

A Note on Radicals of Semiring of Matrices

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Abstract:

In this article we introduce and investigate radicals of semiring of matrices. We establish that if R is a radical class which is (right or left)-hereditary and (right or left)-strong, then R has the property that the R -radical of the semiring of matrices of order n over a semiring R is equal to the semiring of $n \times n$ -matrices over the semiring $R(R)$.

Keywords: Semirings, Radical classes, Semiring of Matrices.

1. Introduction

Throughout this article semirings will be associative, not necessarily with unity element and radicals in the sense of Kurosh Amitsur as defined in [9]. In this article we have introduced and investigated radicals of semiring of matrices and of polynomial semirings.

In this article, we have shown that if R is a radical class which is (right or left)-hereditary and (right or left)-strong, then R has the property that the R -radical of the semiring of matrices of order n over a semiring R is equal to the semiring of matrices of order n over the semiring $R(R)$.

The interrelation and independence of polynomial extensibility of radical and semisimple classes and of the Amitsur property are investigated for associative rings in [12]. In this article, we have tried to generalize some results for semirings. Throughout this article \mapsto stands for onto homomorphism and \triangleleft stands for an ideal of a semiring R . For details of semiring theory and more on radical theory for associative semirings the readers are referred to [1], [2], [6], [8] and [10].

2. Radical of Semiring of Matrices.

Definition 2.1. The additive semigroup $(R, +)$ of a semiring R will be denoted by R^+ and for an abelian semigroup R^+ , we may define always a semiring R^0 with zero multiplication, called a zero-semiring by the rule $xy = 0$ for all $x, y \in R$.

Definition 2.2. A radical R is called an A -radical if for any semiring $R \in R$ and any additive homomorphism $f: R \rightarrow S$ such that $f(R)$ is a subsemiring of S also $f(S) \in R$.

Definition 2.3. A semiring R is said to be simple if it has no proper ideals.

Definition 2.4. If M is the class of all simple semirings with unity, then U_M is called the Brown McCoy radical class.

Proposition 2.5. Let ρ be a regular class of semirings. The upper radical U_ρ is hereditary if and only if ρ satisfies the condition (I) : if $0 \neq I \triangleleft R$ and there is a $I \mapsto T$ such that $0 \neq T \in \rho$, then there exists an $R \mapsto S$ such that $0 \neq S \in \rho$.

Proof. Suppose that U_ρ is hereditary. If $0 \neq I \triangleleft R$ and there is a $I \mapsto T$ such that $0 \neq T \in \rho$, then there exists an $R \mapsto S$ such that $0 \neq S \in \rho$. Conversely, by (J), $I \notin U_\rho \Rightarrow R \notin U_\rho$. But then $R \in U_\rho \Rightarrow I \in U_\rho$, hence U_ρ is hereditary.

Proposition 2.6. *If K is a subtractive ideal in an ideal I of a semiring R and I/K is a semiring with unity element, then K is an ideal in R .*

Proof. Let $e + K$ be the identity in I/K . Then any $a \in R$ and any $k \in K$. Then we have $ak \in I$ as $K \triangleleft I$. Therefore $ak + K = (e + K)(ak + K) = e(ak) + K = (ea)k + K = K$. $ak + K = K \Rightarrow ak + k_1 = k_2 \in K$ for some $k_1, k_2 \in K \Rightarrow ak \in K$ (K is subtractive). Similarly we get $ka \in K$ and $K \triangleleft R$.

Proposition 2.7. *If ρ is any regular class of semirings with unity element, then the upper radical U_ρ is hereditary.*

Proof. We must show that ρ satisfies condition (I) of Proposition 2.5 that I is an ideal in a semiring R such that I has a non-zero homomorphic image $C = I/K$ (K is subtractive) in ρ . Then I/K has a unity and by Proposition 2.6, K is an ideal of R . Since I/K has a unity, it must be a direct summed of R/K . Then R/K can be mapped homomorphically to I/K and this is in ρ and non-zero. Hence (I) holds and U_ρ is hereditary.

Theorem 2.8. *Brown McCoy radical is hereditary.*

Theorem 2.9. *Dorroh's Extension Theorem : Every semiring R can be embedded as an ideal into a semiring R with unity element.*

Proof. On the set $R' = \{ (a, n) / a \in R \text{ and } n \in \mathbb{Z}^+ \cup \{0\} \}$. Define $(a, n) + (a', n') = (a + a', n + n')$ and $(a, n)(a', n') = (aa' + n'a + nb', nn')$. R' is a semiring with unity $(0, 1)$ and $R \cong (R, 0) \triangleleft R'$

Note : The ring R' is referred to be the Dorroh's extension.

We shall use the following notations. If R is a semiring and n is a positive integer. $R^{(n)}$ denotes the semiring of matrices of order n over R . For $i, j \in \{1, 2, 3, \dots, n\}$, $R^{(i, j)}$ denotes the subsemiring of $R^{(n)}$ consisting of all matrices with elements from R in the (i, j) th position and with 0's elsewhere. For $i \in \{1, 2, 3, \dots, n\}$, we define $R^{(i)}$ as the right ideal $\sum_{j=1}^n R^{(i, j)}$ of $R^{(n)}$

and we define L_i as the left ideal $\sum_{k=1}^n R^{(k, i)}$ of $R^{(n)}$. If $x \in R$ and J is a non-empty subset of $\{1, 2, 3, \dots, n\}$ with $i \in J$, then $X_j^{(i)}(x)$ denotes the $n \times n$ matrix with x in the (i, j) -th position for all $j \in J$ and with 0's elsewhere. Then $X^{(i)} = \bigcup_{x \in R} X_j^{(i)}(x)$ is a left ideal of the semiring $R^{(n)}$.

Moreover $R \cong X_J(i, x)$ under the natural map.

If R is the Brown McCoy radical, then R is hereditary and satisfies $R(R^{(n)}) = (R(R))^{(n)}$.

Proposition 2.10. *If $I \triangleleft R$, then $I^{(n)}$. Moreover if I is a subtractive ideal, then so is $I^{(n)}$.*

Proposition 2.11. *If R is a semiring with unity element and $K \triangleleft R^{(n)}$, then $K = I^{(n)}$ with some $I \triangleleft R$.*

Proposition 2.12. *If R is a radical, then $R(R^{(n)}) = I^{(n)}$ for some ideal I of R and for every semiring R .*

Proof. For any radical R and for any semiring R , if $I \triangleleft R$, then $R(I) \triangleleft R$. If $I \triangleleft R$, then $I^{(n)} \triangleleft R^{(n)} \Rightarrow R(I^{(n)}) \triangleleft R^{(n)} \Rightarrow R(I^{(n)}) \subseteq R(R^{(n)})$. In particular, $R(R^{(n)})$ is an ideal in $R^{(n)}$. Therefore $R(R^{(n)}) = I^{(n)}$ for some ideal I in R' , where R' is Dorroh's extension. But $I^{(n)} \subseteq R^{(n)} \Rightarrow I \subseteq R \Rightarrow I \triangleleft R$.

Theorem 2.13. *Let R be a radical class, let R be a semiring, and let n be a positive integer. The following statements are equivalent.*

- (1) *If $R \in R$, then $R^{(n)} \in R$. (2) $(R(R))^{(n)} \subseteq R(R^{(n)})$.*

Proof. Now $R(R) \in R$ so that by (1) $(R(R))^{(n)} \in R$. Since $(R(R))^{(n)}$ is an ideal in $R^{(n)}$, hence $(R(R))^{(n)} \subseteq R(R^{(n)})$. Now $R \in R$ implies that $R(R) = R$. So that $R^{(n)} = (R(R))^{(n)}$. By (2) $(R(R))^{(n)} \subseteq R(R^{(n)})$. Hence $R^{(n)} = R(R^{(n)})$ and $R^n \in R$.

Theorem 2.14. *Let R be a radical class, let R be a semiring, and let n be a positive integer. The following statements are equivalent.*

1. If $R^{(n)} \in R$, then $R \in R$.

2. $R(R^{(n)}) \subseteq (R(R))^{(n)}$

Proof. By Proposition 2.12 $R(R^{(n)}) = I^{(n)}$, for some ideal I of R . From (1) we have $I \in R$. Hence $I \subseteq R(R)$ and so $R(R^{(n)}) = I^{(n)} \subseteq (R(R))^{(n)}$. Now $R^{(n)} \in R$ implies that $R(R^{(n)}) = R^{(n)}$. Thus by (2), $R^{(n)} = R(R^{(n)}) \subseteq (R(R))^{(n)}$ and so $R^{(n)} = (R(R))^{(n)}$. Hence $R(R) = R$ and $R \in R$.

Theorem 2.15. *Let R be a strong radical class. Then $R \in R$ implies $R^{(n)} \in R$.*

Proof. Let $n > 1$, let $i \in \{1, 2, 3, \dots, n\}$ be fixed and let $j \in \{1, 2, 3, \dots, n\}$ with $i \neq j$. Set $J = \{i, j\}$. Then $X_j^{(i)} \in R$. Since R is strong, therefore $X_j^{(i)} \subseteq R(R^{(i)})$. Setting $K = \{i\}$. We like wise obtain $X_k^{(i)} \subseteq R(R^{(i)})$. Hence $X_j^{(i)} + X_k^{(i)} \subseteq R(R^{(i)})$. Since $i \neq j$ and j was otherwise arbitrary, then $R_i \subseteq R(R^{(i)})$. Now $R^{(i)}$ is R -right ideal of $R^{(n)}$ so that, since R is strong, we have $R^{(i)} \subseteq R(R^{(n)})$. This is true for $j \in \{1, 2, 3, \dots, n\}$. We obtain $\sum_{i=1}^n R^{(i)} \subseteq R(R^{(n)})$. Hence $R^{(n)} = R(R^{(n)})$ and $R^{(n)} \in R$.

Theorem 2.16. *Let R be a hereditary (or right hereditary) radical class. Then $R^{(n)} \in R$ implies $R \in R$.*

Theorem 2.17. *Let R be a hereditary and left-strong (right strong) radical class. Then $R \in R$ implies $R^+ \in R$.*

Theorem 2.18. *Let R be a hereditary and left-strong (right strong) radical class. Then $R \in R$ implies $R^{(n)} \in R$.*

Theorem 2.19. *Let R be a radical class which is (left or right)-hereditary and (left or right) -strong. Then $R(R^{(n)}) = (R(R^{(n)}))^n$.*

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