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On the Chromatic Restrained Domination Number of Strong Product of Graphs

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Abstract:

Let $G = (V, E)$ be a graph. A subset D of V is said to be a chromatic restrained dominating set (or crd-set) if D is a restrained dominating set and $\chi(< D >) = \chi(G)$. The minimum cardinality taken over all minimal chromatic restrained dominating sets is called the chromatic restrained domination number of G and is denoted by $\gamma_r^c(G)$. In this paper, we obtain the chromatic restrained domination number for the strong product of some standard graphs.

Keywords : Domination, Restrained Domination, Chromatic Number, Strong Product.

AMS Subject Classification : 05C15, 05C69

1. Introduction

All the graphs $G = (V, E) = (n, m)$ considered here are simple, finite and undirected, with neither loops nor multiple edges. For $D \subseteq V$, the subgraph induced by D is denoted by $\langle D \rangle$. A k-vertex-coloring of a graph, or simply a *k-coloring*, is an assignment of k-colors to its vertices. The coloring is proper if no two adjacent vertices are assigned the same color. A coloring in which k-colors are used is a k-coloring. A graph is k-colorable if it has a proper k-coloring. The minimum k for which a graph

G is k-colorable is called its *chromatic number*, and denoted by $\chi(G)$. Graph Theory terminologies which are not defined here can be seen in [1] and [2].

A set $D \subseteq V$ of vertices in a graph G is called a *dominating set* if every vertex $u \in V$ is either an element of D or is adjacent to an element of D. The minimum cardinality taken over all minimal dominating sets is called the *domination number* of G and is denoted by $\gamma(G)$. A set $D \subseteq V$ is a *restrained dominating set* if every vertex in $V - D$ is adjacent to a vertex in D and another vertex in $V - D$ [3]. The minimum cardinality taken over all minimal restrained dominating sets is called the restrained domination number of G and is denoted by $\gamma_r(G)$. A set D is a γ_r - set if D is a restrained dominating set of cardinality $\gamma_r(G)$.

Strong product of two graphs G and H is the graph $G \boxtimes H$ whose vertex set is $V(G) \times V(H)$, vertices (u, x) and (v, y) being adjacent if and only if $uv \in E(G)$ and $x = y$ (or) $u = v$ and $xy \in E(H)$ or $uv \in E(G)$ and $xy \in E(H)$ [4].

T. N. Janakiraman and M. Poobalaranjani introduced the concept of chromatic preserving set. A set $D \subseteq V$ is a *chromatic preserving set* or a cp-set if $\chi(< D >) = \chi(G)$ and the minimum cardinality taken over all cp-sets in G is called the *chromatic preserving number* or cp-number of G and is denoted by $cpn(G)$ [5]. A subset D of V is said to be a *dom-chromatic set* (or dc-set)if D is a dominating set and $\chi(< D >) = \chi(G)$. The minimum cardinality taken over all minimal dom-chromatic sets in G is called the dom-chromatic number and is denoted by $\gamma_{ch}(G)$[6]. In this paper, the chromatic restrained domination number on the strong product of some standard graphs are obtained.

2 Main Results

In this section, we obtain the chromatic restrained domination number for the strong product of some standard graphs.

Definition 2.1 Let $G = (V, E)$ be a graph. A subset D of V is said to be a chromatic restrained dominating set (or crd-set) if D is a restrained dominating set and $\chi(< D >) = \chi(G)$. The minimum cardinality taken over all minimal chromatic restrained dominating sets is called the chromatic restrained domination number and is denoted by $\gamma_r^c(G)$. Throughout this paper, we denote the chromatic restrained domination number on the strong product of two graphs G and H by $\gamma_r^c(G \boxtimes H)$.

Theorem 2.2 For $r, s \geq 2, \gamma_r^c(K_r \boxtimes K_s) = rs$.

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Proof. Let $V(K_r) = \{u_1, u_2, u_3, \dots, u_r\}$ and $V(K_s) = \{v_1, v_2, v_3, \dots, v_s\}$. Then, $V(K_r \boxtimes K_s) = \{(u_i, v_j)/1 \le i \le r, 1 \le j \le s\}$ where each vertex in $K_r \boxtimes K_s$ is a full degree vertex and $|V(K_r \boxtimes K_s)| = rs$. Also, $\chi(K_r \boxtimes K_s) = rs$ as each vertex can be given different colors. Let D be a $\gamma_r^c -$ set of $K_r \boxtimes K_s$. For any proper subset S of $V(K_r \boxtimes K_s)$, $\chi(\langle S\rangle) < rs$. Thus, $D = V(K_r \boxtimes K_s)$ is the only chromatic restrained dominating set of $K_r \boxtimes K_s$. Therefore, $\gamma_r^c(K_r \boxtimes K_s) = |D| = rs$.

Theorem 2.3 For $s \ge 4$, $\gamma_r^c(K_r \boxtimes P_s) = \left\lceil \frac{s-4}{3} \right\rceil + 2r$.

Proof. Let $V(K_r) = \{u_1, u_2, u_3, \dots, u_r\}$ and $V(P_s) = \{v_1, v_2, v_3, \dots, v_s\}$. Then $V(K_r \boxtimes P_s) = \{(u_i, v_j)/1 \le i \le r, 1 \le j \le s\}$ with cardinality rs. Also, $K_r \boxtimes P_s$ contains r rows and s columns $V_1, V_2, V_3, \dots, V_s$. Since $\chi(K_r) = r$ and $\chi(P_s) = 2$, each column V_i, i is odd can be colored with r colors and each V_j, j is even can be colored with another r colors. Thus, $\chi(K_r \boxtimes P_s) = 2r$.

Case (i): $s \equiv 0(mod\ 3)$

Let $D = \{(u_1, v_{3k-1})/1 \le k \le \frac{s}{3}\}$ where $|D| = \frac{s}{3}$. Then, D is a dominating set and there does not exists any isolated vertex in $\langle V - D\rangle$. Thus, D is a restrained dominating set and $\gamma_r(K_r \boxtimes P_s) \le |D| = \frac{s}{3}$. Since, $\gamma(P_s) = \frac{s}{3}$ and each vertex in column V_j is adjacent to all the vertices in V_{j-1}, V_j and V_{j+1}, $\gamma_r(K_r \boxtimes P_s) \ge \frac{s}{3}$. Thus, $\gamma_r(K_r \boxtimes P_s) = \frac{s}{3}$. But, every minimum restrained dominating set is independent and so, $\chi(\langle D\rangle) = 1 \ne \chi(K_r \boxtimes P_s)$. Thus, D is not a chromatic restrained dominating set of $K_r \boxtimes P_s$. Consider $D_1 = D \cup \{(u_i, v_2), (u_j, v_3)/2 \le i \le r, 1 \le j \le r\}$. Since $(u_1, v_2) \in D$, $\langle\{(u_i, v_2), (u_j, v_3)/1 \le i, j \le r\}\rangle$ is a complete subgraph on $2r$ vertices and so, $\chi(\langle D_1\rangle) = 2r = \chi(K_r \boxtimes P_s)$. Clearly, D_1 is a restrained dominating set. Thus, D_1 is a chromatic restrained dominating set of $K_r \boxtimes P_s$ and $\gamma_r^c(K_r \boxtimes P_s) \le |D_1| = \frac{s}{3} + 2r - 1 = \frac{s-3}{3} + 2r = \left\lceil \frac{s-4}{3} \right\rceil + 2r$. Then, it remains to show that, $\gamma_r^c(K_r \boxtimes P_s) \ge \left\lceil \frac{s-4}{3} \right\rceil + 2r$. Since, $\chi(K_r \boxtimes P_s) = 2r$ and $K_r \boxtimes P_s$ contains induced subgraph which is complete on $2r$ vertices, any minimum chromatic restrained dominating set must contain those $2r$ vertices which are the vertices of two adjacent columns. Let them be V_2 and V_3, so that, all the vertices of V_1 and V_4 are adjacent to the vertices of V_2 and V_3. From the remaining $s - 4$ columns, choose a vertex of each column $V_{3(k+1)}, 1 \le k \le \left\lceil \frac{s-4}{3} \right\rceil$ which is adjacent to all the vertices of $V_{3(k+1)} - 1, V_{3(k+1)}$ and $V_{3(k+1)} + 1$. Thus, $\gamma_r^c(K_r \boxtimes P_s) \ge \left\lceil \frac{s-4}{3} \right\rceil + 2r$. Therefore, $\gamma_r^c(K_r \boxtimes P_s) = \left\lceil \frac{s-4}{3} \right\rceil + 2r$.

Case (ii): $s \equiv 1(mod\ 3)$

Let $D_2 = \{(u_1, v_{3k-1})/1 \leq k \leq \left\lfloor\frac{s}{3}\right\rfloor\} \cup \{(u_1, v_s)\}$ where $|D_2| = \left\lfloor\frac{s}{3}\right\rfloor + 1 = \left\lceil\frac{s}{3}\right\rceil$. Then, every vertex of $V - D_2$ is adjacent to at least one vertex of D_2 and at least one another vertex in $V - D_2$. Thus, D_2 is a restrained dominating set and $\gamma_r(K_r \boxtimes P_s) \leq |D_2| = \left\lceil\frac{s}{3}\right\rceil$. Since, $\gamma(P_s) = \left\lceil\frac{s}{3}\right\rceil$ and each vertex in column V_j is adjacent to all the vertices in V_{j-1}, V_j and V_{j+1}, $\gamma_r(K_r \boxtimes P_s) \geq \left\lceil\frac{s}{3}\right\rceil$. Therefore, $\gamma_r(K_r \boxtimes P_s) = \left\lceil\frac{s}{3}\right\rceil$. Consider $D_3 = V_2 \cup V_3 \cup \{(u_1, v_{3(k+1)})/1 \leq k \leq \frac{s-4}{3}\}$. Clearly, D_3 is a restrained dominating set and $\langle V_2 \cup V_3\rangle$ is a complete graph on $2r$ vertices. Thus, $\chi(\langle D_3\rangle) = 2r = \chi(K_r \boxtimes P_s)$ and so, D_3 is a chromatic restrained dominating set of $K_r \boxtimes P_s$. Then, $\gamma_r^c(K_r \boxtimes P_s) \leq |D_3| = \left\lceil\frac{s-4}{3}\right\rceil + 2r$. Since $\chi(K_r \boxtimes P_s) = 2r$, any minimum chromatic restrained dominating set must contain all the r vertices of two adjacent columns. Let them be V_2 and V_3 which is adjacent to all the vertices of V_1 and V_4. Again from the remaining $s - 4$ columns, choose a vertex from columns $V_{3(k+1)}, 1 \leq k \leq \frac{s-4}{3}$. Thus, we get a minimum chromatic restrained dominating set of $K_r \boxtimes P_s$ and $\gamma_r^c(K_r \boxtimes P_s) \geq \left\lceil\frac{s-4}{3}\right\rceil + 2r$. Therefore, $\gamma_r^c(K_r \boxtimes P_s) = \left\lceil\frac{s-4}{3}\right\rceil + 2r$.

Case (iii): $s \equiv 2(mod\ 3)$

Clearly, D_2 is a restrained dominating set of $K_r \boxtimes P_s$ and $\gamma_r(K_r \boxtimes P_s) = \left\lceil\frac{s}{3}\right\rceil$. Also, $D_1 = D_2 \cup \{(u_i, v_2), (u_j, v_3)/2 \leq i \leq r, 1 \leq j \leq r\}$ is a chromatic restrained dominating set of $K_r \boxtimes P_s$ and so, $\gamma_r^c(K_r \boxtimes P_s) = \left\lceil\frac{s-4}{3}\right\rceil + 2r$.

Theorem 2.4 For $r, s \geq 2$, $\gamma_r^c(K_r \boxtimes K_{1,s}) = 2r$.

Proof. Let $V(K_r) = \{u_1, u_2, u_3, \ldots, u_r\}$ and $V(K_{1,s}) = \{v_0, v_1, v_2, v_3, \ldots, v_s\}$ where v_0 is the full degree vertex of $K_{1,s}$. Then, $V(K_r \boxtimes K_{1,s}) = \{(u_i, v_j)/1 \leq i \leq r, 0 \leq j \leq s\}$ and $|V(K_r \boxtimes K_{1,s})| = (s+1)r$. Clearly, $K_r \boxtimes K_{1,s}$ contains r rows and $s+1$ columns $(V_1, V_2, V_3, \ldots, V_{s+1})$ where the induced subgraph of $K_r \boxtimes K_{1,s}$ formed from all the vertices of two columns V_1 and $V_i, i \neq 1$ is a complete subgraph on $2r$ vertices. Then, $\chi(\langle V_1 \cup V_2\rangle) = 2r$ and all the remaining vertices can be colored using r colors used for coloring the column V_2, since there does not exists adjacency between columns $V_2, V_3, \ldots, V_{s+1}$. Thus, $\chi(K_r \boxtimes K_{1,s}) = 2r$. Clearly, $D = \{(u_1, v_0)\}$ is a restrained dominating set, as (u_1, v_0) is a full degree vertex of $K_r \boxtimes K_{1,s}$. Therefore, $\gamma_r(K_r \boxtimes K_{1,s}) = 1$. But, $\chi(\langle D\rangle) = 1 \neq$

$\chi(K_r \boxtimes K_{1,s})$, and so D is not a chromatic restrained dominating set. Consider $D_1 = V_1 \cup V_2$, where $\chi(\langle D_1 \rangle) = 2r = \chi(K_r \boxtimes K_{1,s})$. Since $(u_1, v_0) \in D_1$, D_1 is also a restrained dominating set. Thus, D_1 is a chromatic restrained dominating set of $K_r \boxtimes K_{1,s}$ and $\gamma_r^c(K_r \boxtimes K_{1,s}) \leq |D_1| = |V_1| + |V_2| = 2r$. Since $\chi(K_r \boxtimes K_{1,s}) = 2r$, any minimum chromatic restrained dominating set must contain a minimum of $2r$ vertices. Therefore, $\gamma_r^c(K_r \boxtimes K_{1,s}) \geq 2r$. Hence, $\gamma_r^c(K_r \boxtimes K_{1,s}) = 2r$.

Theorem 2.5 For $r, s, m \geq 2$, $\gamma_r^c(K_m \boxtimes K_{r,s}) = 2m$.

Proof. Let $V(K_m) = \{u_1, u_2, u_3, \dots, u_m\}$ and $V(K_{r,s}) = \{v_1, v_2, v_3, \dots, v_r, v_{r+1}, v_{r+2}, \dots, v_{r+s}\}$. Then, $V(K_m \boxtimes K_{r,s}) = \{(u_i, v_j)/1 \leq i \leq m, 1 \leq j \leq r+s\}$. Clearly, $K_m \boxtimes K_{r,s}$ contains m rows and $r+s$ columns where $V_1, V_2, \dots, V_{r+s}$ denotes the columns. Also, the induced subgraph of $K_m \boxtimes K_{r,s}$ formed from all the vertices of two columns, one among the columns $V_1, V_2, \dots, V_r$ and another among the columns $V_{r+1}, V_{r+2}, \dots, V_{r+s}$ is a complete subgraph on $2m$ vertices. Clearly, the columns $V_1, V_2, \dots, V_r$ can be colored with m colors and the remaining columns $V_{r+1}, V_{r+2}, \dots, V_{r+s}$ can be colored with another m colors. Thus, $\chi(K_m \boxtimes K_{r,s}) = 2m$. Consider a vertex from one of the columns $V_1, V_2, \dots, V_r$ and another vertex from one of the columns $V_{r+1}, V_{r+2}, \dots, V_{r+s}$. So, let $D = \{(u_1, v_r), (u_1, v_{r+1})\}$. Clearly, D is a restrained dominating set and $\gamma_r(K_m \boxtimes K_{r,s}) = 2$. But, $\chi(\langle D \rangle) = 2 \neq \chi(K_m \boxtimes K_{r,s})$ and so, D is not a chromatic restrained dominating set of $K_m \boxtimes K_{r,s}$. Consider $D_1 = V_r \cup V_{r+1}$. Then, D_1 is a restrained dominating set and $\chi(\langle D_1 \rangle) = |V_r| + |V_{r+1}| = 2m = \chi(K_m \boxtimes K_{r,s})$. Thus, D_1 is a chromatic restrained dominating set of $K_m \boxtimes K_{r,s}$ and $\gamma_r^c(K_m \boxtimes K_{r,s}) \leq |D_1| = 2m$. Since $\chi(K_m \boxtimes K_{r,s}) = 2m$, any minimum chromatic restrained dominating set must contain at least $2m$ vertices. Therefore, $\gamma_r^c(K_m \boxtimes K_{r,s}) \geq 2m$. Hence, $\gamma_r^c(K_m \boxtimes K_{r,s}) = 2m$.

Theorem 2.6 For $s \geq 5$ and s is odd, $\gamma_r^c(K_r \boxtimes W_s) = 3r$.

Proof. Let $V(K_r) = \{u_1, u_2, u_3, \dots, u_r\}$ and $V(W_s) = \{v_0, v_1, v_2, \dots, v_{s-1}\}$. Then, $V(K_r \boxtimes W_s) = \{(u_i, v_j)/1 \leq i \leq r, 0 \leq j \leq s-1\}$ where (u_1, v_0) is the full degree vertex in $K_r \boxtimes W_s$. Also, $K_r \boxtimes W_s$ consists of r rows and s columns, where $V_1, V_2, \dots, V_s$ denotes the columns. Clearly, V_1 can be colored with r colors, V_2 can be colored with another r colors , V_3 can be colored with another r colors and the remaining vertices can be colored with one among those $3r$ colors. Then, $\chi(K_r \boxtimes W_s) = 3r$. Clearly, $D = \{(u_1, v_0)\}$ is a restrained dominating set of $K_r \boxtimes W_s$. Then, $\gamma_r(K_r \boxtimes W_s) = 1$. But, $\chi(\langle D \rangle) = 1 \neq \chi(K_r \boxtimes W_s)$. This implies that, D is not a chromatic restrained dominating set of $K_r \boxtimes W_s$. Let $D_1 = V_1 \cup V_2 \cup V_3 = \{(u_1,$

$v_0), (u_2, v_0), \dots, (u_r, v_0), (u_1, v_1), (u_2, v_1), \dots, (u_r, v_1), (u_1, v_2), (u_2, v_2), \dots, (u_r, v_2)\}$. Since the induced subgraph formed from all the vertices of V_1, V_2 and V_3 is a complete graph on $3r$ vertices, $\chi(\langle D_1 \rangle) = 3r = \chi(K_r \boxtimes W_s)$. Also, D_1 is a restrained dominating set. Therefore, D_1 is a chromatic restrained dominating set of $K_r \boxtimes W_s$ and $\gamma_r^c(K_r \boxtimes W_s) \leq |D_1| = 3r$. Since $\chi(K_r \boxtimes W_s) = 3r$, any minimum chromatic restrained dominating set of $K_r \boxtimes W_s$ must contain a minimum of $3r$ vertices and so $\gamma_r^c(K_r \boxtimes W_s) \geq 3r$. Hence, $\gamma_r^c(K_r \boxtimes W_s) = 3r$.

Theorem 2.7 For any $r, s \geq 2$, $\gamma_r^c(K_{1,r} \boxtimes P_s) = \left\lceil \frac{s-4}{3} \right\rceil + 4$.

Proof. Let $V(K_{1,r}) = \{u_0, u_1, u_2, u_3, \dots, u_r\}$ where u_0 is the full degree vertex of $K_{1,r}$ and $V(P_s) = \{v_1, v_2, v_3, \dots, v_s\}$. Then, $V(K_{1,r} \boxtimes P_s) = \{(u_i, v_j)/0 \leq i \leq r, 1 \leq j \leq s\}$ and $|V(K_{1,r} \boxtimes P_s)| = (r+1)s$. Clearly, $K_{1,r} \boxtimes P_s$ consists of $r+1$ rows and s columns denoted as $V_1, V_2, V_3, \dots, V_s$. Now, the first row can be colored with two colors and the remaining r rows can be colored with extra two colors since no two vertices belonging to different rows (among those r rows) are adjacent. This implies that, $\chi(K_{1,r} \boxtimes P_s) = 4$.

Case (i): $s \equiv 0 (mod\ 3)$

Let $D_1 = \{(u_0, v_{3k-1})/1 \leq k \leq \frac{s}{3}\}$ where $|D_1| = \frac{s}{3}$. Then, D_1 is a restrained dominating set since D_1 is a dominating set and $\langle V - D_1 \rangle$ has no vertices of degree one. Thus, $\gamma_r(K_{1,r} \boxtimes P_s) \leq |D_1| = \frac{s}{3}$. Since $\gamma(K_{1,r} \boxtimes P_s) = \frac{s}{3}$, $\gamma_r(K_{1,r} \boxtimes P_s) \geq \frac{s}{3}$. Therefore, $\gamma_r(K_{1,r} \boxtimes P_s) = \frac{s}{3}$. Since D_1 is an independent set, $\chi(\langle D_1 \rangle) = 1 \neq \chi(K_{1,r} \boxtimes P_s)$ and so, D_1 is not a chromatic restrained dominating set. Consider $D_2 = D_1 \cup \{(u_0, v_3), (u_1, v_2), (u_1, v_3)\}$. Then, $\langle D_2 \rangle$ contains a complete subgraph on four vertices $(u_0, v_2), (u_0, v_3), (u_1, v_2)$ and (u_1, v_3). Thus, $\chi(\langle D_2 \rangle) = 4 = \chi(K_{1,r} \boxtimes P_s)$. Also, D_2 is a restrained dominating set. This implies that, D_2 is a chromatic restrained dominating set and $\gamma_r^c(K_{1,r} \boxtimes P_s) \leq |D_2| = |D_1| + 3 = \frac{s}{3} + 3$. Suppose there exists a chromatic restrained dominating set S_1 such that $|S_1| < \frac{s}{3} + 3$. Then $|D_1| < |S_1| < \frac{s}{3} + 3 = |D_1| + 3$ and the only possible case for cardinality of S_1 is either $\frac{s}{3} + 1$ or $\frac{s}{3} + 2$. But, there does not exists a chromatic restrained dominating set with cardinality $\frac{s}{3} + 1$ or $\frac{s}{3} + 2$. Therefore, $\gamma_r^c(K_{1,r} \boxtimes P_s) = \frac{s}{3} + 3 = \left\lceil \frac{s-4}{3} \right\rceil + 4$.

Case (ii): $s \equiv 1 (mod\ 3)$

Let $D_3 = \{(u_0, v_{3k-1})/1 \le k \le \frac{s-1}{3}\} \cup \{(u_0, v_s)\}$ where $|D_3| = \frac{s-1}{3} + 1 = \left\lceil\frac{s}{3}\right\rceil$. Then, D_3 is a dominating set and $\langle V - D_3\rangle$ has no isolated vertices. Thus, D_3 is a restrained dominating set and $\gamma_r(K_{1,r} \boxtimes P_s) \le |D_3| = \left\lceil\frac{s}{3}\right\rceil$. Since any minimum dominating set must contain the first vertex of each column $V_{3j-1}, 1 \le j \le \frac{s-1}{3}$ together with the first vertex of column V_3 or the first vertex of each column V_{3j+1}, $1 \le j \le \frac{s-1}{3}$ together with the first vertex of column V_1 or the first vertex of each column V_{3j}, $1 \le j \le \frac{s-1}{3}$ together with the first vertex of column V_1 and so on, $\gamma(K_{1,r} \boxtimes P_s) = \frac{s-1}{3} + 1 = \left\lceil\frac{s}{3}\right\rceil$. Thus, $\gamma_r(K_{1,r} \boxtimes P_s) \ge \left\lceil\frac{s}{3}\right\rceil$ and so, $\gamma_r(K_{1,r} \boxtimes P_s) = \left\lceil\frac{s}{3}\right\rceil$. Since D_3 is independent, $\chi(\langle D_3\rangle) = 1 \ne \chi(K_{1,r} \boxtimes P_s)$. This implies that, D_3 is not a chromatic restrained dominating set. Consider $D_4 = \{(u_0, v_2), (u_0, v_3), (u_1, v_2), (u_1, v_3), (u_0, v_{3k})/2 \le k \le \frac{s-1}{3}\}$. Then, D_4 is a restrained dominating set since the columns V_1, V_2, V_3 and V_4 are dominated by the vertices $(u_0, v_2), (u_0, v_3), (u_1, v_2), (u_1, v_3)$ and all the remaining vertices are adjacent to one of the vertex in $\{(u_0, v_{3k})/2 \le k \le \frac{s-1}{3}\}$. Also, $\langle D_4\rangle$ contains K_4 as an induced subgraph and so, $\chi(\langle D_4\rangle) = 4$. Therefore, D_4 is a chromatic restrained dominating set of $K_{1,r} \boxtimes P_s$ and $\gamma_r^c(K_{1,r} \boxtimes P_s) \le |D_4| = \frac{s-1}{3} + 3 = \left\lceil\frac{s-4}{3}\right\rceil + 4$. Suppose there exists a chromatic restrained dominating set S_2 such that $|S_2| < \frac{s-1}{3} + 3$. Then $|D_3| < |S_2| < \frac{s-1}{3} + 3 = |D_3| + 2$ and the only possible cardinality of S_2 is $\frac{s-1}{3} + 2$. But there does not exists a chromatic restrained dominating set with cardinality $\frac{s-1}{3} + 2$. Therefore, $\gamma_r^c(K_{1,r} \boxtimes P_s) \ge \frac{s-1}{3} + 3$. Hence, $\gamma_r^c(K_{1,r} \boxtimes P_s) = \frac{s-1}{3} + 3 = \left\lceil\frac{s-4}{3}\right\rceil + 4$.

Case (iii): $s \equiv 2(mod\ 3)$

Let $D_5 = \{(u_0, v_{3k-1})/1 \le k \le \left\lceil\frac{s}{3}\right\rceil\}$ with cardinality $\left\lceil\frac{s}{3}\right\rceil$. Since D_5 is a dominating set and $\langle V - D_5\rangle$ has no isolated vertices, D_5 is a restrained dominating set. Thus, $\gamma_r(K_{1,r} \boxtimes P_s) \le |D_5| = \left\lceil\frac{s}{3}\right\rceil$. Since $\gamma(K_{1,r} \boxtimes P_s) = \left\lceil\frac{s}{3}\right\rceil$, $\gamma_r(K_{1,r} \boxtimes P_s) \ge \left\lceil\frac{s}{3}\right\rceil$. Therefore, $\gamma_r(K_{1,r} \boxtimes P_s) = \left\lceil\frac{s}{3}\right\rceil$. But every minimum restrained dominating set is independent and so, $\chi(\langle D_5\rangle) = 1 \ne \chi(K_{1,r} \boxtimes P_s)$. This indicates that, D_5 is not a chromatic restrained dominating set. Consider $D_6 = D_5 \cup \{(u_0, v_3), (u_1, v_2), (u_1, v_3)\}$. Then $\langle D_6\rangle$ contains K_4 as an induced subgraph and so, $\chi(\langle D_6\rangle) = 4 = \chi(K_{1,r} \boxtimes P_s)$. Also, D_6 is a restrained dominating set. Therefore, D_6 is a chromatic restrained dominating set of $K_{1,r} \boxtimes P_s$. Thus, $\gamma_r^c(K_{1,r} \boxtimes$

$P_s) \leq |D_6| = \left\lceil \frac{s}{3} \right\rceil + 3 = \left\lceil \frac{s-4}{3} \right\rceil + 4$. Since $\chi(K_{1,r} \boxtimes P_s) = 4$ and any $K_{1,r} \boxtimes P_s$ contains K_4 as an induced subgraph, any chromatic restrained dominating set must contain all the four vertices of K_4 so that, $\chi(K_{1,r} \boxtimes P_s) = 4$. Let it be $(u_0, v_2), (u_0, v_3), (u_1, v_2), (u_1, v_3)$ which dominates all the vertices in columns V_1, V_2, V_3, V_4. From the remaining $s-4$ columns, choosing the first vertex of each column $V_{3k+2}, 1 \leq k \leq \left\lceil \frac{s-4}{3} \right\rceil$, we get a minimum chromatic restrained dominating set. Thus, $\gamma_r^c(K_{1,r} \boxtimes P_s) \geq \left\lceil \frac{s-4}{3} \right\rceil + 4$. Therefore, $\gamma_r^c(K_{1,r} \boxtimes P_s) = \left\lceil \frac{s-4}{3} \right\rceil + 4$.

Theorem 2.8 Let $r, s \geq 2$. Then $\gamma_r^c(K_{1,r} \boxtimes K_{1,s}) = 4$.

Proof. Let $V(K_{1,r}) = \{u_0, u_1, u_2, \ldots, u_r\}$ and $V(K_{1,s}) = \{v_0, v_1, v_2, \ldots, v_s\}$. Then, $V(K_{1,r} \boxtimes K_{1,s}) = \{(u_i, v_j)/0 \leq i \leq r, 0 \leq j \leq s\}$ where (u_0, v_0) is the full degree vertex of $K_{1,r} \boxtimes K_{1,s}$. Clearly, $D = \{(u_0,\ v_0)\}$ is a restrained dominating set and $\gamma_r(K_{1,r} \boxtimes K_{1,s}) = 1$. Let $V_1,\ V_2,\ V_3, \ldots, V_{r+1}$ denotes the $r+1$ rows of $K_{1,r} \boxtimes K_{1,s}$ where V_1 can be colored with two colors and the remaining r rows can be colored with another two colors. Thus, $\chi(K_{1,r} \boxtimes K_{1,s}) = 4$. But, $\chi(\langle D \rangle) = 1 \neq \chi(K_{1,r} \boxtimes K_{1,s})$ and so, D is not a chromatic restrained dominating set. Let $D_1 = \{(u_0, v_0), (u_0, v_1), (u_1, v_0), (u_1, v_1)\}$ where $\langle D_1 \rangle = K_4$. This implies that, $\chi(\langle D_1 \rangle) = 4$ and D_1 is also a restrained dominating set. Thus, D_1 is a chromatic restrained dominating set of $K_{1,r} \boxtimes K_{1,s}$ and $\gamma_r^c(K_{1,r} \boxtimes K_{1,s}) \leq |D_1| = 4$. Since $\chi(K_{1,r} \boxtimes K_{1,s}) = 4$, any minimum chromatic restrained dominating set must contain at least four vertices and so, $\gamma_r^c(K_{1,r} \boxtimes K_{1,s}) \geq 4$. Therefore, $\gamma_r^c(K_{1,r} \boxtimes K_{1,s}) = 4$.

Theorem 2.9 For any $m, r, s \geq 2$, $\gamma_r^c(K_{1,m} \boxtimes K_{r,s}) = 4$.

Proof. Let $V(K_{1,m}) = \{u_0, u_1, u_2, \ldots, u_m\}$ and $V(K_{r,s}) = \{v_1,\ v_2,\ v_3, \ldots,\ v_r,\ v_{r+1},\ v_{r+2}, \ldots, v_{r+s}\}$ where u_0 is the full degree vertex of $K_{1,m}$. Now, $V(K_{1,m} \boxtimes K_{r,s}) = \{(u_i, v_j)/0 \leq i \leq m, 1 \leq j \leq r + s\}$ and $|V(K_{1,m} \boxtimes K_{r,s})| = (m+1)(r+s)$. Also, there exists $m+1$ rows and $r+s$ columns in $K_{1,m} \boxtimes K_{r,s}$. Clearly, the first row can be colored with two colors and the remaining m rows can be colored with another two colors since there does not exists adjacency between any two vertices belonging to those m different rows. Thus, $\chi(K_{1,m} \boxtimes K_{r,s}) = 4$. Let $D = \{(u_0, v_r), (u_0, v_{r+1})\}$. Then D is a restrained dominating set as D is a dominating set and every vertex in $V - D$ is adjacent to at least one another vertex in $V - D$. Thus, $\gamma_r(K_{1,m} \boxtimes K_{r,s}) \leq 2$. Since there does not exists a full degree vertex in $K_{1,m} \boxtimes K_{r,s}$, $\gamma_r(K_{1,m} \boxtimes K_{r,s}) < 2$ is impossible. Therefore, $\gamma_r(K_{1,m} \boxtimes K_{r,s}) = 2$. But $\chi(\langle D \rangle) = 2$ and so, D is not a chromatic restrained dominating set of $K_{1,m} \boxtimes K_{r,s}$. Consider $D_1 = D \cup$

$\{(u_1, v_r), (u_1, v_{r+1})\}$. Then, $\langle D_1 \rangle$ is a complete graph on four vertices and $\chi(\langle D_1 \rangle) = 4 = \chi(K_{1,m} \boxtimes K_{r,s})$. Also, D_1 is a restrained dominating set. Thus, D_1 is a chromatic restrained dominating set and $\gamma_r^c(K_{1,m} \boxtimes K_{r,s}) \le |D_1| = 4$. Since $\chi(K_{1,m} \boxtimes K_{r,s}) = 4$, $\gamma_r^c(K_{1,m} \boxtimes K_{r,s}) \ge 4$. Therefore, $\gamma_r^c(K_{1,m} \boxtimes K_{r,s}) = 4$.

Theorem 2.10 For any $r \ge 3, s \ge 4$, $\gamma_r^c(K_{1,r} \boxtimes W_s) = \begin{cases} 6 \ if \ \ s \ is \ odd \\ 2s \ if \ \ s \ is \ even \end{cases}$.

Proof. Let $V(K_{1,r}) = \{u_0, u_1, u_2, \dots, u_r\}$ and $V(W_s) = \{v_0, v_1, v_2, \dots, v_{s-1}\}$ where u_0 and v_0 are the full degree vertices of $K_{1,r}$ and W_s respectively. Then, $V(K_{1,r} \boxtimes W_s) = \{(u_i, v_j)/0 \le i \le r, 0 \le j \le s-1\}$ where $|V(K_{1,r} \boxtimes W_s)| = (r+1)s$. Also, $deg(u_0, v_0) = (r+1)s - 1$. Clearly, $K_{1,r} \boxtimes W_s$ contains $r+1$ rows $(V_1, V_2, V_3, \dots, V_{r+1})$ and s columns.

Case (i): s is odd

Then the first row of $K_{1,r} \boxtimes W_s$ can be colored with three colors and the second row can be colored with another three colors. Since there does not exists adjacency between any two vertices belonging to different rows of $V_2, V_3, \dots, V_{r+1}$, all the r rows can be colored with three colors. Then, $\chi(K_{1,r} \boxtimes W_s) = 6$. Since (u_0, v_0) is the full degree vertex, $D = \{(u_0, v_0)\}$ is a restrained dominating set of $K_{1,r} \boxtimes W_s$ and $\gamma_r(K_{1,r} \boxtimes W_s) = 1$. But $\chi(\langle D \rangle) = 1$ and so, D is not a chromatic restrained dominating set. Let $D_1 = \{(u_0, v_0), (u_0, v_1), (u_0, v_2), (u_1, v_0), (u_1, v_1), (u_1, v_2)\}$ where $\langle D_1 \rangle = K_6$ and $|D_1| = 6$. Then, $\chi(\langle D_1 \rangle) = 6 = \chi(K_{1,r} \boxtimes W_s)$ and D_1 is a restrained dominating set. Therefore, D_1 is a chromatic restrained dominating set and $\gamma_r^c(K_{1,r} \boxtimes W_s) \le |D_1| = 6$. Since $\chi(K_{1,r} \boxtimes W_s) = 6$, any chromatic restrained dominating set must contain at least six vertices. Thus, $\gamma_r^c(K_{1,r} \boxtimes W_s) \ge 6$. Therefore, $\gamma_r^c(K_{1,r} \boxtimes W_s) = 6$.

Case (ii): s is even

Then the first row V_1 can be colored with four colors. Since, some of the vertices in V_1 and V_2 are adjacent, the second row V_2 can be colored by introducing three more colors. Also, the remaining rows $V_3, V_4, \dots, V_{r+1}$ can be colored by assigning the same colors as in V_2. Thus, $\chi(K_{1,r} \boxtimes W_s) = 7$. Let $D_2 = \{(u_0, v_0), (u_0, v_1), (u_0, v_2), \dots, (u_0, v_{s-1}), (u_1, v_0), (u_1, v_1), (u_1, v_2), \dots, (u_1, v_{s-2})\}$. Then $\chi(\langle D_2 \rangle) = 7 = \chi(K_{1,r} \boxtimes W_s)$. But D_2 is not a restrained dominating set since $(u_1, v_{s-1}) \in V - D_2$ has no adjacent vertex in $V - D_2$. So, consider $D_3 = D_2 \cup \{(u_1, v_{s-1})\}$. Then, $\chi(\langle D_3 \rangle) = 7$ and D_3 is a restrained dominating set. Thus, D_3 is a chromatic restrained dominating set of $K_{1,r} \boxtimes W_s$ and

$\gamma_r^c K_{1,r} \boxtimes W_s) \leq |D_3| = |D_2| + 1 = 2s$. Since s is even and $\chi(K_{1,r} \boxtimes W_s) = 7$, any minimum chromatic restrained dominating set of $K_{1,r} \boxtimes W_s$must contain at least $2s$ vertices and so, $\gamma_r^c(K_{1,r} \boxtimes W_s) \geq 2s$. Therefore, $\gamma_r^c(K_{1,r} \boxtimes W_s) = 2s$.

3. Conclusion

In this article, the chromatic restrained domination number on the strong product of certain standard graphs are obtained. A promising avenue for future research is to investigate the bounds on the strong product of graphs and identify the extremal graphs that define the upper and lower limits of the chromatic restrained domination number in such products.

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