

Results on Grundy Number of Fan Graph Families

K. Annathurai¹, P. Periasamy, V. Sankar Raj³

¹ Department of Mathematics, Thiruvalluvar College, Papanasam - 627425,
 Tamil Nadu, India. Email: kannathuraitvcmaths@gmail.com

² Department of Mathematics, Research Scholar, Register Number 21124012091019,
 Manonmaniam Sundaranar University, Tirunelveli - 627012, Tamilnadu,
 India. Email: ppskmttc2013@gmail.com

³ Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli
 – 627012, Tamilnadu, India. Email: sankarrajev@gmail.com

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Abstract:

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A Grundy k -coloring of a graph G is a proper k -coloring of vertices in G using colors $\{1, 2, \dots, k\}$ such that for any two colors i and j , $i < j$, any vertex colored j is adjacent to some vertex colored i . The First-Fit or Grundy chromatic number (or simply Grundy number) of a graph G , denoted by $\Gamma(G)$, is the largest integer k , such that there exists a Grundy k -coloring for G . It can be easily seen that $\Gamma(G)$ equals to the maximum number of colors used by the greedy (or First-Fit) coloring of G . In this paper, we obtain the Grundy chromatic number of fan graph, line graph of fan graph, middle graph of fan graph and total graph of fan graph.

Keywords: Grundy chromatic number, Fan graph, line graph, middle graph, Total graph.

1 Introduction

The notion of Grundy colorings was introduced by Patrick Michael Grundy in 1939. Author has dealing with combinatorial games contained ideas that led to the concept of Grundy colorings of graphs. A Grundy coloring [4, 6, 16] of a graph G is a proper vertex coloring of G (whose colors, as usual, are positive integers) having the property that for every two colors i and j with $i < j$, every vertex colored j has a neighbor colored i . Consequently, every Grundy coloring is a complete coloring. The 4-coloring of the tree T_1 of Figure 1 is a Grundy 4-coloring and is therefore a complete 4-coloring as well. However, the complete 3-coloring of T_2 shown in Figure 1.1 is not a Grundy 3-coloring. Recall that a greedy coloring [1] c of a graph G is obtained from an ordering $\phi : v_1, v_2, \dots, v_n$ of the vertices of G in some manner, by defining $c(v_1) = 1$, and

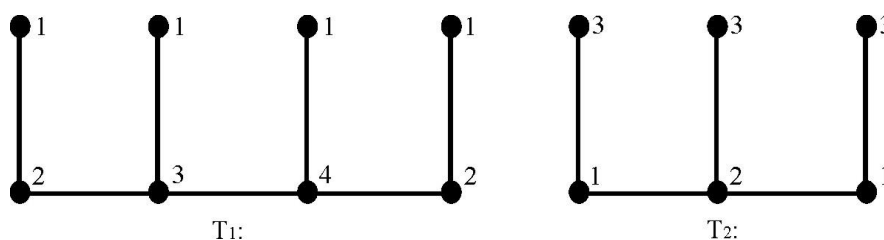


Figure 1.1: Complete and Grundy colorings

once colors have been assigned to V_1, V_2, \dots, V_p for some integer p with $1 \leq p \leq n$, $c(V_{p+1})$ is defined as the smallest color not assigned to any neighbor of V_{p+1} belonging to the set $\{V_1, V_2, \dots, V_p\}$. The coloring c so produced is then a Grundy coloring of G . (i.e.), every greedy coloring is a Grundy coloring. The maximum positive integer k for which a graph G has a Grundy k -coloring is denoted by $\Gamma(G)$ and is called the Grundy chromatic number of G or more simply the Grundy number of G . If the Grundy number of a graph G is k , then in any Grundy k -coloring of G (using the colors $1, 2, \dots, k$), every vertex v of G colored k must be adjacent to a vertex colored i for each integer i with $1 \leq i \leq k$. Thus $\Delta(G) \geq \deg(v) \geq (k - 1)$ and so

$$\Gamma(G) \leq \Delta(G) + 1 \quad (1)$$

for every graph G . Since every Grundy coloring of a graph G is a proper coloring, it follows that,

$$\chi(G) \leq \Gamma(G).$$

The Grundy number of a graph was perhaps introduced for the first time by Christen and Selkow [2]. In [3], Erdős et., al. proved that the Grundy number of a graph is in fact the same as a chromatic number of a graph which was defined and studied independently by Simmons [12]. In [8] the authors studied the Grundy number of hypercubes and determined the exact values. From computational point of view, polynomial time algorithms for determining the Grundy number have been given for trees in [7] and for partial k -trees in [13]. In a manuscript [5] the NP-completeness of determining the Grundy number of general graphs has been proved. Therefore, they gave an affirmative answer to the problem 10.4 posed in the graph coloring problem book [9] which asks whether determining the Grundy chromatic number of graphs is an NP-complete problem.

2 Preliminaries

All graphs we consider are simple and finite.

Definition 2.1 The generalized fan graph $F_{n,m}$ is defined as the graph join $K_n + P_m$, where K_n is the empty graph with n vertices and P_m is the path graph with m vertices. The case $n = 1$ corresponds to the usual Fan Graphs, while $n = 2$ corresponds to the Double Fan graphs, etc. Number of edges of the Generalized Fan graph $F_{n,m}$ with $(n + m)$ vertices is $nm + (m - 1)$.

Definition 2.2 The line graph of G denoted by $L(G)$ is the graph with vertices are the edges of G with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent.

Definition 2.3 Middle graph $M(G)$ of a graph G is defined with the vertex set $V(G) \cup E(G)$, in which two elements are adjacent if and only if either both are adjacent edges in G or one of the elements is a vertex and the other one is an edge incident to the vertex in G .

Definition 2.4 Total graph $T(G)$ of a graph G defined with the vertex set $V(G) \cup E(G)$, in which two elements are adjacent if and only if one of the following holds true (i) both are adjacent edges or vertices in G (ii) one is a vertex and other is an edge incident to it in G .

3 Main Results

We begin by presenting the existing results concerning the Grundy chromatic number of fan graph, line graph of fan graph, central graph of fan graph, middle graph of fan graph and total graph of fan graph.

Theorem 3.1

For any positive integer m and $n \geq 2$, then $\Gamma(F_{m,n})$ is 3, if $n = 2, 3$ and is 4, if $n \geq 4$.

Proof:

Let us consider the Fan Graph of the form $F_{m,n}$, where $m \geq 1$ and $n \geq 2$.

The Grundy chromatic number of $F_{m,n}$ be denoted as $\Gamma(F_{m,n})$.

Let $V(F_{m,n})$ represent the vertices of $F_{m,n}$.

That is, $V(F_{m,n})$ is U_i ($1 \leq i \leq m$) and V_j ($1 \leq j \leq n$).

We define a mapping, $\sigma : V(F_{m,n}) \rightarrow \mathbb{N}$ as follows:

Case (i): For $n = 2, 3$.

For $1 \leq i \leq m, 1 \leq j \leq 3$,

$$\sigma(U_i) = 3.$$

$$\sigma(v_j) = 1, \text{ if } j \equiv 1 \pmod{2} \quad \text{and } 2, \text{ if } j \equiv 0 \pmod{2}$$

Since $\Gamma(F_{n,m}) \leq 3$.

Assume that $\Gamma(F_{n,m}) > 3$.

For $n = 2$, there exist a clique of order 3. It is obviously the color is 3.

For $n = 3$,

Suppose, the vertices U_i ($1 \leq i \leq m$) received any one of the color 1 to 4 and remaining three color are assigned to the vertices V_j ($1 \leq j \leq 3$).

This arises a contradiction of Grundy chromatic number definition.

Hence $\Gamma(F_{m,n}) \leq 3$.

Therefore, $\Gamma(F_{m,n}) = 3, (m \geq 1; 2 \leq n \leq 3)$.

Case (ii): For $n \geq 4$.

For $1 \leq i \leq m$,

$$\sigma(U_i) = 4.$$

Subcase (i): For $n \equiv 0 \pmod{3}$.

$$\sigma(V_j) = (\text{For } 1 \leq j \leq n-2) \quad 1, \text{ if } j \equiv 1 \pmod{3} \quad 3, \text{ if } j \equiv 2 \pmod{3} \quad \text{and} \quad 2, \text{ if } j \equiv 0 \pmod{3}$$

$$\sigma(V_{n-1}) = 2; \quad \sigma(V_n) = 1.$$

Subcase (ii): For $n \equiv 2 \pmod{3}$.

$$\sigma(V_j) = (\text{For } 1 \leq j \leq n-1) \quad 1, \text{ if } j \equiv 1 \pmod{3} \quad 3, \text{ if } j \equiv 2 \pmod{3} \quad \text{and} \quad 2, \text{ if } j \equiv 0 \pmod{3}$$

$$\sigma(V_n) = 2.$$

Subcase (iii): For $n \equiv 1 \pmod{3}$.

$$\sigma(v_j) = (\text{For } 1 \leq j \leq n) \quad 1, \text{ if } j \equiv 1 \pmod{3} \quad 3, \text{ if } j \equiv 2 \pmod{3} \quad \text{and} \quad 2, \text{ if } j \equiv 0 \pmod{3}$$

Since $\Gamma(F_{n,m}) \leq 4$. Assume that $\Gamma(F_{n,m}) > 4$,

Suppose, the vertices U_i ($1 \leq i \leq m$) received any one of the color 1 to 5 and remaining four color are assigned to the vertices v_j ($1 \leq j \leq n$).

This arises a contradiction of Grundy chromatic number definition.

Hence $\Gamma(F_{m,n}) \leq 4$.

Therefore, $\Gamma(F_{m,n}) = 4, n \geq 4$.

Theorem 3.2

For any positive integer m and $n \geq 3$, then $\Gamma [L(F_{m,n})] = n + 1, n \geq m$.

Proof:

Let us consider the Fan Graph of the form $F_{m,n}$, where $m \geq 1$ and $n \geq 3$.

Let $V(F_{m,n}) = \{U_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\}$ represent the vertices of $F_{m,n}$.

By the definition of line graph,

let $V [L(F_{m,n})] = \sum_{i=1}^m \{e_{i,j} : 1 \leq j \leq n\} \cup \{x_{j,j+1} : 1 \leq j \leq n-1\}$.

We define a mapping, $\sigma : V [L(F_{m,n})] \rightarrow N$ as follows:

For $1 \leq i \leq m, 1 \leq j \leq n$,

$$\sigma(e_{i,j}) = i + j, i \equiv i \pmod{m}; j \equiv j \pmod{n}$$

$$\sigma(x_{j,j+1}) = 1, 1 \leq j \leq n-1.$$

Since above color partition $\Gamma(F_{n,m}) \leq n + 1$.

Assume that $\Gamma(F_{n,m}) > n + 1$.

Since there exist a clique of order n .

By the above Equation $\Gamma [L(F_{m,n})] \leq n + 1$. This arises a contradiction.

Therefore, $\Gamma [L(F_{m,n})] = n + 1$.

Theorem 3.3

For any positive integer m and $n \geq 3$, then $\Gamma [M(F_{m,n})] = n + 2, n \geq m$.

Proof:

Let us consider the Fan Graph of the form $F_{m,n}$, where $m \geq 1$ and $n \geq 3$.

Let $V(F_{m,n}) = \{U_i : 1 \leq i \leq m\} \cup \{V_j : 1 \leq j \leq n\}$ represent the vertices of $F_{m,n}$.

By the definition of middle graph,

let $V [M(F_{m,n})] = V(F_{m,n}) \cup \sum_{i=1}^m \{e_{i,j} : 1 \leq j \leq n\} \cup \{x_{j,j+1} : 1 \leq j \leq n-1\}$.

We define a mapping, $\sigma : V [M(F_{m,n})] \rightarrow N$ as follows: For $1 \leq i \leq m, 1 \leq j \leq n$,

$$\sigma(e_{i,j}) = i + j + 1, i \equiv i \pmod{m}; j \equiv j \pmod{n}$$

$$\sigma(U_i) = 1;$$

$$\begin{aligned}\sigma(V_j) &= 2; \\ \sigma(e_{j,j+1}) &= 1, 1 \leq j \leq n-1.\end{aligned}$$

Since above color partition $\Gamma(F_{n,m}) \leq n+2$.

Assume that $\Gamma(F_{n,m}) > n+2$.

Since there exist a clique of order $n+1$.

By Equation (1), $\Gamma[M(F_{m,n})] \leq n+2$. This arises a contradiction.

Therefore, $\Gamma[M(F_{m,n})] = n+2$.

Theorem 3.4

For any positive integer m and $n \geq 3$, then $\Gamma[T(F_{m,n})] = n+2, n \geq m$.

Proof:

Let us consider the Fan Graph of the form $F_{m,n}$, where $m \geq 1$ and $n \geq 3$.

Let $V(F_{m,n}) = \{U_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\}$ represent the vertices of $F_{m,n}$.

By the definition of total graph,

let $V[T(F_{m,n})] = V(F_{m,n}) \cup \sum_{i=1}^m \{e_{i,j} : 1 \leq j \leq n\} \cup \{x_{j,j+1} : 1 \leq j \leq n-1\}$.

We define a mapping, $\sigma : V[T(F_{m,n})] \rightarrow N$ as follows: For $1 \leq i \leq m, 1 \leq j \leq n$,

$$\begin{aligned}\sigma(e_{i,j}) &= i+j+1, i \equiv i \pmod{m}; j \equiv j \pmod{n} \\ \sigma(U_i) &= 1; \\ \sigma(V_j) &= 2; \\ \sigma(e_{j,j+1}) &= 1, 1 \leq j \leq n-1.\end{aligned}$$

Since the above color partition $\Gamma(F_{n,m}) \leq n+2$.

Assume that $\Gamma(F_{n,m}) > n+2$. Since there exist a clique of order $n+1$.

$\Gamma[T(F_{m,n})] \leq n+2$. This arises a contradiction.

Therefore, $\Gamma[T(F_{m,n})] = n+2$.

Conclusion

In this paper we discussed about the Grundy chromatic number of fan graph, line graph of fan graph, middle graph of fan graph and total graph of fan graph. This paper motivates the reader to do further research in Grundy coloring of some other graphs and also other colorings of different types of graphs.

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