Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No.10S Y (2025)

Three-Dimensional Nonlinear Fredholm Integral Equation of The Second Kind Solved by Comparing Fibonacci Collocation Method and Hermite-Galerkin Method

M. H. Ahmed and S. H. Aljahdali

Taibah University, College of Science, Department of Mathematics, P.O.Box 344, Madinah,3002, Saudi Arabia.

Article History:

Received: *01/01/2025*

Revised: *06/02/2025*

Accepted: *10/03/2025*

Abstract: In this article, we present the numerical solutions of three-dimensional nonlinear Fredholm integral equations using two different polynomial-based approaches. The first method employs Fibonacci polynomials within the collocation method, while the second method applies Hermite polynomials through the Galerkin projection technique. A comparative study between the two proposed approaches is carried out, and the obtained results are also contrasted with those available in related works. The numerical results are illustrated in a series of tables and figures, demonstrating the efficiency and accuracy of the methods.

Introduction: The three-dimensional nonlinear Fredholm integral equations (3D-NLFIEs) of the second kind are fundamental tools in modelling a wide range of problems in applied mathematics, physics, and engineering. These equations naturally arise in mathematical models of electromagnetic scattering, quantum mechanics, heat transfer, and population dynamics [6, 12, 14]. In three dimensions, the nonlinear structure combined with the multi-variable kernel makes analytical solutions intractable, which emphasizes the importance of developing efficient and accurate numerical methods. It is worth mentioning that several recent studies have applied second-kind Fredholm integral equations to advanced fields such as electromagnetic wave propagation in dielectric gratings. For example, the paper [15] presents a volume integral equation (VIE) formulation combined with the Galerkin method, which enables stable and accurate solutions for the scattering phenomena in multilayer dielectric gratings. This work highlights the significance of numerical techniques based on integral equations in addressing complex and practical problems in physics and applied mathematics, which is in line with our research focus on developing numerical approaches for solving nonlinear Fredholm integral equations [9].

Objectives: This paper consists of five sections: an introduction and definition of the 3D-NLFIEs, a statement of objectives, a review of Fibonacci and Hermite polynomials with the application of the Fibonacci collocation and Hermite–Galerkin methods, results including existence and uniqueness proofs and numerical comparisons, and finally a discussion analysing the accuracy and efficiency of the methods.

Methods: We employed the Fibonacci collocation method and the Hermite–Galerkin method to solve 3D-NLFIEs.

Results: Established the existence and uniqueness of the solution for the 3D-NLFIE and obtained numerical solutions using MATLAB by applying both methods. A comparative

analysis was then conducted between the two methods, and the results were reported in previous studies [7,8].

Conclusions: The results demonstrated that the Fibonacci collocation method consistently outperformed the Hermite–Galerkin method for different values of N, achieving higher accuracy and lower numerical errors across most test points. This highlights the effectiveness of the Fibonacci approach in improving convergence and solution quality compared to the Hermite–Galerkin method under the same number of N.

Keywords: Three-dimensional nonlinear Fredholm integral equations; Fibonacci polynomials; Hermite polynomials; Collocation method; Galerkin method

1. Introduction

A general form of the three-dimensional nonlinear Fredholm integral equation of the second kind can be written as

$$u(x_1, y_1, z_1) - \lambda \int_a^b \int_c^d \int_e^f k(x_1, y_1, z_1, s_1, t_1, r_1)\, Q\big(u(s_1, t_1, r_1)\big) ds_1\, dt_1\, dr_1 = g(x_1, y_1, z_1),$$

$$\forall\ (x_1, y_1, z_1) \in D \tag{1}$$

where u is the unknown function. In domain D is usually taken as unit cube $[0,1]^3$, $k(x_1, y_1, z_1, s_1, t_1, r_1)$ is the kernel function and g is a given source function In domain $D \times D$ and D. Depending on the properties of the kernel (continuous, symmetric, degenerate, or singular), the complexity of both analysis and computation varies significantly [12]. Several numerical approaches have been developed for solving 3D-NLFIEs. Basseem [1] implemented the degenerate kernel technique to approximate solutions of three-dimensional nonlinear integral equa tions. Kazemi et al. [8] introduced a Haar wavelet-based procedure for solving such equations, while Maleknejad et al. [10] employed Bernstein polynomials for the numerical solution of two-dimensional nonlinear Fredholm integral equations, which was later extended to higher dimensions. These contributions demonstrate the continuous effort to establish reliable approximation schemes for three-dimensional nonlinear integral equations. In this study, we employ and compare two polynomial-based approximation techniques: the Fibonacci Collocation Method and the Hermite–Galerkin Method. Fibonacci polynomials, derived from general izations of the Fibonacci sequence, have shown promising results in solving various types of integral and differential equations. Hermite polynomials, with their well-known orthogonality properties, are widely applied in Galerkin-type schemes for integral equations. We further compare our results with three other numerical methods available in the literature [1,7,8].

2. Objectives

The remainder of this paper is organized as follows. Section 1 presents an introduction and a formal definition of the 3D-NLFIEs. Section 2 outlines the content and purpose of each section of the study. In section 3 reviews the Fibonacci and Hermite polynomials, discusses some of their key properties, and demonstrates the application of the Fibonacci collocation method and the Hermite–Galerkin method to the 3D-NLFIEs. Section 4 establishes the existence and uniqueness of the solution for the nonlinear three-dimensional Fredholm integral equation and applies the two methods to several illustrative examples, followed by a comparison with results reported in previous studies. Finally, Section 5 provides a critical analysis of the obtained results, highlighting the efficiency and accuracy of the proposed methods in comparison with other techniques.

3. Methods

3.1 Fibonacci Polynomials.

[2–5] Leonardo of Pisa, commonly known as Fibonacci, was an Italian mathematician from the 13^{th} century. In 1202, he posed and solved a problem related to the growth of a rabbit population under idealized conditions. The resulting sequence is now known as the Fibonacci numbers:

$$0,1,1,2,3,5,8,\cdots$$

This sequence gained widespread attention for its elegant structure and natural occurrence in various scientific phenomena. In 1883, Belgian mathematician Eugene Charles Catalan and German mathematician E. Jacobsthal studied broader classes of polynomials inspired by Fibonacci numbers. These are known as *Fibonacci polynomials*, typically denoted by $F_n(x_1)$.This class of polynomials plays a significant role in both theoretical and applied mathematics. Nu- merous properties and identities involving $F_n(x_1)$ have been studied due to their wide applicability. Fibonacci polynomials also appear in the solution of second-order ordinary differential equations:

$$\frac{d^2y_1}{dx_1^2} - x_1\frac{dy_1}{dx_1} - y_1 = 0,$$

whose solution for $n \in N_0$is:

$$F_n(x_1) = \frac{1}{\alpha - \beta}(\alpha^n - \beta^n),$$

where

$$\alpha = \frac{x_1 + \sqrt{x_1^2 + 4}}{2}, \ \beta = \frac{x_1 - \sqrt{x_1^2 + 4}}{2}$$

An explicit representation of Fibonacci polynomials is given as follows, :

$$F_{n+1}(x_1) = \sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}x_1^{n-2k}, n \geq 0.$$

The first few terms are:

$$F_1(x_1) = 1,$$

$$F_2(x_1) = x_1,$$

$$F_3(x_1) = x_1^2 + 1,$$

$$F_4(x_1) = x_1^3 + 2x_1,$$

$$F_5(x_1) = x_1^4 + 3x_1^2 + 1,$$

In what follows, we present some fundamental properties of Fibonacci polynomials:

Properties of Fibonacci Polynomials.

(Recurrence Relation) Fibonacci polynomials satisfy the recurrence relation:

$$F_{n+2}(x_1) = x_1F_{n+1}(x_1) + F_n(x_1),$$

with:

$$F_1(x_1) = 1, \ F_2(x_1) = x_1.$$

Notably: If $x_1 = 1$ then $F_n(1) = F_{n+1}$, if $x_1 = 0$ then $F_{2n}(0) = 0$ for $n = 1,2,...$

(Extrema) The extrema of $F_n(x_1)$ over $[a, b]$ are:

$$x_{1k} = a + \frac{b-a}{n+1}k, \ k = 1,2,\dots,n+1.$$

(Generating Function) The generating function is given by:

$$\sum_{n=0}^{\infty} F_n(x_1) z_1^n = \frac{1}{1 - x_1 z_1 - z_1^2}.$$

(Evenness and Oddness) $F_1(x_1)$ is even when n is odd, and odd when n is even.

3.2 Hermite Polynomials [11,13,17]**.**

Hermite polynomials, widely used in both pure and applied mathematics, have a rich historical back ground. While a form of these polynomials was introduced by Pierre-Simon Laplace as early as 1810, it was Charles Hermite who later provided a formal definition for a broader class of what are now known as generalized Hermite polynomials—an extension that remains relatively underexplored compared to the classical case . Hermite polynomials have gained significant importance in recent decades due to their applications in quantum mechanics, engineering, physics, and various other scientific disciplines. Their mathematical utility and orthogonality properties make them essential tools in the analysis of complex systems. The classical Hermite polynomials arise as solutions to the second-order linear differential equation defined on the interval $(-\infty, \infty)$, given by:

$$\frac{d^2 y_1}{dx_1^2} - 2x_1 \frac{dy_1}{dx_1} + 2ny_1 = 0,$$

Where $n \geq 0$. The solution corresponding to a given n is denoted by $H_n(x_1)$ and is referred to as Hermite polynomial of degree n. As with other families of classical orthogonal polynomials, Hermite polynomials can be introduced and derived through various equivalent formulations, including differential equations, generating functions, and recurrence relations.In what follows, we present some fundamental properties of Hermite polynomials .

Properties of the Hermite Polynomials. (Rodrigues' Formula) Hermite polynomials $H_n(x_1)$ can be expressed using Rodrigues' formula, which involves repeated differentiation of the Gaussian function. The formula is given by:

$$H_n(x_1) = (-1)^n e^{x_1^2} \frac{d^n}{dx_1^n}\left(e^{-x_1^2}\right),$$

For $n = 0,1,2,3, \dots$.

The first few Hermite polynomials in terms of $x_1$are:

$$H_0(x_1) = 1,$$
$$H_1(x_1) = 2x_1,$$
$$H_2(x_1) = 4x_1^2 - 2,$$
$$H_3(x_1) = 8x_1^3 - 12x_1,$$
$$H_4(x_1) = 16x_1^4 - 48x_1^2 + 12,$$
$$H_5(x_1) = 32x_1^5 - 160x_1^3 + 120,$$
$$H_6(x_1) = 64x_1^6 - 480x_1^4 + 720x_1^2 - 120,$$

(Recurrence Relation) The fundamental recurrence relation of $H_n(x_1)$ is:

$$H_{n+2}(x_1) = 2x_1 H_{n+1}(x_1) - 2nH_n(x_1),$$

For $n = 0,1,2,3, \dots$, which can generate $H_n(x_1)$ efficiently and easily with the initial terms:

$$H_0(x_1) = 1, H_1(x_1) = 2x_1.$$

(Orthogonality) Hermite polynomials $H_n(x_1)$ form an orthogonal set on the interval $[-\infty, \infty]$ with respect to the weight function $e^{-x_1^2}$. This means:

$$\langle H_m(x_1), H_n(x_1)\rangle = \int_{-\infty}^{\infty} e^{-x_1^2} H_m(x_1) H_n(x_1) dx_1 = \begin{cases} 0\,, & m \neq n \\ 2^n \pi^{\frac{1}{2}} n!\,, & m = n \end{cases},$$

(Generating Function) The generating function for $H_n(x_1)$ is

$$e^{-z_1^2 + 2z_1 x_1} = \sum_{n=0}^{\infty} H_n(x_1)\, \frac{z_1^n}{n!}.$$

Oddness and Evenness It is clear that the polynomial $H_n(x_1)$ is even or odd in x_1, according to whether n is even or odd.

3.3 Fibonacci Collocation Method: Consider the which is represented the approximate solution:

$$u_N(x_1, y_1, z_1) \approx \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} \sum_{k=1}^{N+1} c_{ijk} F_i(x_1) F_j(y_1) F_k(z_1). \tag{2}$$

Where $F_i\,(x_1), F_j(y_1)$ and $F_k(z_1)$ are Fibonacci polynomial of degree i, j, k, c_{ijk} the coefficients are determined choose the collocation points (x_n, y_m, z_l) by dividing the intervals $[a, b], [c, d]$ and $[e, f]$ into $N + 1$, respectively:

$$x_{1n} = a + \frac{(b-a)}{N+1} n, \qquad y_{1m} = c + \frac{(d-c)}{N+1}\, m, \qquad z_{1l} = e + \frac{(f-e)}{N+1}\, l, \qquad n, m, l = 1,2, \dots, N+1. \tag{3}$$

First substituting Eq.(2) into three-dimensional nonlinear Fredholm integral equations (1) at use the collocation points (x_n, y_m, z_l):

$$\sum_{i=1}^{N+1} \sum_{j=1}^{N+1} \sum_{k=1}^{N+1} c_{ijk} F_i(x_1) F_j(y_1) F_k(z_1)$$
$$- \lambda \int_a^b \int_c^d \int_e^f k(x_1, y_1, z_1, s_1, t_1, r_1)\, Q \left(\sum_{i=1}^{N+1} \sum_{j=1}^{N+1} \sum_{k=1}^{N+1} c_{ijk} F_i(x_1) F_j(y_1) F_k(z_1)\,. \right) ds_1\, dt_1\, dr_1$$
$$= g(x_1, y_1, z_1). \tag{4}$$

Then the coefficients c_{ijk} are determined by solving the nonlinear system of matrix iteratively:

$$A \;\cdot c = B$$

Where A is a matrix of size $(N+1)^3 \times (N+1)^3$, B is a matrix of size $(N+1)^3 \times 1$ and c is a matrix of size $(N+1)^3 \times 1$ defined by $\left[c_{111}\; c_{112} \dots c_{(N+1)^3}\right]^T$. Thus we have obtained a nonlinear system of equations which contains $(N+1)^2$ unknown coefficients. solving this system to obtain the values of these coefficients, we arrive at the approximate solution $u_N(x_1\;, y_1, z_1)$.

3.4 Hermite–Galerkin Method. .Consider the which is represented the approximate solution:

$$u_N(x_1, y_1, z_1) \approx \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} \sum_{k=1}^{N+1} c_{ijk} H_i(x_1) H_j(y_1) H_k(z_1). \tag{5}$$

Where $H_i\,(x_1), H_j(y_1)$ and $H_k(z_1)$ are Hermite polynomial, c_{ijk} the coefficients are determined.
First substituting Eq.(5) into three-dimensional nonlinear Fredholm Integral Equations Eq.(1) and applying the Galerkin condition leads to the following equation:

$$\int_a^b \int_c^d \int_e^f \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} \sum_{k=1}^{N+1} c_{ijk} H_i(x_1) H_j(y_1) H_k(z_1) H_q(x_1) H_p(y_1) H_v(z_1)\, dx_1 dy_1 dz_1$$
$$- \int_a^b \int_c^d \int_e^f (W) H_q(x_1) H_p(y_1) H_v(z_1)\, dx_1\, dy_1\, dz_1$$
$$= \int_a^b \int_c^d \int_e^f g(x_1, y_1, z_1) H_q(x_1) H_p(y_1) H_v(z_1)\, dx_1\, dy_1\, dz_1 \tag{6}$$

where

$$W = \lambda \int_a^b \int_c^d \int_e^f k(x_1, y_1, z_1, s_1, t_1, r_1)\ Q\left(\sum_{i=1}^{N+1}\sum_{j=1}^{N+1}\sum_{k=1}^{N+1} c_{ijk} H_i(x_1) H_j(y_1) H_k(z_1).\right) ds_1\, dt_1\, dr_1$$

For all $q, p, v = 1, 2, \dots, N+1$. The coefficients c_{ijk} are determined by solving the nonlinear system of matrix iteratively:

$$A \ \cdot c = B$$

Where A is a matrix of size $(N+1)^3 \times (N+1)^3$, B is a matrix of size $(N+1)^3 \times 1$ and c is a matrix of size $(N+1)^3 \times 1$ defined by $\left[c_{111}\ c_{112}\ \dots c_{(N+1)^3}\ \right]^T$.Thus, we have obtained a nonlinear system of equations which contains $(N+1)^2$ unknown coefficients. solving this system to obtain the values of these coefficients, we arrive at the approximate solution $u_N(x_1\ , y_1, z_1)$.

4. Results

4.1 Existence and Uniqueness of Solution for Three-Dimensional Nonlinear Fredholm Integral Equation

We consider the nonlinear Fredholm integral equation in three dimensions Eq(1):

$$u(x_1, y_1, z_1) - \lambda \int_a^b \int_c^d \int_e^f k(x_1, y_1, z_1, s_1, t_1, r_1)\ Q\big(u(s_1, t_1, r_1)\big) ds_1\, dt_1\, dr_1 = g(x_1, y_1, z_1),$$

Where the domain $D = [a, b] \times [c, d] \times [e, f]$, and Q is a nonlinear function.

Theorem 1. Let $X = C([a, b] \times [c, d] \times [e, f])$ be the Banach space of continuous functions $u(x_1, y_1, z_1)$, equipped with the supremum norm:

$$\|u\| = sup_{(x_1, y_1, z_1) \in [c,d] \times [e,f]} \left|u(x_1, y_1, z_1)\right|.$$

Assume that:

(1) The kernel $k(x_1, y_1, z_1, s_1, t_1, r_1)$ is continuous on $D \times D$,

(2) The function $Q: R \longrightarrow R$ satisfies a Lipschitz condition:

$$|Q(u) - Q(v)| \le L|u - v| \qquad \forall u, v \in R,$$

(3) The constant $C = LM\ < 1$, where

$$M = sup_{(x_1, y_1, z_1) \in \mathrm{D}} \int_a^b \int_c^d \int_e^f |k(x_1, y_1, z_1, s_1, t_1, r_1)| ds_1\, dt_1 dr_1.$$

Then, the integral equation has a unique solution $u^* \in X$.

Proof. Define the operator $T : X\ \to X$ by

$$(Tu)(x_1, y_1, z_1) = g(x_1, y_1, z_1) + \int_a^b \int_c^d \int_e^f k(x_1, y_1, z_1, s_1, t_1, r_1)\ Q\big(u(s_1, t_1, r_1)\big) ds_1\, dt_1\, dr_1$$

Let $u, v \in X$. Then

$$\begin{aligned} &|(Tu)(x_1, y_1, z_1) - (Tv)(x_1, y_1, z_1)| \\ &\quad = \left|\int_a^b \int_c^d \int_e^f k(x_1, y_1, z_1, s_1, t_1, r_1)\left[\ Q\big(u(s_1, t_1, r_1)\big) -\ Q\big(v(s_1, t_1, r_1)\big)\right] ds_1\, dt_1\, dr_1\right| \\ &\quad \le L \int_a^b \int_c^d \int_e^f |k(x_1, y_1, z_1, s_1, t_1, r_1)| \cdot |u(s_1, t_1, r_1) - v(s_1, t_1, r_1)| ds_1\, dt_1 dr_1 \\ &\quad \le L \int_a^b \int_c^d \int_e^f |k(x_1, y_1, z_1, s_1, t_1, r_1)|\, ds_1\, dt_1 dr_1 \cdot \|u - v\|. \end{aligned}$$

Taking the supremum over $(x_1, y_1, z_1) \in D, we\ get$:

$$\|Tu - Tv\| \le LM\|u - v\| = C\|u - v\|.$$

Since $C < 1$, the operator T is a contraction. By Banach's Fixed point Theorem, there exists a unique fixed point $u^* \in X$

Such that $Tu^* = u^*$, i.e, $u^*$is the unique solution of the integral equation. □

4.2 Numerical examples

We present the numerical results for the three-dimensional nonlinear Fredholm integral equations. Tables containing the approximate solution values are provided, along with three-dimensional graphical representations of the solutions obtained using MATLAB. The results are derived by employing the proposed Fibonacci and Hermite-Galerkin methods and further compared with the other methods discussed earlier.

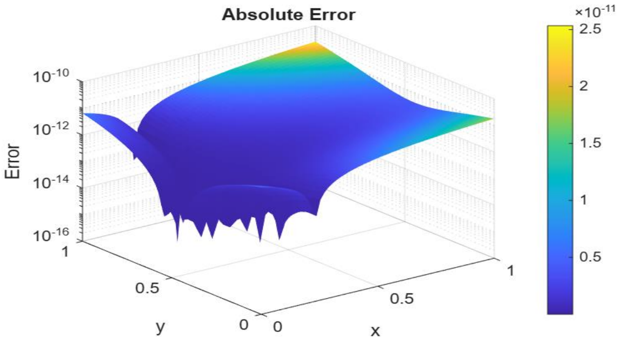

Figure 1. Absolute error of Example 1 by FC method, $N = 2$, and $z_1 = 0.5$.

Example1. Consider the following 3D-NFIE [7,8]:

$$u(x_1, y_1, z_1) = g(x_1, y_1, z_1) + \int_0^1\int_0^1\int_0^1 \frac{1}{100}\, x_1^2 s_1 (t_1^2 + y_1) r_1 z_1 u^2(s_1, t_1, r_1)\ ds_1\, dt_1\, dr_1, \quad (x_1, y_1, z_1) \in [0,1]^3 \quad (7)$$

Where $g(x_1, y_1, z_1) = x_1^2 y_1^2 z_1 - \frac{1}{16800} x_1^2 z_1 - \frac{1}{12000} x_1^2 y_1 z_1$, and exact solution is $u(x_1, y_1, z_1) = x_1^2 y_1^2 z_1$.

The comparison of absolute errors of equation(7) for different values of (x_1, y_1, z_1) ,between Fibonacci collocation (FC), Hermite–Galerkin (H-G) methods with N=2, the Successive Approximations Method at N=10 [8], and the Haar Wavelet Method at 2N=8 [7] ,are presented in Table (1). Figures (1) and (2) show the absolute error distribution obtained by both methods.

TABLE 1. Numerical results for Example 1, $z_1 = 0.5$

(x_1, y_1, z_1)	Fibonacci collocation method $N = 2$	Hermite-Galerkin method $N = 2$	Method of [8] with $N = 10$	Method of [7] with $2N = 8$
(0,0,0)	1.4727×10^{-11}	8.729×10^{-9}	0.	0.
(0.1,0.1,0.1)	3.9781×10^{-12}	2.3279×10^{-9}	1.549×10^{-9}	3.51×10^{-9}
(0.2,0.2,0.2)	7.7772×10^{-13}	5.6042×10^{-11}	1.345×10^{-8}	3.075×10^{-8}
(0.3,0.3,0.3)	3.7868×10^{-13}	4.2694×10^{-10}	4.9181×10^{-8}	1.127×10^{-7}
(0.4,0.4,0.4)	6.0653×10^{-13}	1.6048×10^{-9}	1.2552×10^{-7}	2.884×10^{-7}
(0.5,0.5,0.5)	1.1211×10^{-12}	2.2105×10^{-9}	2.6264×10^{-7}	6.0474×10^{-7}
(0.6,0.6,0.6)	2.6918×10^{-12}	1.7484×10^{-9}	4.8404×10^{-7}	1.116×10^{-6}
(0.7,0.7,0.7)	6.4935×10^{-12}	6.5454×10^{-10}	8.1658×10^{-7}	1.886×10^{-6}
(0.8,0.8,0.8)	1.3411×10^{-11}	3.3782×10^{-11}	1.2905×10^{-6}	2.9856×10^{-6}
(0.9,0.9,0.9)	2.3367×10^{-11}	6.1231×10^{-10}	1.9393×10^{-6}	4.4936×10^{-6}
(1.0,1.0,1.0)	3.4658×10^{-11}	2.3253×10^{-9}	2.8000×10^{-6}	6.4936×10^{-6}

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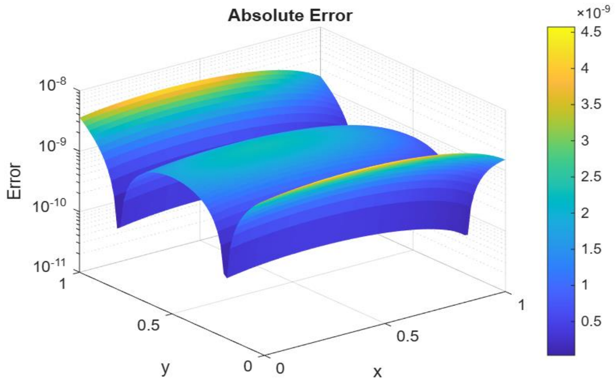

Figure 2. Absolute error of Example 1 by HG method, $N = 2$, and $z_1 = 0.5$.

Example 2. Consider the following 3D-NFIE [8]:

$$u(x_1, y_1, z_1) = g(x_1, y_1, z_1) + \int_0^1 \int_0^1 \int_0^1 x_1^2 y_1 z_1 t_1 r_1 \sin(s_1) u^3(s_1, t_1, r_1)\ ds_1\, dt_1\, dr_1, \quad (x_1, y_1, z_1) \in [0,1]^3 \qquad (8)$$

Where $g(x_1, y_1, z_1) = y_1 z_1 cos(x_1) - \frac{1}{100} x_1^2 y_1 z_1 - \frac{1}{100} x_1^2 y_1 z_1 cos^4(1)$, and the exact solution is

$u(x_1, y_1, z_1) = y_1 + z_1 cos\,(x_1)$.

Table (2) presents the absolute error of equation (8) for different values of (x_1, y_1, z_1), between Fibonacci collocation (FC), Hermite–Galerkin (HG) methods with N=6, the Successive Approximations Method at N=10 [8] Figures (3)–(4) illustrate the absolute errors obtained by both methods.

TABLE 2. Numerical results for Example 2, $z_1 = 0.5$

(x_1, y_1, z_1)	Fibonacci collocation method $N = 4$	Fibonacci collocation method $N = 6$	Hermite-Galerkin method $N = 4$	Hermite-Galerkin method $N = 6$	Method of [8] with $N = 10$
(0,0,0)	3.3736×10^{-10}	2.5886×10^{-7}	2.5387×10^{-2}	1.0534×10^{-1}	0.
(0.1,0.1,0.1)	3.863×10^{-7}	2.6904×10^{-11}	1.367×10^{-3}	2.822×10^{-3}	1.9452×10^{-9}
(0.2,0.2,0.2)	2.6101×10^{-9}	2.9308×10^{-9}	1.5613×10^{-3}	6.1957×10^{-3}	3.1124×10^{-8}
(0.3,0.3,0.3)	3.9605×10^{-7}	1.2532×10^{-8}	2.1344×10^{-3}	2.6484×10^{-3}	1.5756×10^{-7}
(0.4,0.4,0.4)	4.1756×10^{-8}	3.9765×10^{-8}	1.5719×10^{-2}	2.2452×10^{-2}	4.9799×10^{-7}
(0.5,0.5,0.5)	6.1579×10^{-7}	9.7949×10^{-8}	2.4358×10^{-2}	4.4104×10^{-2}	1.2158×10^{-6}
(0.6,0.6,0.6)	2.1135×10^{-7}	2.0196×10^{-7}	1.4499×10^{-2}	1.369×10^{-2}	2.521×10^{-6}
(0.7,0.7,0.7)	6.6599×10^{-7}	3.7446×10^{-7}	6.0005×10^{-4}	2.5634×10^{-3}	4.6706×10^{-6}
(0.8,0.8,0.8)	6.6795×10^{-7}	6.4132×10^{-7}	3.7639×10^{-3}	8.2663×10^{-3}	7.9678×10^{-6}
(0.9,0.9,0.9)	5.3352×10^{-6}	1.0188×10^{-6}	5.2081×10^{-3}	8.7402×10^{-3}	1.2763×10^{-5}
(1.0,1.0,1.0)	1.6307×10^{-6}	1.5608×10^{-6}	7.4386×10^{-3}	1.1347×10^{-1}	1.9452×10^{-5}

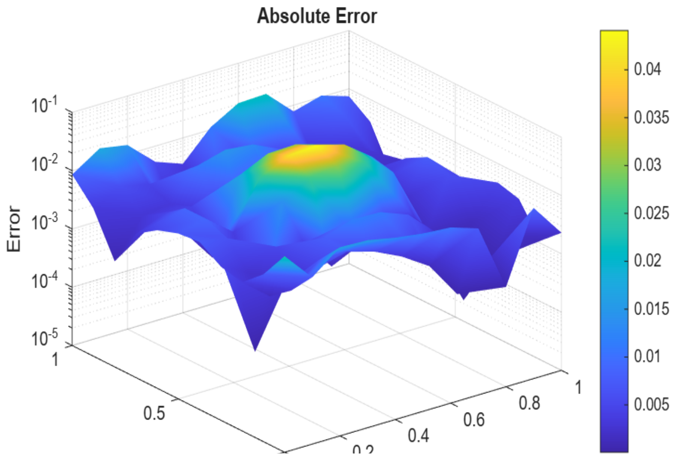

Figure 3. Absolute error of Example 2 by FC method, $N = 6$, and $z_1 = 0.5$.

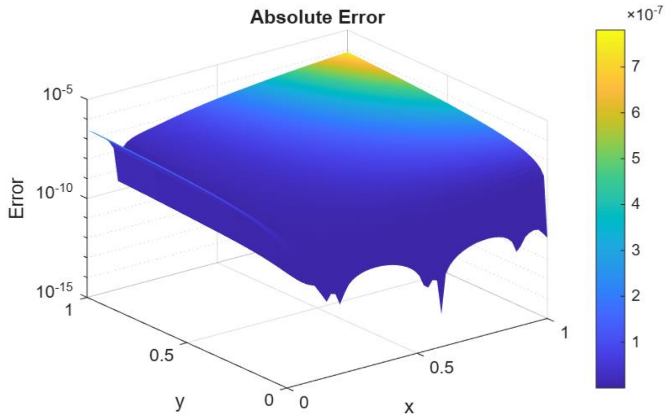

Figure 4. Absolute error of Example 2 by HG method, $N = 6$, and $z_1 = 0.5$.

Example 3. Consider the following 3D-NFIE [7]:

$$u(x_1, y_1, z_1) = g(x_1, y_1, z_1) + \int_0^1 \int_0^1 \int_0^1 x_1^2\, t_1\, cos(z_1)\, sin(s_1) u^3(s_1, t_1, r_1)\; ds_1\, dt_1\, dr_1, \qquad (x_1, y_1, z_1) \in [0,1]^3, \qquad (9)$$

where $g(x_1, y_1, z_1) = y_1 z_1 cos^2(x_1) \; + \frac{1}{140} x_1^2 \; cos\;(z_1)(cos^7(1) - 1)$, and the exact solution is

$u(x_1, y_1, z_1) = y_1 + z_1 cos^2(x_1)$.

Table (3) shows the comparison of the absolute error of equation (9) using Hermite–Galerkin (HG) and Fibonacci collocation (FC) methods for various values of (x_1, y_1, z_1). Figures (5)&(6) illustrate the absolute errors obtained by both methods. A comparison with the Haar Wavelet Method at $2N = 32$ [7].

TABLE 3. Numerical results for Example 3, $z_1 = 0.5$

(x_1, y_1, z_1)	Fibonacci collocation method $N = 6$	Hermite-Galerkin method $N = 6$	Method of [7] with $2N = 32$
(0,0,0)	6.3588×10^{-7}	2.7131×10^{-1}	0.
(0.1,0.1,0.1)	1.0895×10^{-7}	7.1995×10^{-3}	1.1244×10^{-8}
(0.2,0.2,0.2)	2.9425×10^{-9}	1.5717×10^{-2}	4.4304×10^{-8}

(0.3,0.3,0.3)	5.5898×10^{-8}	6.4034×10^{-3}	9.7169×10^{-8}
(0.4,0.4,0.4)	6.3907×10^{-8}	5.7392×10^{-2}	1.6654×10^{-7}
(0.5,0.5,0.5)	1.4021×10^{-8}	1.1352×10^{-1}	2.4794×10^{-7}
(0.6,0.6,0.6)	1.2999×10^{-7}	3.6224×10^{-2}	3.3578×10^{-7}
(0.7,0.7,0.7)	1.9685×10^{-7}	3.5896×10^{-3}	4.2354×10^{-7}
(0.8,0.8,0.8)	4.7301×10^{-8}	1.4785×10^{-2}	5.0391×10^{-7}
(0.9,0.9,0.9)	8.6554×10^{-7}	1.0595×10^{-2}	5.6902×10^{-7}
(1.0,1.0,1.0)	2.3595×10^{-7}	3.1193×10^{-1}	6.1061×10^{-7}

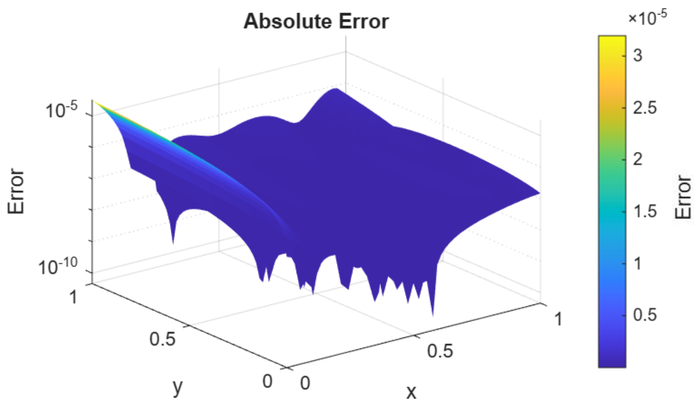

Figure 5. Absolute error of Example 3 by FC method, $N = 6$, and $z_1 = 0.5$.

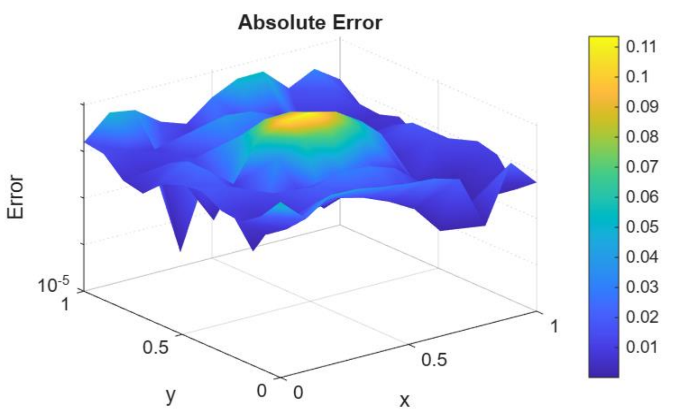

Figure 6. Absolute error of Example 3 by HG method, $N = 6$, and $z_1 = 0.5$.

5. Discussion

In this article, we have obtained the numerical solutions of 3D-NFIEs using two different polynomial-based approaches: the Fibonacci collocation method and the Hermite–Galerkin method. Adetailed comparison between the two techniques has been carried out, and the results demonstrate that the Fibonacci approach provides better accuracy than the Hermite–Galerkin method. Furthermore, when compared with other methods reported in the literature [7,8]. In the third example, the Haar wavelet method reaches its maximum error of 1.1244×10^{-8} at 2N=32 [8],

whereas the Fibonacci method yields a significantly smaller error of 2.9425×10^{-9} at $N = 6$, In the second example, the Fibonacci method also achieves an error of 2.6904×10^{-11} at $N = 6$, which is lower than that obtained by the Successive Approximations method, which records an error of 1.9452×10^{-9} at $N = 10$. In the first example, the Fibonacci method outperforms both methods, achieving the smallest error 3.7868×10^{-13} at $N = 2$. The Fibonacci method still outperforms them, achieving higher accuracy with a smaller number of basis functions N. This highlights the efficiency and effectiveness of the Fibonacci collocation scheme in solving such classes of integral equations. All computations and implementations are performed using MATLAB.

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