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Analyzing Detour Distance and Domination Number in Special Graph Classes

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Abstract:

In this paper, we investigate the relationship between detour distance and domination number within various special classes of graphs. The detour distance between two vertices is defined as the length of the longest path connecting them, and it provides an alternative metric to the traditional geodesic distance in graph theory. By integrating this concept with domination theory, we introduce new structural measures that capture the extended influence of vertices beyond their immediate neighborhoods. We define and analyze the detour domination number, denoted as γDD(G), which represents the minimum cardinality of a set of vertices such that every vertex in the graph lies on a detour path from at least one dominating vertex. We examine the behavior of this parameter in specific graph classes, including complete graphs, paths, cycles, trees, and grid graphs, deriving exact values or bounds where applicable. Additionally, we study the upper detour domination number, investigate its extremal properties, and explore how it contrasts with traditional domination metrics. We also characterize graph classes where the detour domination number equals the classical domination number and where it significantly diverges. The results contribute to a deeper understanding of detour-based influence in graphs, offering new perspectives for applications in network resilience, transport systems, and distributed computing, where long-range reachability and control are essential.

Keywords: Detour Distance, Detour Dominating Set, Domination Number, Upper Detour Domination, Forcing Set in Graphs, Special Graph Classes.

INTRODUCTION

Graph theory, a foundational discipline in discrete mathematics, has evolved to become a powerful tool for modeling a wide variety of systems in science, engineering, social networks, computer science, and operations research. One of the most widely studied problems in graph theory is the domination problem, which seeks a subset of vertices that exerts influence or control over the entire graph. Traditional domination theory focuses on minimum sets of vertices that are adjacent to all other vertices either directly or indirectly. However, in many real-world networks—such as communication systems, transportation grids, biological systems, and social networks—the shortest-path assumption is often unrealistic. This motivates the need for domination models that incorporate longest-path interactions, leading to the development of detour-based domination concepts.

This work introduces and elaborates upon new graph invariants based on detour domination and detour distances. First, we formalize the detour dominating degree set, which includes sets that not only detourdominate the graph but are evaluated based on degree-based metrics of influence. We then introduce the upper detour dominating degree set, which seeks to maximize the local domination measure based on vertex degrees involved in the detour structure. Such sets provide insight into the maximum structural influence that can be exerted through long-distance interactions in the graph. The associated invariant, the upper detour domination number $\gamma_{dd}^{+}(G)$, captures the maximum weighted detour reachability.

PRELIMINARIES

Definition 1: Detour Distance
Let $G=(V,E)$ be a connected graph and let $u,v \in V(G)$.
The detour distance $D(u,v)$ is defined as the length of the longest $u-v$ path in the graph G.
Formally,

$$D(u,v)=\max\{\text{ length }(P)\mid P \text{ is a } u-v \text{ path in } G\}$$

This measure contrasts with the usual distance $d(u,v)$, which considers the shortest path. The detour distance captures long-range connectivity between vertices and is particularly relevant in resilience and influence spread models.

Definition 2: Detour Dominating Set
A subset $S \subseteq V(G)$ is called a detour dominating set if for every vertex $v \in V(G)$, there exists a vertex $u \in S$ such that v lies on a detour path originating from u. That is,
$\forall v \in V(G), \exists u \in S$ such that $v \in V(P_{uv})$, where P_{uv} is a detour path from u to v
This definition implies that S collectively "controls" the graph via its longest internal routes, not just the immediate neighborhoods.

Definition 3: Detour Dominating Degree Set ($\gamma_{dd}(G)$)
A detour dominating degree set $S \subseteq V(G)$ is a detour dominating set with minimum cardinality. The detour domination degree number, denoted by $\gamma_{dd}(G)$, is defined as:

$$\gamma_{dd}(G)=\min\{|S| \mid S \subseteq V(G), S \text{ is a detour dominating set }\}$$

This number reflects the minimum number of vertices required to detour-dominate the entire graph. It serves as a generalization of the classical domination number under the lens of maximum connectivity.

Definition 4: Upper Detour Dominating Degree Set ($\gamma_{dd}^{+}(G)$)
Let G be a graph and let $S \subseteq V(G)$ be a detour dominating set. The set S is called an upper detour dominating degree set if it maximizes a structural parameter called the local detour

domination degree, denoted $ld(s)$. The upper detour domination degree number $\gamma_{dd}^{+}(G)$ is given by:

$$\gamma_{dd}^{+}(u,v) = \max\{ld(s)\} = \max\left\{\deg(u) + \deg(v) + \sum_{w\in N(u,v)} \deg(w)\right\}$$

where:

- $u, v \in S$
- $N(u,v)$ is a neighborhood of vertices lying on the detour path from u to v
- $\deg(x)$ denotes the degree of vertex x

This definition captures the strongest detour influence a subset can exert in terms of degree and detour structure.

THEOREMS

Theorem 1: Detour Domination Number in Complete Graphs
Theorem:
Let $K_n - (V, E)$ be a complete graph on $n \geq 2$ vertices. Then the detour domination number $\gamma_D(K_n)$ is 1, i.e.,

$$\gamma_D(K_n) - 1$$

Proof:
In a complete graph K_n, every pair of distinct vertices $u, v \in V$ is connected by a unique edge, so:

$$d(u,v) - 1 \ \forall u \neq v$$

However, the detour distance $D(u,v)$, defined as the length of the longest possible path between u and v without repeating any vertices, is:

$$D(u,v) - n - 1 \ \forall u \neq v$$

since the longest path between any two vertices in K_n must visit all other $n-2$ vertices exactly once before reaching v, utilizing all $n-1$ vertices (including u and v).
Let $S \subseteq V$ be a detour dominating set, i.e., every vertex $v \in V \setminus S$ must lie on a detour path that includes some $u \in S$. Due to the complete connectivity of K_n, from any chosen vertex $u \in V$, there exists a detour path of length $n-1$ that visits every other vertex $v \in V \setminus \{u\}$. Hence, one vertex suffices to detour dominate the entire graph.
Therefore:

$$\gamma_D(K_n) - 1$$

To formalize this in matrix terms, let $A \in \mathbb{R}^{n\times n}$ be the adjacency matrix of K_n, defined by:

$$A - J_n - I_n$$

where J_n is the all-ones matrix and I_n is the identity matrix. The entries of A are given by:

$$A_{ij} = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i - j \end{cases}$$

Let $x - (x_1, x_2, \ldots, x_n)^T$ be the characteristic vector of the detour dominating set $S \subseteq V$, where:

$$x_i - \begin{cases} 1 & \text{if } v_i \in S \\ 0 & \text{otherwise} \end{cases}$$

Then the detour domination constraint requires that for each $j - 1,2, \ldots, n$, there must exist at least one $i \neq j$ such that $x_i - 1$. This is equivalent to:

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} x_i \geq 1$$

However, since K_n allows for each vertex to lie on a detour path from any other vertex, and since every vertex is adjacent to all others, a single vertex suffices for detour domination.

Thus, the integer program becomes:

$$\text{Minimize} \sum_{i=1}^{n} x_i$$

$$\text{Subject to} \sum_{\substack{i=1 \\ i \neq j}}^{n} x_i \geq 1 \text{ for all } j - 1,2, \ldots, n$$

$$x_i \in \{0,1\}, i - 1,2, \ldots, n$$

Setting any one $x_k - 1$ and all others $x_i - 0$ for $i \neq k$ satisfies all constraints, yielding the minimum value:

$$\gamma_D(K_n) - \sum_{i=1}^{n} x_i - 1$$

This completes the proof.

Theorem 2: Detour Domination Number in Path Graphs P_n
Theorem:
Let $P_n - (V, E)$ denote a simple undirected path graph on $n \geq 1$ vertices with consecutive edges. Then the detour domination number $\gamma_D(P_n)$ satisfies:

$$\gamma_D(P_n) - \left\lceil \frac{n}{3} \right\rceil$$

Proof:
Let $V - \{v_1, v_2, \ldots, v_n\}$, and let edges $E - \{(v_i, v_{i+1}) \mid 1 \leq i < n\}$ connect consecutive

vertices. Then P_n is a path of length $n-1$.
The distance $d(v_i, v_j)$ between any two vertices v_i and v_j is:

$$d(v_i, v_j) - |i-j|$$

The detour distance $D(v_i, v_j)$, defined as the length of the longest simple path between v_i and v_j, equals:

$$D(v_i, v_j) - n - 1, \text{ for } (v_i, v_j) - (v_1, v_n)$$

Let $S \subseteq V$ be a detour dominating set such that for every vertex $v_k \in V$, there exists $v_i \in S$ for which v_k lies on a detour path (of length $n-1$) beginning at v_i. That is:

$$\forall v_k \in V, \exists v_i \in S \text{ such that } v_k \in \text{Path}_{\text{detour}}(v_i)$$

We construct such a set S by selecting every third vertex, i.e.,

$$S - \{v_i \in V \mid i \equiv 2(\bmod 3)\}$$

Each such $v_i \in S$ can detour-dominate its neighbors:

$$N_D[v_i] - \{v_{i-1}, v_i, v_{i+1}\} \cap V$$

Thus, the total detour coverage from all selected dominators is:

$$\bigcup_{v_i \in S} N_D[v_i] - V$$

and the size of such a set is:

$$|S| = \left\lceil \frac{n}{3} \right\rceil$$

Now, to formulate the problem algebraically, define indicator variables:

$$x_i = \begin{cases} 1 & \text{if } v_i \in S \\ 0 & \text{otherwise} \end{cases} \text{ for } i - 1,2, \ldots, n$$

Then, for each vertex $v_k \in V$, the domination constraint is:

$$x_{k-1} + x_k + x_{k+1} \geq 1 \; \forall 1 \leq k \leq n$$

with boundary conditions handled by setting:

$$x_0 - x_{n+1} - 0$$

The optimization problem becomes:

$$\begin{aligned} \text{Minimize:} \quad & z - \sum_{i=1}^{n} x_i \\ \text{Subject to:} \quad & x_{i-1} + x_i + x_{i+1} \geq 1, \forall i - 1, \ldots, n \\ & x_0 - x_{n+1} - 0 \\ & x_i \in \{0,1\}, \forall i \end{aligned}$$

This is an instance of a binary integer programming problem.
Thus, the solution gives the detour domination number:

$$\gamma_D(P_n) - \min\left\{\sum_{i=1}^{n} x_i \mid x_{i-1} + x_i + x_{i+1} \geq 1, x_i \in \{0,1\}, i - 1, \dots, n\right\}$$

In compact set-theoretic optimization form, the detour domination number can also be written as:

$$\gamma_D(P_n) = \min_{x\in\{0,1\}^a}\left\{\sum_{i=1}^{n} x_i \mid \bigcup_{i:x_i-1} N_D[i] - V(P_n)\right\}$$

where:

$$N_D[i] - \{i-1, i, i+1\} \cap [1, n]$$

This completes the proof.

Theorem 3: Detour Domination Number in Cycle Graphs C_n

Theorem:
For the cycle graph C_n on $n \geq 3$ vertices, the detour domination number satisfies:

$$\gamma_{DD}(C_n) - \left\lceil\frac{n}{3}\right\rceil$$

Proof:
Let $C_n - (V, E)$ be the cycle graph with vertex set $V - \{v_1, v_2, \dots, v_n\}$ and edge set defined by:

$$E - \{(v_i, v_{i+1}) \mid 1 \leq i < n\} \cup \{(v_n, v_1)\}$$

The distance between any two vertices v_i and v_j on the cycle is given by:

$$d(v_i, v_j) - \min(|i-j|, n - |i-j|)$$

The detour distance (longest simple path) between v_i and v_j is:

$$D(v_i, v_j) - \left\lfloor\frac{n}{2}\right\rfloor$$

To construct a detour dominating set $S \subseteq V$, we require that for all $v \in V$, there exists some $v_k \in S$ such that $v \in \text{Path}_{\text{detour}}(v_k)$. Since each detour in the cycle is symmetric and spans up to half the cycle, it suffices to place dominators every 3 vertices.

Construct S as:

$$S - \{v_k \in V \mid k \equiv 1 (\text{mod} 3)\}$$

Using modular arithmetic on indices, each dominator $v_k \in S$ covers:

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$$N_D[v_k] - \{v_{k-1}, v_k, v_{k+1}\} \text{ with indices modulo } n$$

Therefore, the union of closed detour neighborhoods satisfies:

$$\bigcup_{v_k \in S} N_D[v_k] - V$$

and the size of such a set is:

$$|S| - \left\lceil \frac{n}{3} \right\rceil$$

To express this algebraically, define indicator variables:

$$x_i - \begin{cases} 1 & \text{if } v_i \in S \\ 0 & \text{otherwise} \end{cases} \quad \forall i - 1, \dots, n$$

The domination constraint for each vertex $v_j \in V$ is:

$$x_{j-1} + x_j + x_{j+1} \geq 1, \text{ for } j - 1, \dots, n$$

with cyclic (modular) indexing:

$$x_0 - x_n, x_{n+1} - x_1$$

$$\begin{aligned} \text{Minimize:} \quad & \sum_{i=1}^{n} x_i \\ \text{Subject to:} \quad & x_{i-1} + x_i + x_{i+1} \geq 1 \ \forall i - 1, \dots, n \\ & x_i \in \{0,1\} \ \forall i \end{aligned}$$

Thus,

$$\gamma_{DD}(C_n) - \min\left\{\sum_{i=1}^{n} x_i \mid x_{i-1} + x_i + x_{i+1} \geq 1, x_i \in \{0,1\}\right\} - \left\lceil \frac{n}{3} \right\rceil$$

Theorem 4: Detour Domination in Trees via Branch Decomposition
Theorem:
Let $T - (V, E)$ be a tree of order n, rooted at a vertex r, and let it branch into k major paths $B_1, B_2, \dots, B_k$, with each branch B_i of height h_i. Then,

$$\gamma_{DD}(T) \leq \sum_{i=1}^{k} \left\lceil \frac{h_i}{2} \right\rceil$$

Proof:
In a tree, the path between any two vertices is unique. Let $d(u, v)$ denote the number of edges in the unique $u \to v$ path. Then,

$$D(u, v) - d(u, v), \forall u, v \in V(T)$$

Let the root be r, and decompose the tree into disjoint branches from r as:

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$$T - \bigcup_{i=1}^{k} B_i$$

Each branch B_i is a path of height h_i. To detour-dominate each B_i, we select vertices at intervals of 2 along the path from root to leaves. Thus, each branch requires:

$$|S_i| = \left\lceil \frac{h_i}{2} \right\rceil \text{ dominators}$$

Summing over all branches:

$$\gamma_{DD}(T) \leq \sum_{i=1}^{k} \left\lceil \frac{h_i}{2} \right\rceil$$

Define binary variables $x_v \in \{0,1\}$ for all $v \in V(T)$ where:

$$x_v - \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{otherwise} \end{cases}$$

Each vertex $u \in V$ must lie on a longest path from some dominator:

$$\exists v \in V(T) \text{ with } x_v - 1 \text{ such that } u \in P_{v,u}^{\max}$$

Encode this as constraints:

$$\sum_{\substack{v \in V \\ u \in P_{v,u}^{\max}}} x_v \geq 1, \forall u \in V$$

Optimization problem:

$$\begin{aligned} \text{Minimize:} \quad & \sum_{v \in V} x_v \\ \text{Subject to:} \quad & \sum_{v : u \in I_{\text{toa}}^{\max}} x_v \geq 1, \forall u \in V \\ & x_v \in \{0,1\}, \forall v \in V \end{aligned}$$

Hence,

$$\gamma_{DD}(T) \leq \sum_{i=1}^{k} \left\lceil \frac{h_i}{2} \right\rceil$$

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