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SOME PROPERTIES OF DIFFERENTIAL EQUATIONS OF HIGHER-ORDER q-FROBENIUS-TANGENT POLYNOMIALS

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Abstract. The classical q-Frobenius-tangent polynomials dealt within this paper contains many application in various areas. We construct new types of differential equations for q-Frobenius-tangent polynomials using q-derivatives, find some properties and several difference equations of these polynomials.

1. Introduction

In recent years, numerous researchers have explored the Bernoulli, Euler, Genocchi, Frobenius-Euler, and tangent polynomials in their classical, generalized, and unified forms to investigate their properties, relationships, and applications [5, 6, 7, 9, 10]. Building on these studies, Jackson introduced the q-Bernoulli, q-Euler, and q-Genocchi polynomials [1, 2], while Kang [6, 8] and Kang and Kim [7] examined generalized q-tangent polynomials. Kang and Khan [5] studied q-Frobenius-Euler polynomials, Nisar *et al.* [13] introduced q-Frobenius-tangent polynomials, and Ryoo and Kang [15, 16] investigated the q-differential equation forms of Euler and Genocchi polynomials. These works have uncovered numerous properties, relationships, and applications in fields such as umbral calculus, p-adic analysis, and combinatorics.

Let $\sigma \in \mathbb{R}$, $p(\psi)$ and $g(\psi)$ are continuous function, the equation of Bernoulli polynomials as

$$\frac{d\phi}{d\psi} + p(\psi)\phi - g(\psi)\phi^{\sigma} = 0, \tag{1.1}$$

Let $\sigma = 0$, we will get linear equation and it is not nonlinear equation. If $\eta = \phi^{1-\sigma}$ in (1.1), we get differential equation of Bernoulli polynomials.

$$\frac{d\eta}{d\psi} + (1-\sigma)p(\psi)\eta = (1-\sigma)g(\psi),$$

Putting $\sigma = 0$, the equation (1.1) gives the differential equation of the Frobenius-tangent polynomials as follows.

$$\frac{d}{d\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon}(\psi;\eta) + \frac{1}{1-\eta}{}_{\mathbb{F}}\mathbb{T}_{\upsilon}(\psi;\eta) + \frac{1}{1-\eta}{}_{\mathbb{F}}\mathbb{T}_{0}(\psi;\eta) - \psi^{\upsilon} = 0, \tag{1.2}$$

where ${}_{\mathbb{F}}\mathbb{T}_{\upsilon}(\psi;\eta)$ is the Frobenius-tangent polynomials are as follows

$$\sum_{\upsilon=0}^{\infty} {}_{\mathbb{F}}\mathbb{T}_{\upsilon}(\psi;\eta)\frac{\varphi^{\upsilon}}{\upsilon!} = \frac{1-\eta}{e^{(1-\eta)\varphi} - \eta}e^{\psi\varphi}. \tag{1.3}$$

The corresponding Frobenius-tangent numbers have also been derived by ${}_{\mathbb{F}}\mathbb{T}_{\upsilon}(\eta) = {}_{\mathbb{F}}\mathbb{T}_{\upsilon}(0;\eta)$.

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By the above method, the first order differential equation of q-Bernoulli differential can written as $D_q\phi + p(\psi)\phi - g(\psi)\phi^{\sigma} = 0$ in q-calculus. Again, if $\sigma = 0$, the equation (1.1) will give the first order q-differential equation of q-Frobenius-tangent polynomials as

$$D^{(1)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta) + (1-\eta)^{-1}\left({}_{\mathbb{F}}\mathbb{T}_{0,q}(\psi;\eta) + {}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)\right) - \psi^{\upsilon} = 0, \tag{1.4}$$

and D_q is called the q-derivative and ${}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)$ is the q-Frobenius-tangent polynomials.

Let $\upsilon \in \mathbb{Z}_0$ and $\eta \in \mathbb{Z}$, the q-Frobenius-tangent polynomials are defined by (see [7])

$$\frac{1-\eta}{e_q((1-\eta)\varphi) - \eta} e_q(\psi\varphi) = \sum_{\upsilon=0}^{\infty} {}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}. \tag{1.5}$$

The corresponding q-Frobenius-tangent numbers have also been derived by ${}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\eta) = {}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(0;\eta)$.
It is worthy note that if $q \to 1$ then (1.5) becomes (1.2).
The main purpose of this paper is to establish higher-order differential equations for the q-Frobenius-tangent polynomials as defined by (1.5). Building on this concept, we will apply the theory of q-calculus throughout the paper. Let us begin by introducing some definitions from q-calculus theory.

The shifted factorial $(\phi)_{\upsilon}$ in term of q-analogue is given by [3, 4]

$$(\phi;q)_0 = 1, (\phi;q)_{\upsilon} = \prod_{\sigma=0}^{\upsilon-1}(1-q^{\sigma}\phi), \upsilon \in \mathbb{N}.$$

The factorial function in q-calculus theory given by

$$[\phi]_q = \frac{1-q^{\phi}}{1-q}, q \in \mathbb{C} - \{1\}; \phi \in \mathbb{C},$$

$$[\upsilon]_q! = \prod_{\sigma=1}^{\upsilon}[\sigma]_q = [1]_q[2]_q\cdots[\upsilon]_q = \frac{(q;q)_{\upsilon}}{(1-q)^{\upsilon}}, q \neq 1; \upsilon \in \mathbb{N},$$

$$[0]_q! = 1, q \in \mathbb{C}; 0 < q < 1.$$

The definition q-binomial coefficient of Gauss $\left(\begin{array}{c}\upsilon\\ \psi\end{array}\right)_q$ is given by

$$\left(\begin{array}{c}\upsilon\\ \psi\end{array}\right)_q = \frac{[\upsilon]_q!}{[\nu]_q![\upsilon-\nu]_q!} = \frac{(q;q)_{\upsilon}}{(q;q)_{\nu}(q;q)_{\upsilon-\nu}}, \ \nu = 0, 1, \cdots, \upsilon.$$

The function $(\psi+\phi)_q^{\upsilon}$ is given by

$$(\psi+\phi)_q^{\upsilon} = \sum_{\nu=0}^{\upsilon}\left(\begin{array}{c}\upsilon\\ \nu\end{array}\right)_q q^{\nu(\nu-1)/2}\psi^{\upsilon-\nu}\phi^{\nu}, \upsilon \in \mathbb{N}_0. \tag{1.6}$$

The definition of exponential function in q-calculus theory is given by

$$e_q(\psi) = \sum_{\upsilon=0}^{\infty}\frac{\psi^{\upsilon}}{[\upsilon]_q!} = \frac{1}{((1-q)\psi;q)_{\infty}}, 0 <\mid q \mid< 1; \mid \psi \mid<\mid 1-q \mid^{-1}, \tag{1.7}$$

For $\psi \neq 0$, the definition of q-derivative $D_{q,\psi}f(\psi)$ as

$$D_{q,\psi}f(\psi) = D_qf(\psi) = \frac{f(\psi) - f(q\psi)}{(1-q)\psi}, \tag{1.8}$$

and $D_qf(0) = f^{'}(0)$.

Here the function f is differentiable at zero, and it is obvious that $D_q\psi^{\upsilon} = [\upsilon]_q\psi^{\upsilon-1}$. Let us point out that $D^{(\nu)}_{q,\psi}f(\psi)$ converges to $f^{(\nu)}(\psi)$ as q goes to 1. By(1.8), the some formulae of q-derivative.

(i)

$$D_q\left(f(\psi)g(\psi)\right) = q(\psi)D_qf(\psi) + f(q\psi)D_qg(\psi) = f(\psi)D_qg(\psi) + g(q\psi)D_qf(\psi),$$

(ii)

$$\begin{aligned} D_q\left(\frac{f(\psi)}{g(\psi)}\right) &= \frac{g(q\psi)D_qf(\psi) - f(q\psi)D_qg(\psi)}{g(\psi)g(q\psi)} \\ &= \frac{g(\psi)D_qf(\psi) - f(\psi)D_qg(\psi)}{g(\psi)g(q\psi)}, \end{aligned}$$

(iii) for any constant a and b,

$$D_q\left(af(\psi) + bg(\psi)\right) = aD_qf(\psi) + bD_qg(\psi).$$

The q-Bernoulli $B_{\upsilon,q}(\psi)$, the q-Euler $\mathbb{E}_{\upsilon,q}(\psi)$ and q-Genocchi polynomials $\mathbb{G}_{\upsilon,q}(\psi)$ are defined by (see [11, 12, 14]):

$$\frac{\varphi}{e_q(\varphi)-1}e_q(\psi\varphi) = \sum_{\upsilon=0}^{\infty}\mathbb{B}_{\upsilon,q}(\psi)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}\ (\mid\varphi\mid < 2\pi), \tag{1.9}$$

$$\frac{2}{e_q(\varphi)+1}e_q(\psi\varphi) = \sum_{\upsilon=0}^{\infty}\mathbb{E}_{\upsilon,q}(\psi)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}\ (\mid\varphi\mid < \pi), \tag{1.10}$$

$$\frac{2\varphi}{e_q(\varphi)+1}e_q(\psi\varphi) = \sum_{\upsilon=0}^{\infty}\mathbb{G}_{\upsilon,q}(\psi)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}\ (\mid\psi\mid < \pi), \tag{1.11}$$

respectively.

Clearly, we have

$$\mathbb{B}_{\upsilon,q} = \mathbb{B}_{\upsilon,q}(0), \mathbb{E}_{\upsilon,q} = \mathbb{E}_{\upsilon,q}(0), \mathbb{G}_{\upsilon,q} = \mathbb{G}_{\upsilon,q}(0).$$

The main purpose of this paper, we find some differential equation for q-analogue of Frobenius-tangent numbers and polynomials. Based on these polynomials, we construct some differential equation of these polynomials. Also, we derive differential equations associated with symmetric properties.

2. Differential equations of q-analogue of Frobenius-tangent polynomials

This section is dedicated to deriving some fundamental higher-order q-differential equations for q-Frobenius-tangent polynomials through the use of q-calculus theory. Through the application of q-derivatives, we will obtain several related differential equations that connect to the q-analogue of Frobenius-tangent polynomials, based on definition (1.5). Furthermore, we will establish a q-differential equation that captures the symmetric property of these polynomials via q-derivatives.

Theorem 2.1. A solutions of the following differential equation

$$(i)_F\mathbb{T}_{\upsilon-\nu,q}(\psi;\eta) = \frac{[\upsilon-\nu]_q!}{[\upsilon]_q!}D^{(\nu)}_{q,\psi}{}_F\mathbb{T}_{\upsilon,q}(\psi;\eta), \tag{2.1}$$

$$(ii)_F\mathbb{T}_{\upsilon-\nu,q}(q^{-1}\psi;\eta) = \frac{q^{\nu}[\upsilon-\nu]_q!}{[\upsilon]_q!}D^{(\nu)}_{q,\psi}{}_F\mathbb{T}_{\upsilon,q}(q^{-1}\psi;\eta). \tag{2.2}$$

Proof. Using (1.5) and (1.8), we note that

$$D^{(1)}_{q,\psi}\sum_{\upsilon=0}^{\infty}\mathbb{T}_{\upsilon,q}(\psi;\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}=\frac{1-\eta}{e_q((1-\eta)\varphi)-\eta}D^{(1)}_{q,\psi}e_q(\psi\varphi)=\sum_{\upsilon=0}^{\infty}[\upsilon]_q\mathbb{T}_{\upsilon-1,q}(\psi;\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}. \tag{2.3}$$

From the equation (2.3), we get

$$D^{(1)}_{q,\psi}\mathbb{T}_{\upsilon,q}(\psi;\eta)=[\upsilon]_q\mathbb{T}_{\upsilon-1,q}(\psi;\eta).$$

In similar method, we find

$$D^{(2)}_{q,\psi}\mathbb{T}_{\upsilon,q}(\psi;\eta)=[\upsilon]_q[\upsilon-1]_q\mathbb{T}_{\upsilon-2,q}(\psi;u).$$

Therefore, we have

$$D^{(\nu)}_{q,\psi}\mathbb{T}_{\upsilon,q}(x;\eta)=[\upsilon]_q[\upsilon-1]_q\cdots[\upsilon-(\nu-1)]_q\mathbb{T}_{\upsilon-1,q}(\psi;\eta).$$

Hence, we find the desired result at once.

(ii) Similarly, we can proof of Theorem 2.1 (ii), so we omit the proof. □

Theorem 2.2. The differential equation of the q-Frobenius tangent polynomials as follows

$$\frac{(1-\eta)^{\upsilon-1}}{[\upsilon]_q!}D^{(\upsilon)}_{q,\psi}{}_F\mathbb{T}_{\upsilon,q}(\psi;\eta)+\frac{(1-\eta)^{\upsilon-2}}{[\upsilon-1]_q!}D^{(\upsilon-1)}_{q,\psi}{}_F\mathbb{T}_{\upsilon,q}(\psi;\eta)+\frac{(1-\eta)^{\upsilon-3}}{[\upsilon-2]_q!}D^{(\upsilon-2)}_{q,\psi}{}_F\mathbb{T}_{\upsilon,q}(\psi;\eta)$$

$$+\cdots+\frac{(1-\eta)^{3}}{[4]_q!}D^{(4)}_{q,\psi}{}_F\mathbb{T}_{\upsilon,q}(\psi;\eta)+\frac{(1-\eta)^{2}}{[3]_q!}D^{(3)}_{q,\psi}{}_F\mathbb{T}_{\upsilon,q}(\psi;\eta)$$

$$+\frac{(1-\eta)}{[2]_q!}D^{(2)}_{q,\psi}{}_F\mathbb{T}_{\upsilon,q}(\psi;\eta)+D^{(1)}_{q,\psi}{}_F\mathbb{T}_{\upsilon,q}(\psi;\eta)+(1-\eta)^{-1}({}_F\mathbb{T}_{0,q}(\psi;\eta)-\eta{}_F\mathbb{T}_{\upsilon,q}(\psi;\eta)-\psi^{\upsilon}=0.$$

Proof. By using (2.1), we see that

$$\begin{aligned}(1-\eta)e_q(\psi\varphi)&=\sum_{\upsilon=0}^{\infty}{}_F\mathbb{T}_{\upsilon,q}(\psi;\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}(e_q((1-\eta)\varphi)-\eta)\\&=\sum_{\upsilon=0}^{\infty}{}_F\mathbb{T}_{\upsilon,q}(\psi;\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}\left(\sum_{\nu=0}^{\infty}(1-\eta)^{\nu}\frac{\varphi^{\nu}}{[\nu]_q!}-\eta\right)\\&=\sum_{\upsilon=0}^{\infty}\left(\sum_{\nu=0}^{\upsilon}\binom{\upsilon}{\nu}_q(1-u)^{\nu}{}_F\mathbb{T}_{\upsilon-\nu,q}(\psi;\eta)-\eta{}_F\mathbb{T}_{\upsilon,q}(\psi;\eta)\right)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}.\end{aligned} \tag{2.4}$$

and

$$(1-\eta)e_q(\psi\varphi)=(1-\eta)\sum_{\upsilon=0}^{\infty}\psi^{\upsilon}\frac{\varphi^{\upsilon}}{[\upsilon]_q!}. \tag{2.5}$$

Therefore, by (2.4) and (2.5), we get

$$\sum_{\nu=0}^{\upsilon}\binom{\upsilon}{\nu}_q(1-\eta)^{\nu}{}_F\mathbb{T}_{\upsilon-\nu,q}(\psi;\eta)=(1-\eta)\psi^{\upsilon}. \tag{2.6}$$

Taking the $\nu-th$ derivative of above equation, we obtain

$$\sum_{\nu=0}^{\upsilon}\frac{(1-\eta)^{\nu-1}}{[\nu]_q!}D^{(\nu)}_{q,\psi}{}_F\mathbb{T}_{\upsilon,q}(\psi;\eta)=\eta(1-\eta)^{-1}{}_F\mathbb{T}_{\upsilon,q}(\psi;\eta)+(1-\eta)\psi^{\upsilon}=0.$$

Hence, we find the desired result at once. □

Corollary 2.1. As q approaches 1 in Theorem 2.2, we derive.

$$\frac{(1-\eta)^{\upsilon-1}}{\upsilon!}\frac{d^{\upsilon}}{d\psi^{\upsilon}}{}_F\mathbb{T}_{\upsilon}(\psi;\eta)+\frac{(1-\eta)^{\upsilon-2}}{[\upsilon-1]!}\frac{d^{\upsilon-1}}{d\psi^{\upsilon-1}}{}_F\mathbb{T}_{\upsilon}(\psi;\eta)+\frac{(1-\eta)^{\upsilon-3}}{[\upsilon-2]!}\frac{d^{\upsilon-2}}{d\psi^{\upsilon-2}}{}_F\mathbb{T}_{\upsilon}(\psi;\eta)$$

$$+\cdots+\frac{(1-\eta)^{3}}{4!}\frac{d^{4}}{d\psi^{4}}{}_F\mathbb{T}_{\upsilon}(\psi;\eta)+\frac{(1-\eta)^{2}}{3!}\frac{d^{3}}{d\psi^{3}}{}_F\mathbb{T}_{\upsilon}(\psi;\eta)$$

$$+\frac{(1-\eta)}{2!}\frac{d^{2}}{d\psi^{2}}{}_F\mathbb{T}_{\upsilon}(\psi;\eta)+\frac{d}{d\psi}{}_F\mathbb{T}_{\upsilon}(\psi;\eta)+(1-\eta)^{-1}({}_F\mathbb{T}_0(\psi;\eta)-\eta{}_F\mathbb{T}_{\upsilon}(\psi;\eta)-\psi^{\upsilon}=0.$$

Theorem 2.3. Let $\upsilon\geq 0$. Then

$$\frac{{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(1;\eta)-\eta\mathbb{T}_{\upsilon,q}(\eta)}{[\upsilon]_q!}D^{(\upsilon)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)+\frac{{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(1;\eta)-\eta{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\eta)}{[\upsilon-1]_q!}D^{(\upsilon-1)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)+$$

$$\cdots+\frac{{}_{\mathbb{F}}\mathbb{T}_{2,q}(1;\eta)-\eta{}_{\mathbb{F}}\mathbb{T}_{2,q}(\eta)}{[2]_q!}D^{(2)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(1;\eta)+({}_{\mathbb{F}}\mathbb{T}_{1,q}(1;\eta)-\eta{}_{\mathbb{F}}\mathbb{T}_{1,q}(\eta))D^{(1)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)$$

$$+({}_{\mathbb{F}}\mathbb{T}_{0,q}(1;\eta)-\eta{}_{\mathbb{F}}\mathbb{T}_{0,q}(\eta)-(1-\eta)){}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)=0.$$

Proof. From (2.1), we have

$$\sum_{\upsilon=0}^{\infty}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;u)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}=\frac{1-\eta}{e_q((1-\eta)\varphi)-\eta}e_q(\psi\varphi)$$

$$=\frac{1}{1-\eta}\left(\frac{1-\eta}{e_q((1-\eta)\varphi)-\eta}e_q((1-\eta)\varphi)-\eta\frac{1-\eta}{e_q((1-\eta)\varphi)-\eta}\right)\frac{1-\eta}{e_q((1-\eta)\varphi)-\eta}e_q(\psi\varphi).$$

$$(1-\eta)\sum_{\upsilon=0}^{\infty}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}=\sum_{\upsilon=0}^{\infty}\left(\sum_{\nu=0}^{\upsilon}\binom{\upsilon}{\nu}_q\left({}_{\mathbb{F}}\mathbb{T}_{\nu,q}(1;\eta)-\eta{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(\eta)\right){}_{\mathbb{F}}\mathbb{T}_{\upsilon-\nu,q}(\psi;\eta)\right)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}$$

$$\sum_{\nu=0}^{\upsilon}\binom{\upsilon}{\nu}_q\left({}_{\mathbb{F}}\mathbb{T}_{\nu,q}(1;\eta)-\eta{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(\eta)\right){}_{\mathbb{F}}\mathbb{T}_{\upsilon-\nu,q}(\psi;\eta)-(1-\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)=0. \tag{2.7}$$

Replacing ${}_{\mathbb{F}}\mathbb{T}_{\upsilon-\nu,q}(\psi;\eta)$ with $D^{(\nu)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)$ in equation (2.7), we have

$$\sum_{\nu=0}^{\upsilon}\frac{\left({}_{\mathbb{F}}\mathbb{T}_{\nu,q}(1;\eta)-\eta{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(\eta)\right)}{[\nu]_q!}D^{(\nu)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)-(1-\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)=0.$$

Hence, we find the desired result at once. ☐

Corollary 2.2. As q approaches 1 in Theorem 2.3, we derive

$$\frac{{}_{\mathbb{F}}\mathbb{T}_{\upsilon}(1;\eta)-\eta\mathbb{T}_{\upsilon}(\eta)}{\upsilon!}\frac{d^{\upsilon}}{dx^{\upsilon}}{}_{\mathbb{F}}\mathbb{T}_{\upsilon}(\psi;\eta)+\frac{{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(1;\eta)-\eta{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(\eta)}{(\upsilon-1)!}\frac{d^{\upsilon-1}}{d\psi^{\upsilon-1}}{}_{\mathbb{F}}\mathbb{T}_{\upsilon}(\psi;\eta)+$$

$$\cdots+\frac{{}_{\mathbb{F}}\mathbb{T}_{2}(1;\eta)-\eta{}_{\mathbb{F}}\mathbb{T}_{2}(\eta)}{2!}\frac{d^{2}}{d\psi^{2}}{}_{\mathbb{F}}\mathbb{T}_{\upsilon}(\psi;\eta)+({}_{\mathbb{F}}\mathbb{T}_{1}(1;\eta)-\eta\mathbb{T}_{1}(\eta))\frac{d}{d\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon}(\psi;\eta)$$

$$+({}_{\mathbb{F}}\mathbb{T}_{0}(1;\eta)-\eta{}_{\mathbb{F}}\mathbb{T}_{0}(\eta)-(1-\eta)){}_{\mathbb{F}}\mathbb{T}_{\upsilon}(\psi;\eta)=0.$$

Theorem 2.4. Let $\upsilon\geq 0$. Then

$$\sum_{\nu=0}^{\upsilon-1}\frac{(1-\eta)^{\upsilon-\nu-1}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(\eta)}{[\upsilon-\nu-1]_q![\nu]_q!}D^{(\upsilon-1)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)+\sum_{\nu=0}^{\upsilon-2}\frac{(1-\eta)^{\upsilon-\nu-2}q{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(\eta)}{[\upsilon-\nu-2]_q![\nu]_q!}D^{(\upsilon-2)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)+\cdots$$

$$+\sum_{\nu=0}^{2}\frac{(1-\eta)^{2-\nu}q^{\upsilon-3}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(\eta)}{[2-\nu]_q![\nu]_q!}D^{(2)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)+\sum_{\nu=0}^{1}\frac{(1-\eta)^{1-\nu}q^{\upsilon-2}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(\eta)}{[1-\nu]_q![\nu]_q!}D^{(1)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)$$

$$+\left(q^{\upsilon-1}{}_{\mathbb{F}}\mathbb{T}_{0,q}(\eta)-q^{\upsilon}\psi\right){}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)+{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(q\psi;\eta)=0.$$

Proof. Let $q\psi \to \psi$ in (1.5), we have

$$D_{q,\varphi}\sum_{\upsilon=0}^{\infty}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(q\psi;\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}=e_q(q\varphi\psi)D_{q,\varphi}\left(\frac{1-\eta}{e_q((1-\eta)\varphi)-\eta}\right)+\frac{1-\eta}{e_q((1-\eta)\varphi)-\eta}D_{q,\varphi}e_q(q\varphi\psi)$$

$$=\sum_{\upsilon=0}^{\infty}q^{\upsilon}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}\left(q\psi-\sum_{\upsilon=0}^{\infty}\left(\sum_{\nu=0}^{\upsilon}\binom{\upsilon}{\nu}_q(1-\eta)^{\upsilon-\nu}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(\eta)\right)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}\right)$$

$$=\sum_{\upsilon=0}^{\infty}\left(q^{\upsilon+1}\psi{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\eta)-\sum_{\theta=0}^{\upsilon}\sum_{\nu=0}^{\theta}\binom{\upsilon}{\theta}_q\binom{\theta}{\nu}_q(1-\eta)^{\theta-\nu}q^{\upsilon-\theta}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon-\theta,q}(\psi;\eta)\right)\frac{\varphi^{\upsilon}}{[\upsilon]_q!} \tag{2.8}$$

$$D_{q,\varphi}\sum_{\upsilon=0}^{\infty}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(q\psi;\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}=\sum_{\upsilon=0}^{\infty}[\upsilon]_q q^{\upsilon}\psi{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}-$$
$$\sum_{\upsilon=0}^{\infty}[\upsilon]_q\left(\sum_{\theta=0}^{\upsilon-1}\sum_{\nu=0}^{\theta}\binom{\upsilon-1}{\theta}_q\binom{\theta}{\nu}_q(1-\eta)^{\theta-\nu}q^{\upsilon-\theta-1}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon-\theta-1,q}(\psi;\eta)\right)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}. \tag{2.9}$$

On the other hand, we have

$$\varphi D_{q,\varphi}\sum_{\upsilon=0}^{\infty}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(q\psi;\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}=\sum_{\upsilon=0}^{\infty}[\upsilon]_q{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(q\psi;\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}. \tag{2.10}$$

By (2.9) and (2.10), we have

$$\sum_{\theta=0}^{\upsilon-1}\sum_{\nu=0}^{\theta}\binom{\upsilon-1}{\theta}_q\binom{\theta}{\nu}_q(1-\eta)^{\theta-\nu}q^{\upsilon-\theta-1}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon-\theta-1,q}(\psi;\eta)$$
$$=q^{\upsilon}\psi{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)-{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(q\psi;\eta). \tag{2.11}$$

In Theorem 2.1 (i), we get

$${}_{F}\mathbb{T}_{\upsilon-\nu-1,q}(\psi;\eta)=\frac{[\upsilon-\nu-1]_q!}{[\upsilon-1]_q!}D_{q,\psi}^{(\nu)}{}_{F}\mathbb{T}_{\upsilon-1,q}(\psi;\eta). \tag{2.12}$$

$$\sum_{\theta=0}^{\upsilon-1}\sum_{\nu=0}^{\theta}\binom{\upsilon-1}{\theta}_q\binom{\theta}{\nu}_q(1-\eta)^{\theta-\nu}q^{\upsilon-\theta-1}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon-l-1,q}(\psi;\eta)$$
$$=\sum_{\theta=0}^{\upsilon-1}\sum_{\nu=0}^{\theta}\frac{(1-\eta)^{\theta-\nu}q^{\upsilon-\theta-1}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(\eta)}{[\theta-\nu]_q![\nu]_q!}D_{q,\psi}^{(\theta)}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta). \tag{2.13}$$

Therefore, we acquire at the desired result.

□

Corollary 2.3. As q approaches 1 in Theorem 2.4, we derive

$$\sum_{\nu=0}^{\upsilon-1}\frac{(1-\eta)^{\upsilon-\nu-1}{}_{\mathbb{F}}\mathbb{T}_{\nu}(\eta)}{[\upsilon-\nu-1]![\nu]!}\frac{d^{(\upsilon-1)}}{d\psi^{\upsilon-1}}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(\psi;\eta)+\sum_{\nu=0}^{\upsilon-2}\frac{(1-\eta)^{\upsilon-\nu-2}{}_{\mathbb{F}}\mathbb{T}_{\nu}(\eta)}{[\upsilon-\nu-2]![\nu]!}\frac{d^{(\upsilon-2)}}{d\psi^{\upsilon-2}}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(\psi;\eta)+\cdots$$
$$+\sum_{\nu=0}^{2}\frac{(1-\eta)^{2-\nu}{}_{\mathbb{F}}\mathbb{T}_{\nu}(\eta)}{[2-\nu]![\nu]!}\frac{d^2}{d\psi^2}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(\psi;\eta)+\sum_{\nu=0}^{\theta}\frac{(1-\eta)^{\theta-\nu}{}_{\mathbb{F}}\mathbb{T}_{\nu}(\eta)}{[\theta-\nu]![\nu]!}\frac{d}{d\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(\psi;\eta)$$
$$+\left({}_{\mathbb{F}}\mathbb{T}_0(\eta)-\psi\right){}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(\psi;\eta)+{}_{\mathbb{F}}\mathbb{T}_{\upsilon}(\psi;\eta)=0.$$

Theorem 2.5. Let $\upsilon\geq 0$. Then

$$\sum_{\nu=0}^{\upsilon-1}\frac{(1-\eta)^{\upsilon-1}\mathbb{H}_{\nu,q}(\eta)}{[\upsilon-\nu-1]_q![\nu]_q!}D_{q,\psi}^{(\upsilon-1)}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)+\sum_{\nu=0}^{\upsilon-2}\frac{(1-\eta)^{\upsilon-2}q\mathbb{H}_{\nu,q}(\eta)}{[\upsilon-\nu-2]_q![\nu]_q!}D_{q,\psi}^{(\upsilon-2)}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)+\cdots$$

$$+\sum_{\nu=0}^{2}\frac{(1-\eta)^2 q^{\upsilon-3}\mathbb{H}_{\nu,q}(\eta)}{[2-\nu]_q![\nu]_q!}D^{(2)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)+\sum_{\nu=0}^{1}\frac{(1-\eta)q^{\upsilon-2}\mathbb{H}_{\nu,q}(\eta)}{[1-\nu]_q![\nu]_q!}D^{(1)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)$$
$$+\left(q^{-1}\mathbb{H}_{0,q}-\psi\right)q^{\upsilon}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)+{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(q\psi;\eta)=0.$$

Proof. Using (1.5), we have

$$D_{q,\varphi}\sum_{\upsilon=0}^{\infty}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(q\psi;\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}$$
$$=\sum_{\upsilon=0}^{\infty}q^{\upsilon}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}\left(q\psi-\sum_{\upsilon=0}^{\infty}(1-\eta)^{\upsilon}\mathbb{H}_{\upsilon,q}(\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}\sum_{\upsilon=0}^{\infty}(1-\eta)^{\upsilon}\frac{\varphi^{\upsilon}}{[\upsilon]_q!}\right)$$
$$=\sum_{\upsilon=0}^{\infty}\left(q^{\upsilon+1}\psi{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)-\sum_{\theta=0}^{\upsilon}\sum_{\nu=0}^{\theta}\binom{\upsilon}{\theta}_q\binom{\theta}{\nu}_q(1-\eta)^{\theta}q^{\upsilon-\theta}\mathbb{H}_{\nu,q}(\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon-\theta,q}(\psi;\eta)\right)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}.$$

Therefore, we have

$$\sum_{\theta=0}^{\upsilon-1}\sum_{\nu=0}^{\theta}\frac{(1-\eta)^{\theta}q^{\upsilon-\theta-1}\mathbb{H}_{\nu,q}(\eta)}{[\theta-\nu]_q![\nu]_q!}D^{(\theta)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-\theta,q}(\psi;\eta)-q^{\upsilon}\psi{}_{\mathbb{F}}\mathbb{T}_{\upsilon-\theta,q}(\psi;\eta)+{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(q\psi;\eta)=0, \tag{2.14}$$

which is the desired result. □

Corollary 2.4. As q approaches 1 in Theorem 2.5, we derive

$$\sum_{\nu=0}^{\upsilon-1}\frac{(1-\eta)^{\upsilon-1}\mathbb{H}_{\nu}(\eta)}{[\upsilon-\nu-1]![\nu]!}D^{(\upsilon-1)}_{\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(\psi;\eta)+\sum_{\nu=0}^{\upsilon-2}\frac{(1-\eta)^{\upsilon-2}\mathbb{H}_{\nu}(\eta)}{[\upsilon-\nu-2]![\nu]!}D^{(\upsilon-2)}_{\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(\psi;\eta)+\cdots$$
$$+\sum_{\nu=0}^{2}\frac{(1-\eta)^{2}\mathbb{H}_{\nu}(\eta)}{[2-\nu]![\nu]!}D^{(2)}_{\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(\psi;\eta)+\sum_{\nu=0}^{1}\frac{(1-\eta)\mathbb{H}_{\nu}(\eta)}{[1-\nu]!\nu!}D^{(1)}_{\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(\psi;\eta)$$
$$+\left(\mathbb{H}_{0}-\psi\right){}_{\mathbb{F}}\mathbb{T}_{\upsilon-\theta}(\psi;\eta)+{}_{\mathbb{F}}\mathbb{T}_{\upsilon}(\psi;\eta)=0.$$

Theorem 2.6. Let $\upsilon\geq 0$. Then

$$\frac{{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(1-\eta)}{[\upsilon-1]_q!}D^{(\upsilon-1)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)+\frac{q{}_{\mathbb{F}}\mathbb{T}_{\upsilon-2,q}(1-\eta)}{[\upsilon-2]_q!}D^{(\upsilon-2)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)+\cdots$$
$$+\frac{q^{\upsilon-4}{}_{\mathbb{F}}\mathbb{T}_{3,q}(1-\eta)}{[3]_q!}D^{(3)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)+\frac{q^{\upsilon-3}{}_{\mathbb{F}}\mathbb{T}_{2,q}(1-\eta)}{[2]_q!}D^{(2)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)$$
$$+q^{\upsilon-2}{}_{\mathbb{F}}\mathbb{T}_{1,q}(1-\eta)D^{(1)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)+\left(q^{-1}{}_{\mathbb{F}}\mathbb{T}_{0,q}(1-\eta)-\psi\right)q^{\upsilon}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)+{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(q\psi;\eta)=0.$$

Proof. By using (1.5), (1.8) and (2.8), we have

$$D_{q,\varphi}\sum_{\upsilon=0}^{\infty}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(q\psi;\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}$$
$$=\sum_{\upsilon=0}^{\infty}\left(q^{\upsilon+1}\psi{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)-\sum_{\nu=0}^{\upsilon}\binom{\upsilon}{\nu}_q q^{\upsilon-\nu}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(1-\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon-\nu,q}(\psi;\eta)\right)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}. \tag{2.15}$$

On multiplying φ in the above equation, we get

$$\varphi D_{q,\varphi}\sum_{\upsilon=0}^{\infty}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(q\psi;\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}$$
$$=\sum_{\upsilon=0}^{\infty}[\upsilon]_q q^{\upsilon}\psi{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(q^{-1}\psi;\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}$$

$$-\sum_{\upsilon=0}^{\infty}[\upsilon]_q\sum_{\nu=0}^{\upsilon-1}\binom{\upsilon-1}{\nu}_q q^{\upsilon-\nu-1}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(1-\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon-\nu-1,q}(\psi;\eta)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}. \tag{2.16}$$

By (2.15) and (2.16), we attain

$$\begin{aligned}&\sum_{\nu=0}^{\upsilon-1}\binom{\upsilon-1}{\nu}_q q^{\upsilon-\nu-1}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(1-\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon-\nu-1,q}(\psi;\eta)\\ &=q^{\upsilon}\psi{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(q^{-1}\psi;\eta)-q^{\upsilon}\psi{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta)-{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(q\psi;\eta).\end{aligned} \tag{2.17}$$

Applying a relation between $D^{\upsilon}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)$ and ${}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)$ in the left-hand side of (2.17), we obtain

$$\begin{aligned}&\sum_{\nu=0}^{\upsilon-1}\binom{\upsilon-1}{\nu}_q q^{\upsilon-\nu-1}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(1-\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon-\nu-1,q}(\psi;\eta)\\ &=\sum_{\nu=0}^{\upsilon-1}\frac{q^{\upsilon-\nu-1}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(1-\eta)}{[\nu]_q!}D^{(\nu)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\psi;\eta).\end{aligned} \tag{2.18}$$

Hence, complete the proof.

□

Corollary 2.5. As q approaches 1 in Theorem 2.6, we derive

$$\begin{aligned}&\frac{{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(1-\eta)}{[\upsilon-1]!}D^{(\upsilon-1)}_{\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(\psi;\eta)+\frac{{}_{\mathbb{F}}\mathbb{T}_{\upsilon-2}(1-\eta)}{[\upsilon-2]!}D^{(\upsilon-2)}_{\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(\psi;\eta)+\cdots\\ &\quad+\frac{{}_{\mathbb{F}}\mathbb{T}_{3}(1-\eta)}{[3]!}D^{(3)}_{\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(\psi;\eta)+\frac{{}_{\mathbb{F}}\mathbb{T}_{2}(1-\eta)}{2!}D^{(2)}_{\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(\psi;\eta)\\ &+{}_{\mathbb{F}}\mathbb{T}_{1}(1-\eta)D^{(1)}_{\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(\psi;\eta)+\left({}_{\mathbb{F}}\mathbb{T}_{0}(1-\eta)-\psi\right){}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(\psi;\eta)+{}_{\mathbb{F}}\mathbb{T}_{\upsilon}(\psi;\eta)=0.\end{aligned}$$

Theorem 2.7. Let $\upsilon\geq 0$. Then

$$\begin{aligned}&\frac{{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(b^{-1}\zeta;\eta)}{[\upsilon]_q!}D^{(\upsilon)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(a^{-1}\psi;\eta)+\frac{b^{-1}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(b^{-1}\zeta;\eta)}{[\upsilon-1]_q!}D^{(\upsilon-1)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(a^{-1}\psi;\eta)+\cdots\\ &\quad+b^{1-\upsilon}{}_{\mathbb{F}}\mathbb{T}_{1,q}(b^{-1}\zeta;\eta)D^{(1)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(a^{-1}\psi;\eta)+b^{-\upsilon}{}_{\mathbb{F}}\mathbb{T}_{0,q}(b^{-1}\zeta;\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(a^{-1}\psi;\eta)\\ &=\frac{{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(a^{-1}\zeta;\eta)}{[\upsilon]_q!}D^{(\upsilon)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(b^{-1}\psi;\eta)+\frac{a^{-1}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(a^{-1}\zeta;\eta)}{[\upsilon-1]_q!}D^{(\upsilon-1)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(b^{-1}\psi;\eta)+\cdots\\ &\quad+a^{1-\upsilon}{}_{\mathbb{F}}\mathbb{T}_{1,q}(a^{-1}\zeta;\eta)D^{(1)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(b^{-1}\psi;\eta)+a^{-\upsilon}{}_{\mathbb{F}}\mathbb{T}_{0,q}(a^{-1}\zeta;\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(b^{-1}\psi;\eta).\end{aligned}$$

Proof. Let

$$A(\varphi)=\frac{(1-\eta)^2e_q(ab(\psi+\eta)\varphi)}{(e_q((1-\eta)a\varphi)-\eta)(e_q((1-\eta)b\varphi)-\eta)}.$$

Using the definition (1.5) and Cauchy products, then

$$A(\varphi)=\sum_{\upsilon=0}^{\infty}\left(\sum_{\nu=0}^{\upsilon}\binom{\upsilon}{\nu}_q a^{\upsilon-\nu}b^{\nu}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(b^{-1}\zeta;\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon-\nu,q}(a^{-1}\psi;\eta)\right)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}. \tag{2.19}$$

Similarly, we have

$$A(\varphi)=\sum_{\upsilon=0}^{\infty}\left(\sum_{\nu=0}^{\upsilon}\binom{\upsilon}{\nu}_q b^{\upsilon-\nu}a^{\nu}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(a^{-1}\zeta;\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon-\nu,q}(b^{-1}\psi;\eta)\right)\frac{\varphi^{\upsilon}}{[\upsilon]_q!}. \tag{2.20}$$

By (2.19) and (2.20), we have

$$\sum_{\nu=0}^{\upsilon}\binom{\upsilon}{\nu}_q a^{\upsilon-\nu}b^{\nu}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(b^{-1}\zeta;\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon-\nu,q}(a^{-1}\psi;\eta)$$

$$=\sum_{\nu=0}^{\upsilon}\binom{\upsilon}{\nu}_q b^{\upsilon-\nu}a^{\nu}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(a^{-1}\zeta;\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon-\nu,q}(b^{-1}\psi;\eta). \tag{2.21}$$

Applying a relation between $D^{(\upsilon)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)$ and ${}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)$ in (2.21), we have

$$\frac{b^{\nu-\upsilon}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(b^{-1}\zeta;\eta)}{[\nu]_q!}D^{(\nu)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-\nu,q}(a^{-1}\psi;\eta)$$

$$=\frac{a^{\nu-\upsilon}{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(a^{-1}\zeta;\eta)}{[\nu]_q!}D^{(\nu)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-\nu,q}(b^{-1}\psi;\eta).$$

Hence, complete the proof.

□

Corollary 2.6. Letting $a=1$ in Theorem 2.7, we have

$$\frac{{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(b^{-1}\zeta;\eta)}{[\upsilon]_q!}D^{(\upsilon)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)+\frac{b^{-1}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(b^{-1}\zeta;\eta)}{[\upsilon-1]_q!}D^{(\upsilon-1)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)+\cdots$$

$$+b^{1-\upsilon}{}_{\mathbb{F}}\mathbb{T}_{1,q}(b^{-1}\zeta;\eta)D^{(1)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)+b^{-\upsilon}{}_{\mathbb{F}}\mathbb{T}_{0,q}(b^{-1}\zeta;\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(\psi;\eta)$$

$$=\frac{{}_{\mathbb{F}}\mathbb{T}_{\nu,q}(\zeta;\eta)}{[\upsilon]_q!}D^{(\upsilon)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(b^{-1}\psi;\eta)+\frac{a^{-1}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1,q}(\zeta;\eta)}{[\upsilon-1]_q!}D^{(\upsilon-1)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(b^{-1}\psi;\eta)+\cdots$$

$$+{}_{\mathbb{F}}\mathbb{T}_{1,q}(\zeta;\eta)D^{(1)}_{q,\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(b^{-1}\psi;\eta)+{}_{\mathbb{F}}\mathbb{T}_{0,q}(\zeta;\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon,q}(b^{-1}\psi;\eta).$$

Corollary 2.7. As q approaches 1 in Theorem 2.7, we derive

$$\frac{{}_{\mathbb{F}}\mathbb{T}_{\nu}(b^{-1}\zeta;\eta)}{[\upsilon]!}D^{(\upsilon)}_{\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon}(a^{-1}\psi;\eta)+\frac{b^{-1}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(b^{-1}\zeta;\eta)}{[\upsilon-1]!}D^{(\upsilon-1)}_{\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon}(a^{-1}\psi;\eta)+\cdots$$

$$+b^{1-\upsilon}{}_{\mathbb{F}}\mathbb{T}_{1}(b^{-1}\zeta;\eta)D^{(1)}_{\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon}(a^{-1}\psi;\eta)+b^{-\upsilon}{}_{\mathbb{F}}\mathbb{T}_{0}(b^{-1}\zeta;\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon}(a^{-1}\psi;\eta)$$

$$=\frac{{}_{\mathbb{F}}\mathbb{T}_{\nu}(a^{-1}\zeta;\eta)}{[\upsilon]!}D^{(\upsilon)}_{\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon}(b^{-1}\psi;\eta)+\frac{a^{-1}{}_{\mathbb{F}}\mathbb{T}_{\upsilon-1}(a^{-1}\zeta;\eta)}{[\upsilon-1]!}D^{(\upsilon-1)}_{\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon}(b^{-1}\psi;\eta)+\cdots$$

$$+a^{1-\upsilon}{}_{\mathbb{F}}\mathbb{T}_{1}(a^{-1}\zeta;\eta)D^{(1)}_{\psi}{}_{\mathbb{F}}\mathbb{T}_{\upsilon}(b^{-1}\psi;\eta)+a^{-\upsilon}{}_{\mathbb{F}}\mathbb{T}_{0}(a^{-1}\zeta;\eta){}_{\mathbb{F}}\mathbb{T}_{\upsilon}(b^{-1}\psi;\eta).$$

3. Conclusion

We have constructed the q-analogue of Frobenius-tangent polynomials and numbers, and derived several differential equations with these polynomials as solutions. Additionally, we identified differential equations that combine q-Frobenius and q-tangent polynomials. Several properties of the q-analogue of Frobenius-tangent polynomials and numbers were also established. The results obtained in this paper are broadly general and may lead to potential applications in the theory of special functions. Moreover, the main findings are significant, as they allow us to deduce important integral formulas for specific parameter values, which could be particularly useful in laser technology.

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