z + 2\delta^{n+2} z + 4[2S(x_m, x_m, x_{n+4}) + \delta^{n+3} z]$$

$$+S(x_{n+3}, x_{n+3}, x_{n+4})]$$

$$\leq 2\delta^n z + \delta^{n+1} z + 2\delta^{n+2} z + 4\delta^{n+3} z + 8\delta^{n+4} z + \cdots$$

$$= 2\delta^n z + \delta^{n+1} z(1 + 2\delta + 4\delta^2 + 8\delta^3 + \cdots)$$

$$= 2\delta^n z + \delta^{n+1} z(\ 1 + (2\delta)\ + (2\delta)^2 + (2\delta)^3 + \cdots)$$

$$= 2\delta^n z + \delta^{n+1} z\left(\frac{1}{1-2\delta}\right),$$

As $\delta < 1$, it follows that

$$\lim_{n\to\infty} S(x_n, x_n, x_m) = 0.$$

Thus, the sequence $\{x_n\}$ is a Cauchy sequence in X. Completeness of X implies that there exists a point p in X such that $x_n \to p$ as $n \to \infty$.

To show that p is a fixed point of g, we consider

$$S(x_n, x_n, gp) \leq S(x_n, x_n, x_{n+1}) + S(x_n, x_n, x_{n+1}) + S(gp, gp, x_{n+1})$$

$$= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, gp)$$

$$\leq 2\delta^n z + [S(x_{n+1}, x_{n+1}, x_{n+2}) + S(x_{n+1}, x_{n+1}, x_{n+2}) + S(gp, gp, x_{n+2})]$$

$$= 2\delta^n z + [2S(x_{n+1}, x_{n+1}, x_{n+2}) + S(gp, gp, x_{n+2})]$$

$$\leq 2\delta^n z + [2\delta^{n+1} z + S(x_{n+2}, x_{n+2}, gp)]$$

$$\leq 2\delta^n z + 2\delta^{n+1} z + [S(x_{n+2}, x_{n+2}, x_{n+3}) + S(x_{n+2}, x_{n+2}, x_{n+3}) + S(gp, gp, x_{n+3})]$$

$$= 2\delta^n z + 2\delta^{n+1} z + [2S(x_{n+2}, x_{n+2}, x_{n+3}) + S(gp, gp, x_{n+3})]$$

$$\leq 2\delta^n z + 2\delta^{n+1} z + 2\delta^{n+2} z + S(x_{n+3}, x_{n+3}, gp)$$

$$\leq 2\delta^n z + 2\delta^{n+1} z + 2\delta^{n+2} z + [S(x_{n+3}, x_{n+3}, x_{n+4}) + S(x_{n+3}, x_{n+3}, x_{n+4})$$

$$+S(gp, gp, x_{n+4})]$$

$$= 2\delta^n z + 2\delta^{n+1} z + 2\delta^{n+2} z + [2S(x_{n+3}, x_{n+3}, x_{n+4}) + S(gp, gp, x_{n+4})]$$

$$\leq 2\delta^n z + 2\delta^{n+1} z + 2\delta^{n+2} z + 2\delta^{n+3} z + S(x_{n+4}, x_{n+4}, gp)$$

$$= 2\delta^n z + 2\delta^{n+1} z + 2\delta^{n+2} z + 2\delta^{n+3} z + \cdots$$

$$= 2\delta^n z(1 + \delta + \delta^2 + \delta^3 + \cdots)$$

Therefore, $S(x_n, x_n, gp) \leq 2\delta^n z\left(\frac{1}{1-\delta}\right)$,

where $z = S(gx_0, gx_0, gx_1)$. Since $\delta < 1$, as $n \to \infty$. Thus, $\lim_{n\to\infty} S(p, p, gp) = 0$ and $gp = p$. Also, Now to show $fp = p$, we proceed as,

$$S(fp,fp,p) = S(fp,fp,gp)$$
$$\leq \alpha S(p,p,p) + \beta[S(p,p,fp) + S(p,p,gp)] + \gamma[S(p,p,gp) + S(p,p,fp)]$$
$$= \beta[S(p,p,fp) + S(p,p,p)] + \gamma[S(p,p,p) + S(p,p,fp)]$$
$$= (\beta+\gamma)S(p,p,fp)$$
$$= (\beta+\gamma)S(fp,fp,p).$$

Since, $\beta,\gamma > 0$. This is only possible if $S(fp,fp,p) = 0$ and $fp = p$. Therefore, we can conclude that, $fp = gp = p$. Hence, f and g have a common fixed point in $p \in X$.

Uniqueness: Suppose another common fixed point $w \in X$ exists i.e. $fw = gw = w$

$$S(p,p,w) = S(fp,fp,gw)$$
$$\leq \alpha S(p,p,w) + \beta[S(p,p,fp) + S(w,w,gw)] + \gamma[S(p,p,gw) + S(w,w,fp)]$$

Substituting $fp = gp = p$ and $fw = gw = w$, we obtain

$$S(p,p,w) = \alpha S(p,p,w) + \beta[S(p,p,p) + S(w,w,w)] + \gamma[S(p,p,w) + S(w,w,p)]$$
$$= \alpha S(p,p,w) + \gamma[S(p,p,w) + S(p,p,w)]$$
$$= \alpha S(p,p,w) + 2\gamma S(p,p,w)$$
$$= (\alpha + 2\gamma)S(p,p,w).$$

Since, $\alpha,\gamma > 0$, the only possibility is $S(p,p,w) = 0$. Hence, $p = w$. Hence, the common fixed point is unique, and the proof is complete.

Corollary 4.4 Let (X,S) be a complete rectangular S-metric space and $f,g: X \to X$ be a self mappings such that

$S(fx,fx,fy) \leq hS(gx,gx,gy)$, where $h \in [0,1)$.

Also, following conditions holds

- $f(X) \subseteq g(X)$,
- If $g(X)$ is complete.

Then f and g have a unique common fixed point in X.

Proof. The given inequality is a special case of the condition in Theorem (4.2), obtained by choosing $\alpha = h$ and $\beta = 0$. Hence, the result follows directly from Theorem (4.2).

Corollary 4.5 Let (X,S) be a complete rectangular S-metric space and $f,g: X \to X$ be a self mapping satisfies the condition,

$S(fx,fx,fy) \leq q[S(gx,gx,fy) + S(gy,gy,fx)]$, where $q \in [0,\frac{1}{2})$.

Assume further that,

- $f(X) \subseteq g(X)$
- If $g(X)$ is complete subset of X.

Then there exists a unique fixed point in X for f and g.

Corollary 4.6 Let (X, S) be a complete rectangular S-metric space and $f, g: X \to X$ be a self mapping satisfies the condition,

$$S(fx, fx, fy) \leq \beta[S(gx, gx, fx) + S(gy, gy, fy)], \text{ where } q \in [0, \tfrac{1}{2}).$$

Suppose further that,

- $f(X) \subseteq g(X)$,
- If $g(X)$ is complete.

Then f and g have a unique common fixed point in X.

Proof. This inequality results from Theorem (4.2) by setting $\alpha = 0$. The conclusion then directly follows from Theorem(4.2).

Example 4.7 Let $X = \mathbb{R}$ and define the function $S: X \times X \times X \to \mathbb{R}^+$ by $S(x, y, z) = xy + z^2$ for all $x, y \in \mathbb{R}$. Then (X, S) forms a S-metric space.

Now, define the mappings $f: X \to X$ as follows $f(x) = \frac{x}{4}$ and $g(x) = \frac{x}{2}$ for all $x \in [0,1]$. We then compute

$$\begin{aligned}
S(fx, fx, fy) &= S\left(\frac{x}{4}, \frac{x}{4}, \frac{y}{4}\right) \\
&= (\tfrac{x}{4})(\tfrac{x}{4}) + \left(\tfrac{y}{4}\right)^2 \\
&= \tfrac{x^2}{16} + \tfrac{y^2}{16} \\
&= \tfrac{1}{16}(x^2 + y^2) \\
&= \tfrac{1}{4}(\tfrac{1}{4}(x^2 + y^2)) \\
&= \tfrac{1}{4}(S(gx, gx, gy)) \\
&= h\big(S(gx, gx, gy)\big), \text{ where } h = \tfrac{1}{4} < 1.
\end{aligned}$$

Thus f satisfies all the conditions of Corollary (4.4). Therefore, by applying Corollary(4.4). It follows that f and g have a unique common fixed point in X. Clearly, $0 \in X$ is this unique common fixed point.

5. Conclusion

The results presented in this paper contribute to the ongoing development of fixed point theory by extending classical results to the broader context of rectangular S-metric spaces. By introducing new common fixed point theorems for weakly compatible mappings under contractive conditions, we provide a more general framework that encompasses several existing results as special cases. The illustrative examples confirm the applicability of our theorems and highlight the usefulness of rectangular S-metric spaces in analysing complex mapping behaviours. These findings open new avenues for further research in generalized metric spaces and their applications.

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