

Direct Approach to a Generalized Fractionalorder Thermoelastic Problem for a Thickcircular Plate with Lord-Shulman and Classical Coupledthermoelastic Model

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Abstract: In the present article, we discussed the mathematical solution of generalized fractional order thermoelastic problem for a thick circular plate of finite thickness $2b$ occupying the space D defined by $0 \leq r \leq \infty$, $-b \leq z \leq b$. The problem is discussed within the context of the theory of generalized thermoelastic diffusion with one relaxation time. The upper and the lower surfaces of the thick plate are traction free and subjected to an axi-symmetric heat supply. The solution is carried out by using integral transform technique and a direct approach. The most general solutions are obtained and represented graphically using MATHCAD. The validity of this intended model is assessed by comparing it with previously published results. According to the authors this model is explicitly useful for the researchers those studying development in theory of hyperbolic thermoelastic diffusion, in material science and designers of new materials.

Introduction: This paper presents an analytical solution to a generalized fractional order thermoelastic problem in a thick circular plate using both the **Lord-Shulman (LS)** and **Classical Coupled Thermoelasticity (CTE)** models. The analysis considers a finite plate of thickness $2b$, subjected to axisymmetric thermal loading on its traction-free upper and lower surfaces. Employing a direct approach along with integral transforms, we derive general solutions in the Laplace domain without introducing potential functions. Numerical evaluations are carried out using MATHCAD, and results are graphically depicted to highlight variations in temperature, displacement, stress, concentration, and chemical potential. The obtained outcomes show consistency with established results and illustrate the influence of fractional order and time on the thermoelastic behaviour. This model proves valuable for advancing the study of hyperbolic thermoelastic diffusion, material science, and innovative material design.

Objectives: To evaluate numerical solutions for temperature, displacement, stress, concentration, and chemical potential and observe the variations graphically by employing MATHCAD software for two distinct models namely

Lord-Shulman & Classical Coupled Thermoelastic model.

Methods: Concerned problem is solved by employing an effective method of direct approach along with integral transforms without introducing potential function.

Results: Evaluated numerical solutions for the field functions such as temperature, displacement, stress, concentration, and chemical potential.

Conclusions: In this paper, we analytically solved the generalized fractional order thermoelastic problem for a thick circular plate using a direct approach and integral transform techniques. The used approach avoids the use of potential function, offering a simplified and efficient route to obtain exact solution in the Laplace domain. The findings have significant implications for researchers exploring fractional thermoelasticity, particularly in the design of advanced materials and systems subjected to thermal and diffusion effects.

Keywords: Circular Plate, Fractional Order Derivative, Axisymmetric Temperature, Thermoelasticity, Lord-Shulman Model, Classical Coupled Theory.

1. Introduction

Y. Z. Povstenko [1,3], solved some thermoelastic problems based on the equation of heat conduction in 1D as well as 2D with a time-fractional derivative and associated thermal stresses. In four distinct thermoelasticity theories, S. Mukhopadhyay et.al. [2], explored the general thermoelastic interactions in unbounded elastic media and spherical cavities. A. Kar et.al. [4], presented thermoelasticity theories for a hollow sphere with a thermal shock problem.

In the Fractional Calculus technique, H. H. Sherief et.al. [5], introduced the novel coupling between thermoelasticity and widespread thermoelasticity with one relaxation cycle. A. Sur et.al. [6], developed the new theory of thermoelastic distribution of two temperature with new heat conduction equation with fractional order. H. M. Youssef [7], solved the generalized thermoelasticity theory of a half space filled with an elastic material, which has constant elastic parameters in the context of the fractional order derivative. K. R. Gaikwad et.al. [8], presented the thermoelastic problem for the thick circular plate subjected to an interior heat flux under an unsteady-state, the determination of unknown temperature, displacement and thermal stresses on the upper surface of a plate. K. R. Gaikwad et.al. [9], studied the nonhomogeneous heat conduction problem and its thermal deflection due to internal heat generation in a thin hollow circular disk. K. R. Gaikwad [10], analysed the thermoelastic deformation of a thin hollow circular disk due to partially distributed heat supply. K. R. Gaikwad [23], analyzed the problems of heat conduction for temperature and thermal stress. A. Sur et.al. [11], introduced the fractional order generalized thermoelastic functionally graded solid with variable material properties.

E. M. Hussain [12], developed the fractional order thermoelastic problem for an infinitely long solid circular cylinder. W. Raslan [13], addressed the 1D issue utilising the Laplace transform technique of the thermoelasticity fractional order of an infinitely long cylindrical cavity. K. R. Gaikwad [14], developed the thermoelastic mathematical model for circular

sector disk subject to internal heat generation. A. Bayat et.al. [15], analysed the unsteady state thermo-mechanical problem of the FGM thick sphere. W. Raslan [16], resolved a 2D problem of an axi-symmetric temperature distribution fractional thermoelasticity order theory of a thick plate. J. J. Tripathi [17], showed the impact of an axisymmetric supply of heat on the diffusion phenomena of an infinite and finite thick thermoelastic platform and the theory of widespread thermoelastic diffusion with a one-time interval of relaxation. M. A. Ezzat [18], developed a 3D thermoelasticity model with time-dependent thermal shock issue, utilising a fractional thermoelasticity order theory for a half-space.

K. R. Gaikwad et.al. [27], analyzed the various non-integer order problems of thermoelasticity with different approaches and got wonderful results. V. G. Bhandwalkar et.al. [34], proposed the 1D fractional order generalized thermoelastic model for half-space in the context of two models, viz., LS and DPL. In which effect of exponentially varying heat source and the fractional order parameter has been investigated on field variables such as temperature, displacement and stress. V. G. Bhandwalkar et.al. [35], constructed the Caputo-Fabrizio fractional order model to study effects of thermal diffusion in human head skin tissue. Where the accumulation of heat over time by the skin tissue is realistically modeled. In which the infinite memory effects have been avoided with the help of non-singular kernel. K.R. Gaikwad [19], studied two-dimensional steady state problem for thin circular plate under uniform internal energy generation. Finite Hankel transform technique to solve governing heat conduction equation. In 2019 K. R. Gaikwad [20], a 2D problem of thin hollow circular disk is analyzed in the context of fractional order derivative of order $0 < \alpha \leq 2$. By applying Laplace, finite Fourier and Hankel transforms results have been obtained and illustrated graphically using PTC MATHCAD. In 2020 S. G. Khavale et.al. [21], constructed the fractional order model of magneto-thermo-viscoelastic spherical cavity and field variables have obtained. The general solution is derived using Laplace transform technique and state space approach. The time-fractional two-dimensional thermoelastic problem for a thin hollow circular disk has analyzed by K. R. Gaikwad et.al. [22], to study its thermal stresses in non-homogeneous medium under four boundary conditions. The results have been computed and illustrated graphically using PTC MATHCAD software. Also, some contributions to this theory are the work in [24–26, 28–33].

This paper presents an analytical solution to a generalized fractional order thermoelastic problem in a thick circular plate using both the **Lord–Shulman (LS)** and **Classical Coupled Thermoelasticity (CTE)** models. The analysis considers a finite plate of thickness $2b$, subjected to axisymmetric thermal loading on its traction-free upper and lower surfaces. Employing a direct approach along with integral transforms, we derive general solutions in the Laplace domain without introducing potential functions. Numerical evaluations are carried out using MATHCAD, and results are graphically depicted to highlight variations in temperature, displacement, stress, concentration, and chemical potential. The obtained outcomes show consistency with established results and illustrate the influence of fractional order and time on the thermoelastic behaviour. This model proves valuable for advancing the study of hyperbolic thermoelastic diffusion, material science, and innovative material design.

2. Basic Equations & Relations

The field equations for the generalized thermoelastic diffusion in an isotropic medium in the absence of body forces and heat source are as follows:

(1) **Equation of motion [17]:**

$$\rho \ddot{u}_i = \mu u_{i,jj} + (\lambda + \mu) u_{i,jj} - \xi_1 T_{,i} - \xi_2 C_{,i} \quad (1)$$

where ξ_1 and ξ_2 are material constants.

(2) **Equation of heat conduction [7]:**

$$k T_{,ii} = \frac{\partial}{\partial t} \left(1 + \tau_0 \frac{\partial^\alpha}{\partial t^\alpha} \right) (\rho C_E T + T_0 \xi_1 e + T_0 \beta_t C) \quad (2)$$

(3) **Equation of mass diffusion [7]:**

$$D \xi_2 e_{ii} + D \beta_t T_{,ii} + \left(\frac{\partial}{\partial t} + \tau \frac{\partial^2}{\partial t^2} \right) C = D \beta C_{,ii} \quad (3)$$

(4) **Constitutive equations [17]:**

$$\sigma_{ij} = 2\mu \varepsilon_{ij} + \delta_{ij} (\lambda e - \xi_1 (T - T_0) - \xi_2 C), \quad (4)$$

$$P = -\xi_2 e + \beta C - \beta_t (T - T_0) \quad (5)$$

where ε_{ij} , $i, j = 1, 2, 3, \dots$ are the components of the strain tensor given by

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad i, j = 1, 2, 3, \dots \quad (6)$$

3. Two Dimensional Fractional Order Formulation

We take the axis of symmetry as the z axis, and the origin of the system of co-ordinates is at the middle plane between the upper and lower faces of the plate. The problem is studied using the cylindrical polar co-ordinates (r, φ, z) . Due to the rotational symmetry about the z axis of the problem, all quantities are independent of the co-ordinate φ .

For the 2D problem, the displacement vector u has the form $u = (u, 0, w)$. Hence, the components of the strain tensor of Eq. (6) can be written in the form:

$$e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\phi\phi} = \frac{u}{r}, \quad e_{zz} = \frac{\partial w}{\partial z}, \quad e_{rz} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \quad (7)$$

$$e = \frac{u}{r} + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad (8)$$

The Laplacian operator [17]:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad (9)$$

The governing equations (1) to (5) will take the form:

$$\mu \nabla^2 u - \frac{\mu}{r^2} u + (\lambda + \mu) \frac{\partial e}{\partial r} - \xi_1 \frac{\partial T}{\partial r} - \xi_2 \frac{\partial C}{\partial r} = \rho \frac{\partial^2 u}{\partial t^2} \quad (10)$$

$$\mu \nabla^2 w + (\lambda + \mu) \frac{\partial e}{\partial z} - \xi_1 \frac{\partial T}{\partial z} - \xi_2 \frac{\partial C}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2} \quad (11)$$

$$k \nabla^2 T = \frac{\partial}{\partial t} \left(1 + \tau_0 \frac{\partial^\alpha}{\partial t^\alpha} \right) (\rho C_E T + T_0 \xi_1 e + T_0 \beta_t C) \quad (12)$$

$$D \xi_2 \nabla^2 e + D \beta_t \nabla^2 T + \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) C - D \beta \nabla^2 C = 0 \quad (13)$$

$$\sigma_{\phi\phi} = 2\mu e_{\phi\phi} + \lambda e - \xi_1 (T - T_0) - \xi_2 C \quad (14)$$

$$\sigma_{rr} = 2\mu e_{rr} + \lambda e - \xi_1 (T - T_0) - \xi_2 C \quad (15)$$

$$\sigma_{zz} = 2\mu e_{zz} + \lambda e - \xi_1 (T - T_0) - \xi_2 C \quad (16)$$

$$\sigma_{rz} = \mu e_{rz} \quad (17)$$

$$\sigma_{r\phi} = \sigma_{z\phi} = 0 \quad (18)$$

$$P = -\xi_2 e + \beta C - \beta_t (T - T_0) \quad (19)$$

The following are the non-dimensional variables which are expressed as:

$$\tau_1' = c_1^2 \eta \tau_1, \quad P' = \frac{P}{\xi_1}, \quad \theta' = \frac{\xi_1 (T - T_0)}{\lambda + 2\mu}, \quad \sigma'_{ij} = \frac{\sigma_{ij}}{\lambda + 2\mu}.$$

$$r' = c_1 \eta r, \quad z' = c_1 \eta z, \quad u' = c_1 \eta u, \quad w' = c_1 \eta w, \quad t' = c_1^2 \eta t,$$

where,

$$\tau_0' = c_1^2 \eta \tau_0, \quad c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad \eta = \frac{\rho C_E}{k}$$

Putting non-dimensional quantities in equations (10) to (19), the stress components & governing equations adopt the form:

$$\nabla^2 u - \frac{1}{r^2} u + (\xi^2 - 1) \frac{\partial e}{\partial r} - \xi^2 \frac{\partial T}{\partial r} - \xi^2 \frac{\partial C}{\partial r} = \xi^2 \frac{\partial^2 u}{\partial t^2} \quad (20)$$

$$\nabla^2 w + (\xi^2 - 1) \frac{\partial e}{\partial z} - \xi^2 \frac{\partial T}{\partial z} - \xi^2 \frac{\partial C}{\partial z} = \xi^2 \frac{\partial^2 w}{\partial t^2} \quad (21)$$

$$\nabla^2 \theta = \frac{\partial}{\partial t} \left(1 + \tau_0 \frac{\partial^\alpha}{\partial t^\alpha} \right) (\theta + \epsilon e + \epsilon \beta_1 C) \quad (22)$$

$$\nabla^2 e + \beta_1 \nabla^2 \theta + \beta_2 \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) C - \beta_3 \nabla^2 C = 0 \quad (23)$$

$$\sigma_{\phi\phi} = \frac{2}{\xi_2} \frac{u}{r} + \frac{(\xi^2 - 2)}{\xi^2} e - \theta - C \quad (24)$$

$$\sigma_{rr} = \frac{2}{\xi_2} \frac{\partial u}{\partial r} + \frac{(\xi^2 - 2)}{\xi^2} e - \theta - C \quad (25)$$

$$\sigma_{zz} = \frac{2}{\xi_2} \frac{\partial w}{\partial z} + \frac{(\xi^2 - 2)}{\xi^2} e - \theta - C \tag{26}$$

$$\sigma_{rz} = \frac{1}{\xi^2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \tag{27}$$

$$P = -e + \beta_3 C - \beta_1 \theta \tag{28}$$

where,

$$\beta_1 = \frac{\beta_t(\lambda + 2\mu)}{\xi_1 \xi_2}, \beta_2 = \frac{(\lambda + 2\mu)}{D\eta \xi_2^2}, \beta_3 = \frac{\beta(\lambda + 2\mu)}{\xi_2^2}, \varepsilon = \frac{\xi_1^2 T_0}{\rho C_E (\lambda + 2\mu)}$$

4. Initial & Boundary Conditions

- **Initial Conditions:**

The initial conditions are assumed as:

$$(29) \quad \theta = \frac{\partial \theta}{\partial z} = 0, \quad z = \pm b,$$

$$(30) \quad \sigma_{zz} = 0, \quad z = \pm b$$

$$(31) \quad \sigma_{rz} = 0, \quad z = \pm b$$

$$(32) \quad P = 0, \quad z = \pm b$$

- **Boundary Conditions:**

The boundary conditions are assumed as:

$$\frac{\partial \theta}{\partial z} = \pm g_0 F(r, z), \quad z = \pm b, \tag{33}$$

$$\sigma_{zz} = 0, \quad z = \pm b, \tag{34}$$

$$\sigma_{rz} = 0, \quad z = \pm b, \tag{35}$$

$$P = H(t)f(r), \quad z = b, \tag{36}$$

where, $f(r)$ is a known function of r and $H(t)$ is a Heaviside unit step function. Equations (7) to (28) and (29) to (36) constitute the mathematical formulation of the problem under consideration.

5. Geometry of Problem

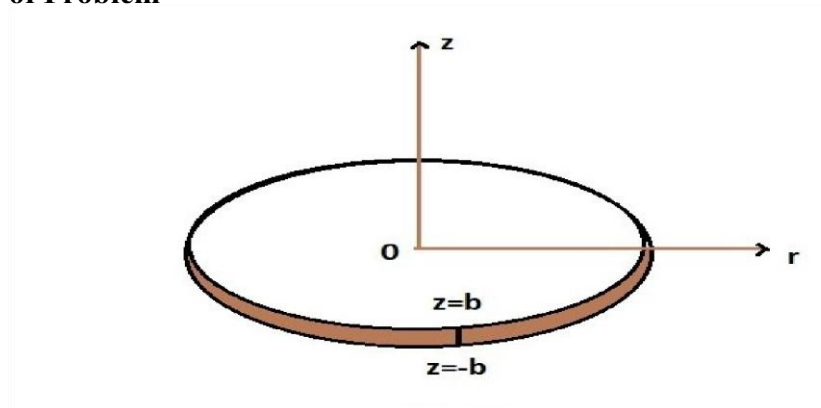


Figure 1. Thick Circular Plate under Axisymmetric Heat Supply

6. Solution using Direct Approach

We use the Laplace transform defined as follows:

$$\bar{f}(r, z, p) = L[f(r, z, t)] = \int_0^{\infty} e^{-pt} f(r, z, t) dt \quad (37)$$

where, p is the Laplace transform parameter.

Applying Laplace transform to equations (20) to (28) and using the homogeneous initial conditions (29) to (32), ones obtain:

$$\nabla^2 \bar{u} - \frac{1}{r^2} \bar{u} + (\xi^2 - 1) \frac{\partial \bar{e}}{\partial r} - \xi^2 \frac{\partial \bar{\theta}}{\partial r} - \xi^2 \frac{\partial \bar{C}}{\partial r} = \xi^2 p^2 \bar{u} \quad (38)$$

$$\nabla^2 \bar{w} + (\xi^2 - 1) \frac{\partial \bar{e}}{\partial z} - \xi^2 \frac{\partial \bar{\theta}}{\partial z} - \xi^2 \frac{\partial \bar{C}}{\partial z} = \xi^2 p^2 \bar{w} \quad (39)$$

$$\nabla^2 \bar{\theta} = \bar{p}(1 + \tau_0 p^\alpha)(\bar{\theta} + \epsilon \bar{e} + \epsilon \beta_1 \bar{C}) \quad (40)$$

$$\nabla^2 \bar{e} + \beta_1 \nabla^2 \bar{\theta} + \beta_2 (p + \tau_0 p^2) \bar{C} - \beta_3 \nabla^2 \bar{C} = 0 \quad (41)$$

$$\sigma_{\phi\phi} = \frac{2}{\xi_2} \frac{\bar{u}}{r} + \frac{(\xi^2 - 2)}{\xi^2} \bar{e} - \bar{\theta} - \bar{C} \quad (42)$$

$$\sigma_{rr} = \frac{2}{\xi_2} \frac{\partial \bar{u}}{\partial r} + \frac{(\xi^2 - 2)}{\xi^2} \bar{e} - \bar{\theta} - \bar{C} \quad (43)$$

$$\sigma_{zz} = \frac{2}{\xi_2} \frac{\partial \bar{w}}{\partial z} + \frac{(\xi^2 - 2)}{\xi^2} \bar{e} - \bar{\theta} - \bar{C} \quad (44)$$

$$\sigma_{rz} = \frac{1}{\xi^2} \left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial r} \right) \quad (45)$$

$$P = -\bar{e} + \beta_3 \bar{C} - \beta_1 \bar{\theta} \quad (46)$$

Simplifying equations (37) and (38), we get,

$$(\nabla^2 - p^2)\bar{e} = \nabla^2\bar{\theta} + \nabla^2\bar{C} = 0 \quad (47)$$

Eliminating \bar{e} and \bar{C} from (39), (40) and (46), we get,

$$(\nabla^6 - A_1\nabla^4 + A_2\nabla^2 - A_3)\bar{\theta} = 0 \quad (48)$$

where,

$$A_1 = -\frac{p}{(\beta_3 - 1)} \{ (p^\alpha \tau_0 + 1)(\beta_1 \varepsilon (\beta_1 + 2) + \beta_3 (\varepsilon + 1) - 1) + \beta_2 (p\tau + 1) + \beta_3 p \}$$

$$A_2 = \frac{p^2}{(\beta_3 - 1)} \{ (p^\alpha \tau_0 + 1)(\beta_1^2 \varepsilon p + \beta_3 p + \beta_2 (p\tau + 1)(\varepsilon + 1)) + \beta_2 p (p\tau + 1) \}$$

$$A_3 = \frac{p^4 \beta_2}{(\beta_3 - 1)} (p\tau_0 + 1)(p\tau + 1)$$

Similarly we can show that the \bar{e} and \bar{C} satisfy the equations,

$$(\nabla^6 - A_1\nabla^4 + A_2\nabla^2 - A_3)\bar{e} = 0 \quad (49)$$

$$(\nabla^6 - A_1\nabla^4 + A_2\nabla^2 - A_3)\bar{C} = 0 \quad (50)$$

Equation (48) can be written as:

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)(\nabla^2 - k_3^2)\bar{\theta} = 0 \quad (51)$$

where, k_1 , k_2 and k_3 are the roots having positive real parts of the characteristic equation given by

$$k^6 - A_1k^4 + A_2k^2 - A_3 = 0 \quad (52)$$

The roots k_1 , k_2 and k_3 are given by:

$$k_1 = \sqrt{\frac{1}{3}(2s_1 \sin(s_2) + A_1)},$$

$$k_2 = \sqrt{\frac{1}{3} \left[A_1 - s_1 \left(\sqrt{3} \cos(s_2) + \sin(s_2) \right) \right]}$$

$$k_3 = \sqrt{\frac{1}{3} \left[A_1 + s_1 \left(\sqrt{3} \cos(s_2) - \sin(s_2) \right) \right]},$$

where,

$$s_1 = \sqrt{A_1^2 - 3A_2}, \quad s_2 = \frac{\sin^{-1}(\nu)}{3}, \quad \nu = -\left(\frac{2A_1^3 - 9A_1A_2 + 27A_3}{2s_1^3} \right).$$

7. Extraction of Field Functions

The solution of equation (51) can be written in the form:

$$\theta = \sum_{i=1}^3 \bar{\theta}_i \quad (53)$$

where, $\bar{\theta}_i$ is a solution of the following equation:

$$(\nabla^2 - k_i^2)\bar{\theta}_i = 0, \quad i = 1,2,3. \quad (54)$$

Taking the Hankel transform of the above equation, we get

$$(D^2 - \gamma^2 - k_i^2)\bar{\theta}_i^* = 0, \quad i = 1,2,3. \quad (55)$$

where,

$$D = \frac{\partial}{\partial z}$$

The solution of equation (55) has the form

$$\bar{\theta}_i^* = \sum_{i=1}^3 B_i(\gamma, p) \cosh(q_i z) \quad (56)$$

where,

$$q_i^2 = \gamma^2 + k_i^2$$

Similarly \bar{e}^* and \bar{C}_i^* can be obtain form (49) & (50) as:

$$\bar{e}^* = \sum_{i=1}^3 C_i(\gamma, p) \cosh(q_i z) \quad (57)$$

$$\bar{C}_i^* = \sum_{i=1}^3 D_i(\gamma, p) \cosh(q_i z) \quad (58)$$

where, $B_i(\gamma, p), C_i(\gamma, p)$ & $D_i(\gamma, p), i = 1,2,3$ are parameters depending on γ & p .

Substituting (56), (57), (58) in (40) and (47) the parameters $C_i(\gamma, p)$ and $D_i(\gamma, p), i = 1,2,3$ can be expressed in terms of $B_i(\gamma, p)$ as:

$$C_i(\gamma, p) = f_i B_i(\gamma, p) \quad (59)$$

$$D_i(\gamma, p) = d_i B_i(\gamma, p) \quad (60)$$

where,

$$f_i = \frac{\{k_i^4 - [p(1 + \tau_0 p^\alpha)(1 - \beta_1 \varepsilon)]k_i^2\}}{\varepsilon p(1 + \tau_0 p^\alpha)[k_i^2(1 + \beta_2) - \beta_1 p^2]}$$

and

$$d_i = \frac{\{k_i^4 - [p(1 + \tau_0 p^\alpha)(1 + \epsilon) + p^2]k_i^2 + p^3(1 + \tau_0 p^\alpha)\}}{\varepsilon p(1 + \tau_0 p^\alpha)[k_i^2(1 + \beta_2) - \beta_1 p^2]}.$$

Applying the inversion of the Hankel transform [14] to equations (56), (57) & (58) we get,

$$\bar{\theta}_i^* = \int_0^\infty \left\{ \sum_{i=1}^3 B_i(\gamma, p) \cosh(q_i z) \right\} \gamma J_0(\gamma r) d\gamma \quad (61)$$

$$\bar{e}_i^* = \int_0^\infty \left\{ \sum_{i=1}^3 C_i(\gamma, p) \cosh(q_i z) \right\} \gamma J_0(\gamma r) d\gamma \quad (62)$$

$$\bar{C}_i^* = \int_0^\infty \left\{ \sum_{i=1}^3 D_i(\gamma, p) \cosh(q_i z) \right\} \gamma J_0(\gamma r) d\gamma \quad (63)$$

Equations (38) to (39) and (61) to (63) then solution for the displacement components in the Laplace transform domain as:

$$\bar{u}(r, z, p) = \int_0^\infty -\gamma^2 J_1(\gamma r) \left[E(\gamma, p) \cosh(qz) + \sum_{i=1}^3 \frac{\lambda_i}{(q_i^2 - q^2)} \cosh(q_i z) \right] d\gamma \quad (64)$$

$$\bar{w}(r, z, p) = \int_0^\infty \gamma^2 J_0(\gamma r) \left[F(\gamma, p) \sinh(qz) + \sum_{i=1}^3 \frac{\lambda_i q_i}{(q_i^2 - q^2)} \cosh(q_i z) \right] d\gamma \quad (65)$$

where, the parameters $E(\gamma, p)$ & $F(\gamma, p)$ depend on γ and s only and

$$q^2 = \gamma^2 + \xi^2 p^2, \quad F(\gamma, p) = \frac{\gamma^2 E(\gamma, p)}{q}, \quad \lambda_i = \{(1 - \xi^2) f_i + \xi^2 (1 + d_i)\} B_i$$

using equations (42) to (46) & the solutions from equations (61) to (65) we obtain the stress components & the chemical potential in the Laplace transform domain as follows:

$$\bar{\sigma}_{\varphi\varphi} = -\frac{2}{\xi^2 r} \int_0^\infty \gamma^2 J_1(\gamma r) \left[E(\gamma, r) \cosh(qz) + \sum_{i=1}^3 \frac{\lambda_i}{(q_i^2 - q^2)} \cosh(q_i z) \right] d\gamma + \bar{G} \quad (66)$$

$$\begin{aligned} \bar{\sigma}_{rr} = & -\frac{2}{\xi^2} \int_0^\infty \gamma^3 E(\gamma, r) \cosh(qz) \left[\frac{1}{\gamma r} J_1(\gamma r) - J_0(\gamma r) \right] d\gamma - \\ & \frac{2}{\xi^2} \int_0^\infty \gamma^3 \left[\frac{1}{\gamma r} J_1(\gamma r) - J_0(\gamma r) \right] \left[\sum_{i=1}^3 \frac{\lambda_i}{(q_i^2 - q^2)} \cosh(q_i z) \right] d\gamma + \bar{G} \end{aligned} \quad (67)$$

$$\bar{\sigma}_{zz} = -\frac{2}{\xi^2} \int_0^\infty \left[\gamma^2 E(\gamma, r) q \cosh(qz) + \sum_{i=1}^3 \frac{\lambda_i q_i^2}{(q_i^2 - q^2)} \cosh(q_i z) \right] \gamma J_0(\gamma r) d\gamma + \bar{G} \quad (68)$$

$$\bar{\sigma}_{rz} = -\frac{1}{\xi^2} \int_0^\infty \left[\left(\frac{\gamma^2 + q^2}{q} \right) E(\gamma, r) \sinh(qz) + 2 \sum_{i=1}^3 \frac{\lambda_i q_i}{(q_i^2 - q^2)} \sinh(q_i z) \right] \gamma^2 J_0(\gamma r) d\gamma \quad (69)$$

$$\bar{P} = \frac{2}{\xi^2} \int_0^\infty \left[\sum_{i=1}^3 \mu_i B_i \cosh(q_i z) \right] \gamma J_0(\gamma r) d\gamma \quad (70)$$

where,

$$\bar{G} = \int_0^\infty \gamma J_0(\gamma r) \left(\sum_{i=1}^3 \zeta_i \cosh(q_i z) \right) d\gamma,$$

$$\mu_i = (-f_i + \beta_3 d_i - \beta_3)$$

$$\zeta_i = \left(\frac{(\xi^2 - 2)}{\xi^2} f_i - d_i - 1 \right) B_i$$

and

8. Numerical Inversion

The solutions of Concentration (C), temperature (θ), Chemical Potential (P), displacement (u) and stresses (σ_{rr} , σ_{rz} & σ_{zz}) are obtained numerically by the inversion of Laplace transform method in time domain [38].

$$f(t) = \frac{e^{\gamma t}}{t} \left[\frac{1}{2} \bar{f}(\gamma) + Re \left(\sum_{k=1}^N (-1)^k \bar{f} \left(\gamma + \frac{ik\pi}{t} \right) \right) \right]$$

9. Physical Constants

These material constants for copper material has been used for numerical calculation as follows [17]:

Table 1. Physical Constants for Copper Material

Physical Constants	Value
Lame's Constant (λ)	$7.76 \times 10^{10} \text{ Nm}^{-2}$
Lame's Constant (μ)	$3.86 \times 10^{10} \text{ kgm}^{-1} \text{ s}^{-2}$
Density (ρ)	8954 kgm^{-3}
Reference temperature (T_0)	293° K
Coefficient of linear thermal expansion (α_t)	$1.78 \times 10^{-5} \text{ K}^{-1}$
Thermal Conductivity (k)	$386 \text{ JK}^{-1} \text{ m}^{-1} \text{ s}^{-1}$
Specific heat per unit mass (C_E)	$383.1 \text{ JK}^{-1} \text{ m}^{-1} \text{ s}^{-1}$
Coefficient of linear diffusion equation (α_c)	$1.78 \times 10^{-5} \text{ K}^{-1}$
Measure of the thermo-diffusion effect (β_t)	$1.2 \times 10^4 \text{ m}^2 \text{ s}^{-2} \text{ K}^{-1}$

Measure of diffusive effect (β)	$0.9 \times 10^6 \approx 1 \text{ m}^5 \text{ kg}^{-1} \text{ s}^{-2}$
Diffusion coefficient (D)	$0.88 \times 10^{-8} \text{ kgsm}^{-3}$
$\eta = \frac{\rho C_E}{k}$	8886.73 sm^{-2}
$\epsilon = \frac{\xi_1^2 T_0}{\rho C_E (\lambda + 2\mu)}$	0.0168 NmJ^{-1}
$\beta_1 = \frac{\beta_t (\lambda + 2\mu)}{\xi_1 \xi_2}$	5.43
$\beta_2 = \frac{(\lambda + 2\mu)}{D \eta \xi_2^2}$	0.533
$\beta_3 = \frac{\beta_t (\lambda + 2\mu)}{\xi_2^2}$	36.24
β^2	4

10. Results & Discussion

In this chapter, we consider a thick circular plate of height $b = 1\text{m}$ and axisymmetric heat is supplied to its upper and lower surface. The procured results have obtained for concentration (C) distribution, Chemical Potential (P) distribution, temperature (θ) distribution, displacement component (u), radial (σ_{rr}) stress, and axial (σ_{zz}) stress. The computations are carried out for distinct values of time $t = 0.05\text{s}, 0.065\text{s}, 0.08\text{s}$.

The results have studied in the context of two different theories Lord-Shulman (LS) and Classical coupled (CTE) theory. The graphs have been plotted along radial direction i.e. in the middle plane when $z = 0$. The changing behaviour of field variables have been studied through graphs in two cases:

- (1) For distinct values of time $t = 0.05\text{s}, 0.065\text{s}, 0.08\text{s}$ when $\alpha = 0.25$, shown in **Figures 2, 4, 6, 8, 10, 12** and
- (2) For distinct values of fractional order parameter $\alpha = 0.25, 0.50, 0.75$ when $t = 0.05\text{s}$, as in **Figures 3, 5, 7, 9, 11, 13**.

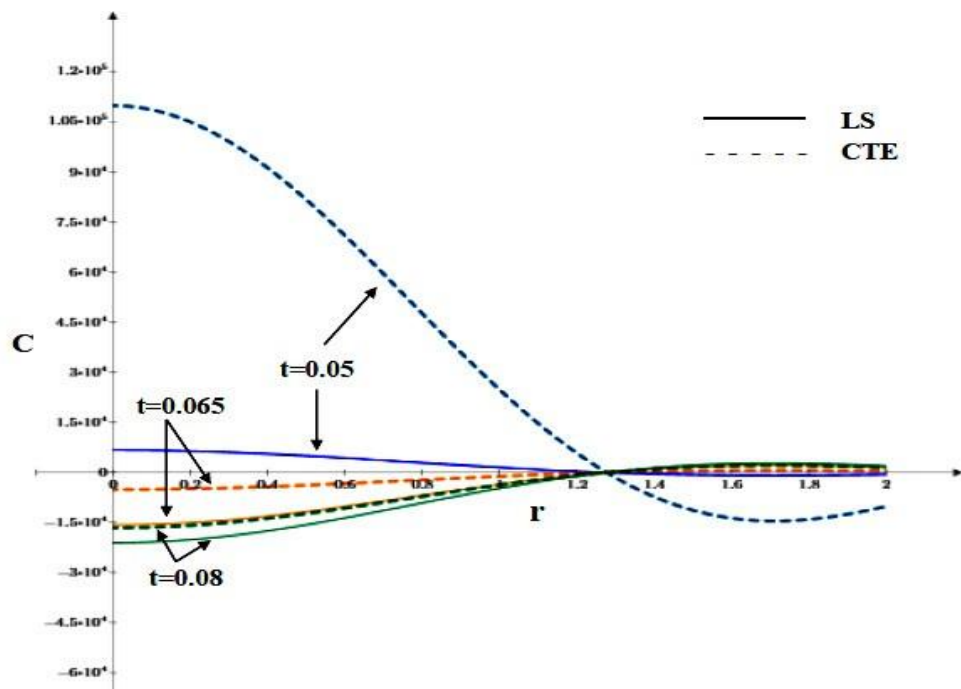


Figure 2. Concentration C distribution for distinct values of time $t(s)$

Figures 2, 4, 6, 8, 10, 12 shows concentration, temperature, chemical potential, displacement, radial stress, and axial stress in the radial direction for $\alpha = 0.25$ order derivative and $t = 0.05s, 0.065s, 0.08s$ in context of both LS and CTE model.

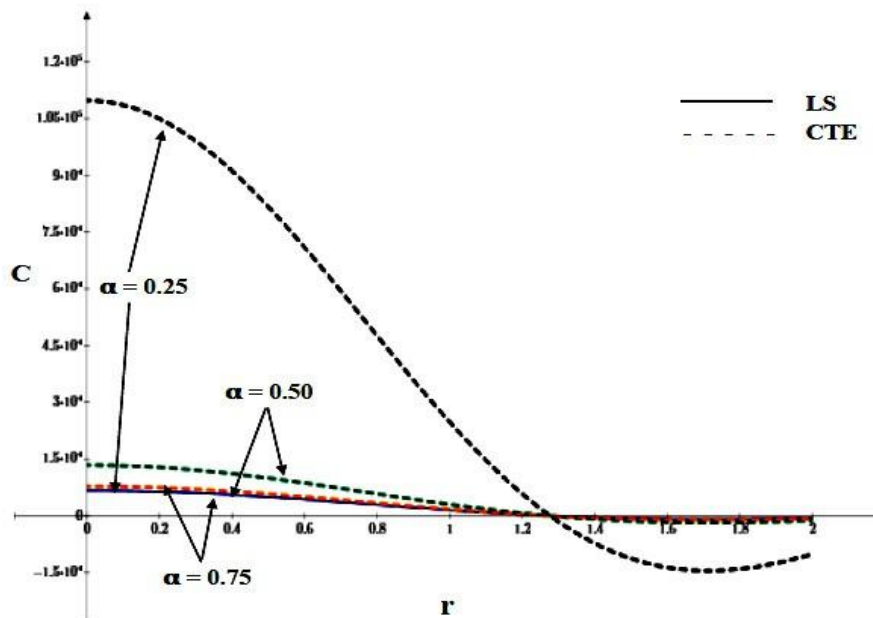


Figure 3. Concentration C distribution for distinct values of α

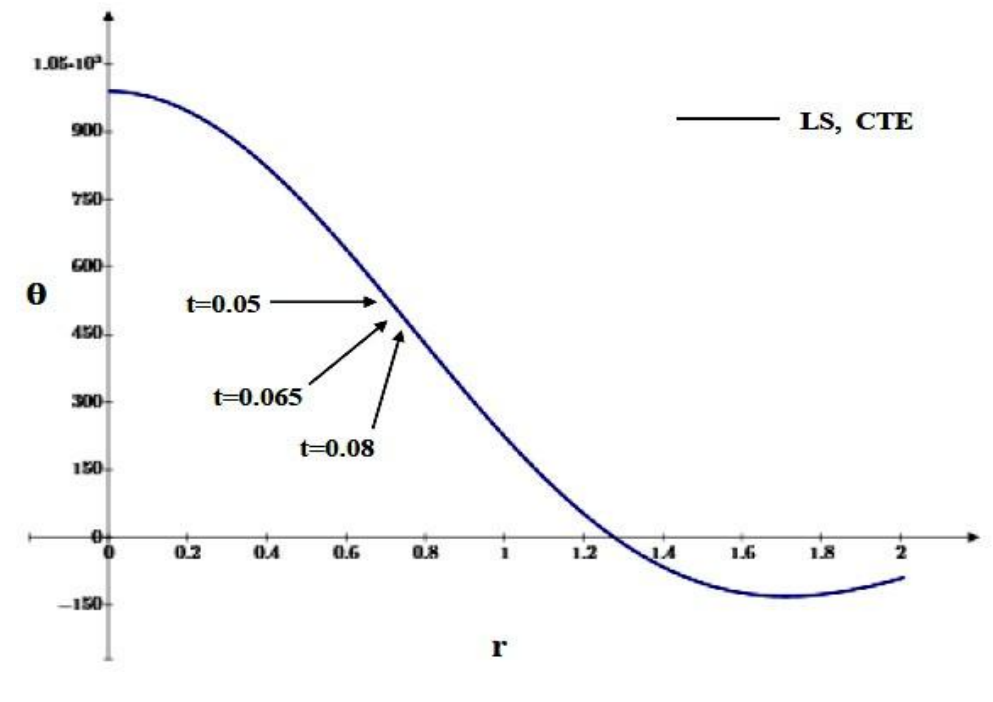


Figure 4. Temperature θ distribution for distinct values of $t(s)$

Figures 3, 5, 7, 9, 11, 13 shows concentration, temperature, chemical potential, displacement, radial stress, and axial stress in the radial direction for fixed time $t = 0.05s$ and distinct values of fractional order parameter $\alpha = 0.05, 0.065, 0.08$.

Figures 2, 6 shows concentration and chemical potential distribution in the middle plane $z = 0$. These figures display second sound effect. Here as time increases, concentration decreases for $0 \leq r \leq 1.3$ then vanishes at $r \approx 1.3$ and for $1.3 \leq r \leq 2.4$, it increases with time.

Figure 4 represents temperature distribution along radial direction. It also exhibits second sound effect. The graph shows a single curve for distinct values of time $t = 0.05s, 0.065s, 0.08s$. Thus θ gives same value as time varies. Here θ decreases for $0 \leq r \leq 1.3$ then vanish at $r \approx 1.3$ and increases for $1.3 \leq r \leq 2.4$.

Figure 5 represents temperature along radial direction for $\alpha = 0.25, 0.50, 0.75$. The graph shows a single curve for distinct values of fractional order parameter $\alpha = 0.25, 0.50, 0.75$ for both LS and CTE model. Here θ decreases for $0 \leq r \leq 1.3$ then vanish at $r \approx 1.3$ and increases for $1.3 \leq r \leq 2.4$.

Figure 3, 7 shows concentration and chemical potential distribution in the middle plane $z = 0$ for distinct α in context of LS and CTE model.

Figure 8 shows displacement component along radial direction for distinct values of time, where the displacement component u decreases gradually, takes maximum value at $r \approx 0.8$ and then slowly increases and vanishes at $r \approx 1.7$.

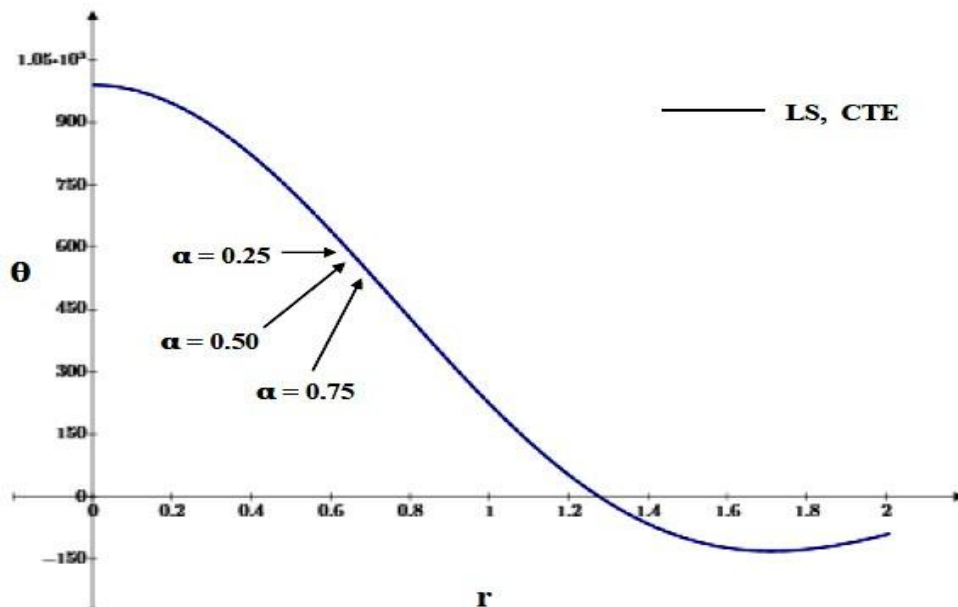


Figure 5. Temperature θ distribution for distinct values of α

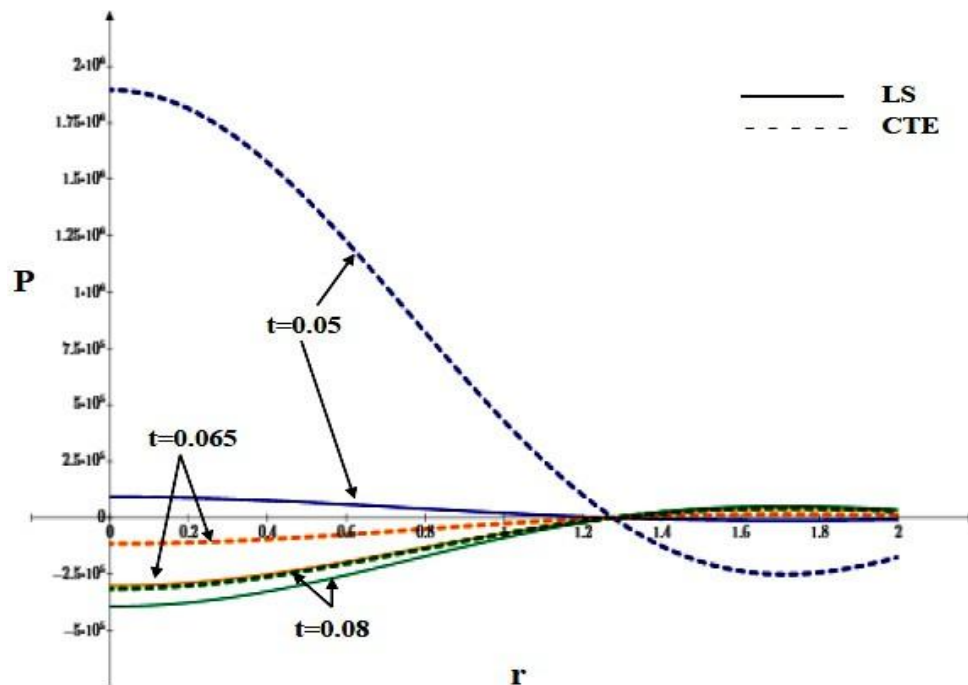


Figure 6. Chemical P Potential for distinct values of $t(s)$

Figure 9 shows displacement component along radial direction for distinct values of fractional order parameter α . In both the cases of LS and CTE model the displacement component u decreases gradually for small values $\alpha = 0.25, 0.50$,

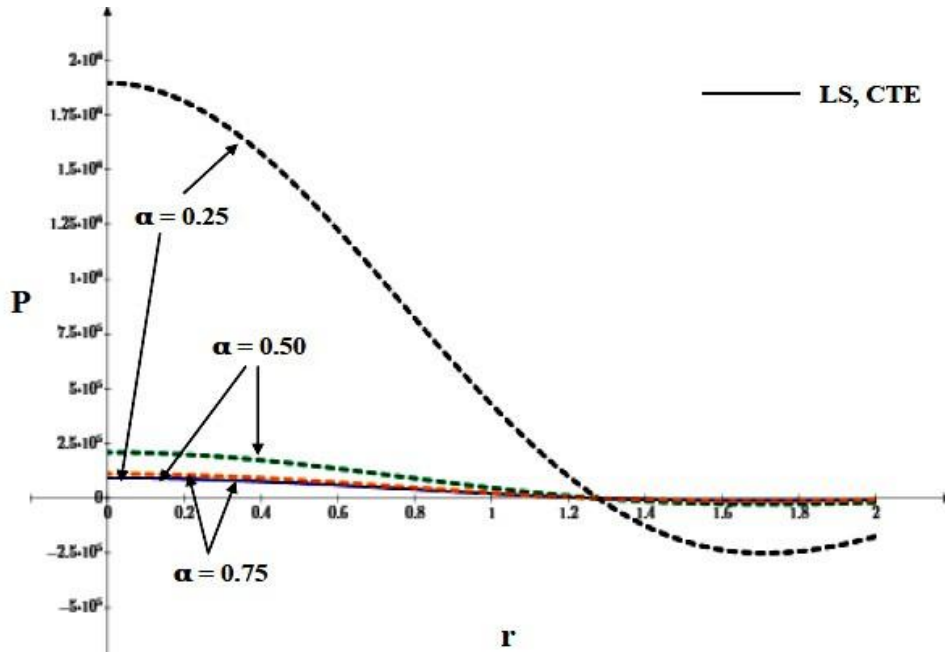


Figure 7. Chemical P Potential for distinct values of α

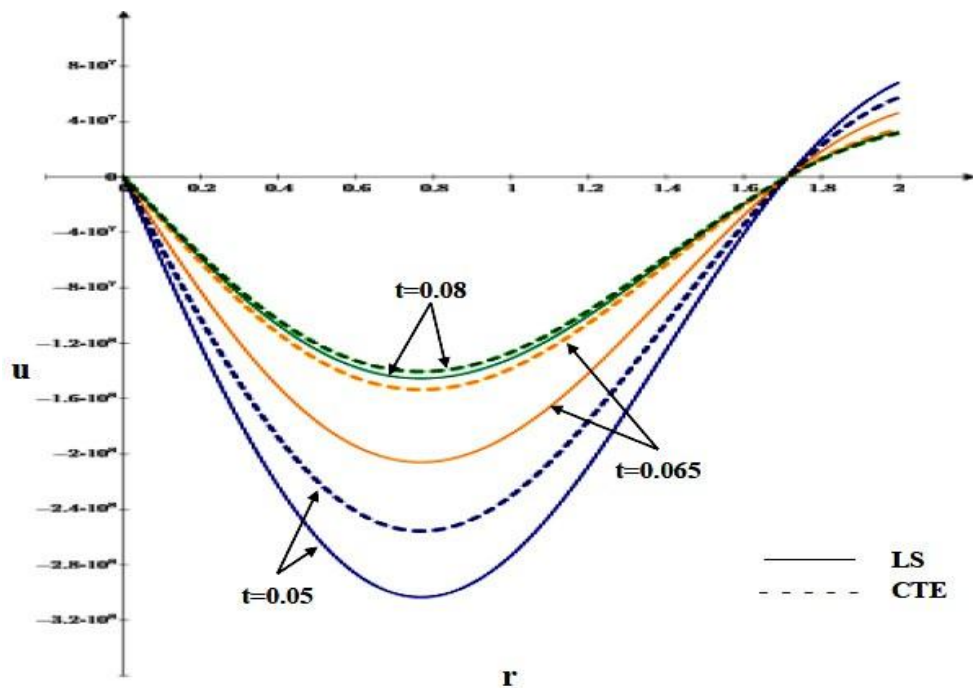


Figure 8. Displacement component u for distinct values of $t(s)$

takes maximum value at $r \approx 0.8$ and then slowly increases and vanish at $r \approx 1.7$. But in CTE model for large value $\alpha = 0.75$, displacement u increases gradually takes maximum at $r \approx 0.8$ and then slowly decreases and vanish at $r \approx 1.7$.

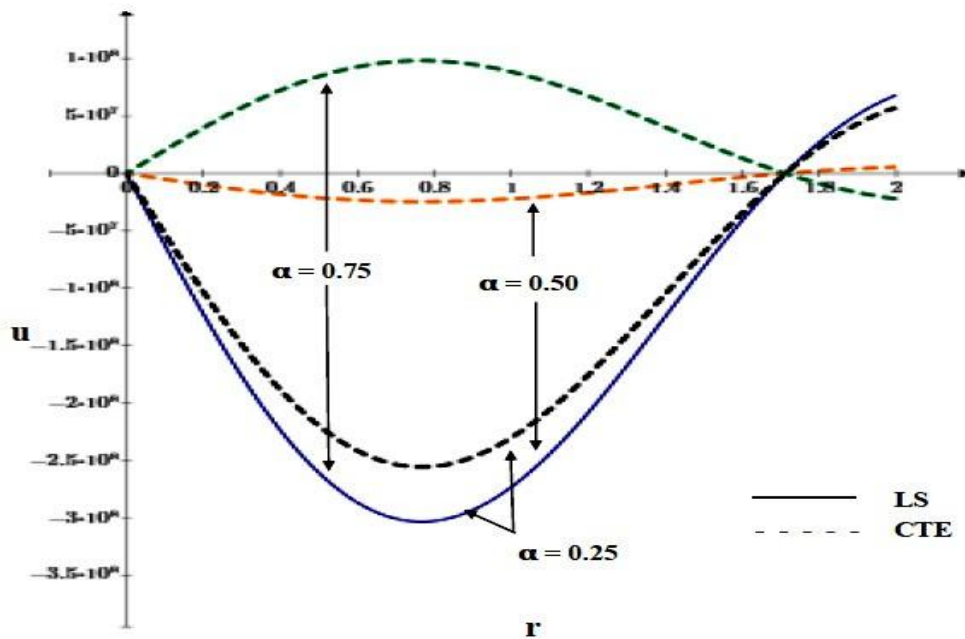


Figure 9. Displacement component u for distinct values of α

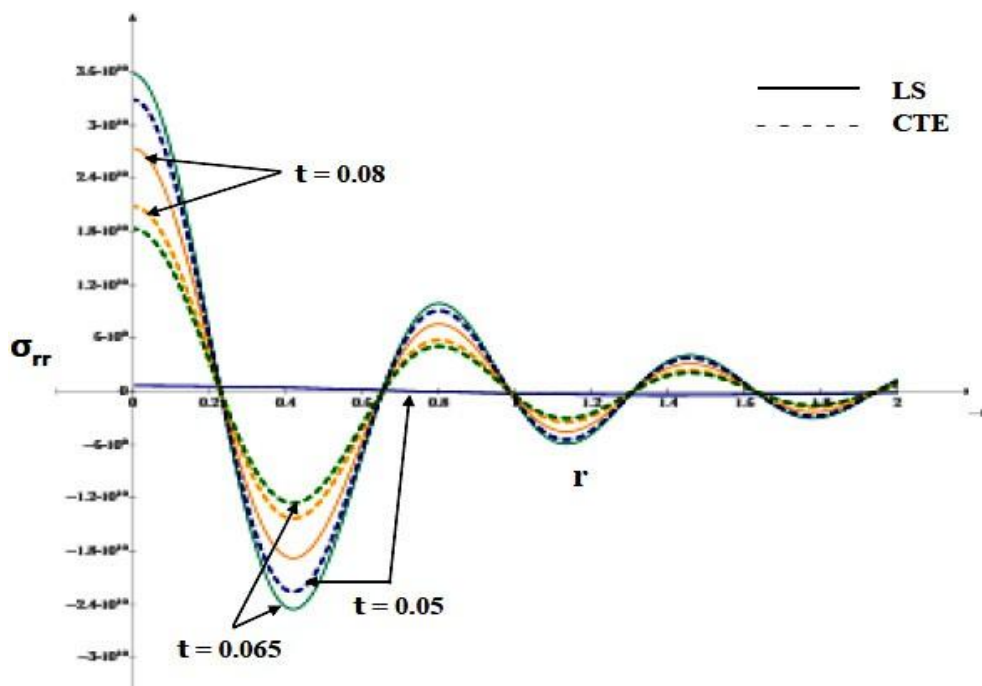


Figure 10. Radial stress σ_{rr} component for distinct values of $t(s)$

Figure 10 shows the oscillatory nature of radial stress σ_{rr} . The nature of radial stress σ_{rr} is oscillating as shown in **figure 11**. While the axial stress is increases with increase in value of α and

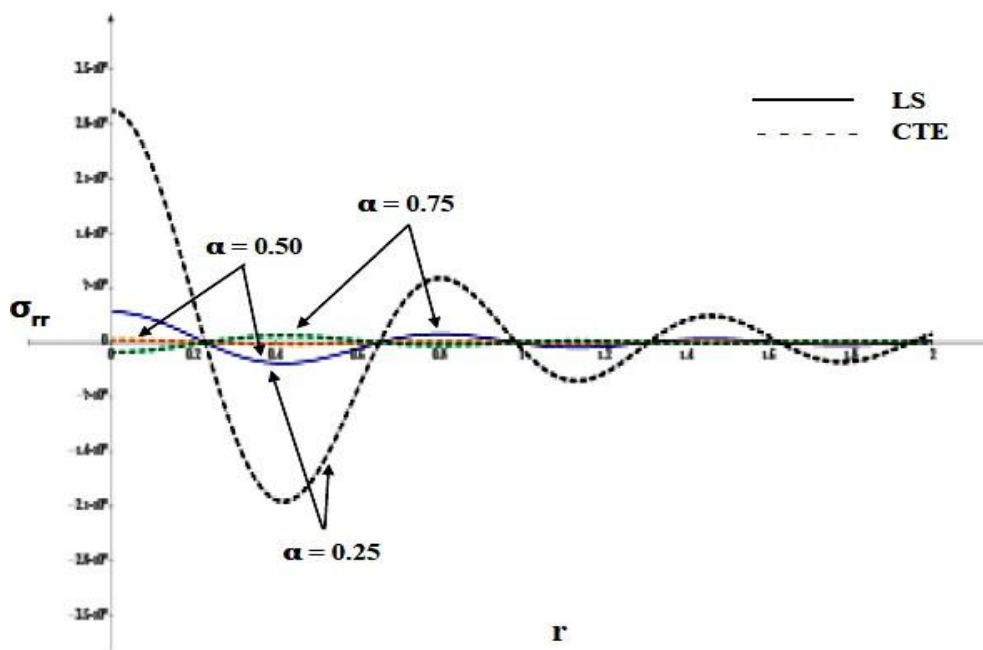


Figure 11. Radial stress σ_{rr} component for distinct values of α

becomes zero at $r = 1$ then after decreases as α increases.

In **figure 12** axial stress σ_{zz} is compressive for $0 \leq r \leq 1$ then after it is tensile for $1 \leq r \leq 2.4$.

Figure 10, 12 represents the radial stress σ_{rr} and the axial stress σ_{zz} in the middle plane $z = 0$. The nature of stresses differ. These stresses shows continuity near the centre of the thick circular plate. Both radial σ_{rr} and axial σ_{zz} stresses becomes zero at $r = 1$.

Figures 11, 13 displays the radial and axial stresses for distinct values of α in axial direction. The nature of graphs in each case is similar for LS and CTE model as shown in **figures 2, 4, 6, 8, 10, 12**.

Also it is clearly seen that the solutions of LS and CTE theory differs for $t = 0.05s, 0.065s, 0.08s$. For large times the results are similar for LS and CTE theory, since the thermal disturbances reach all parts of the medium along with finite speed of propagation. For LS model the variations in the value of fractional order parameter α doesn't affect concentration, temperature, chemical potential, displacement, radial stress, and axial stress as displays in **figures 3, 5, 7, 9, 11, 13**. Thus it gives a single curve for $\alpha = 0.25, 0.50, 0.75$ in each case. For CTE model as value of fractional order parameter α increases the value of

concentration distribution C decreases for $0 \leq r \leq 1.3$, becomes zero at $r \approx 1.3$ and then increases with α .

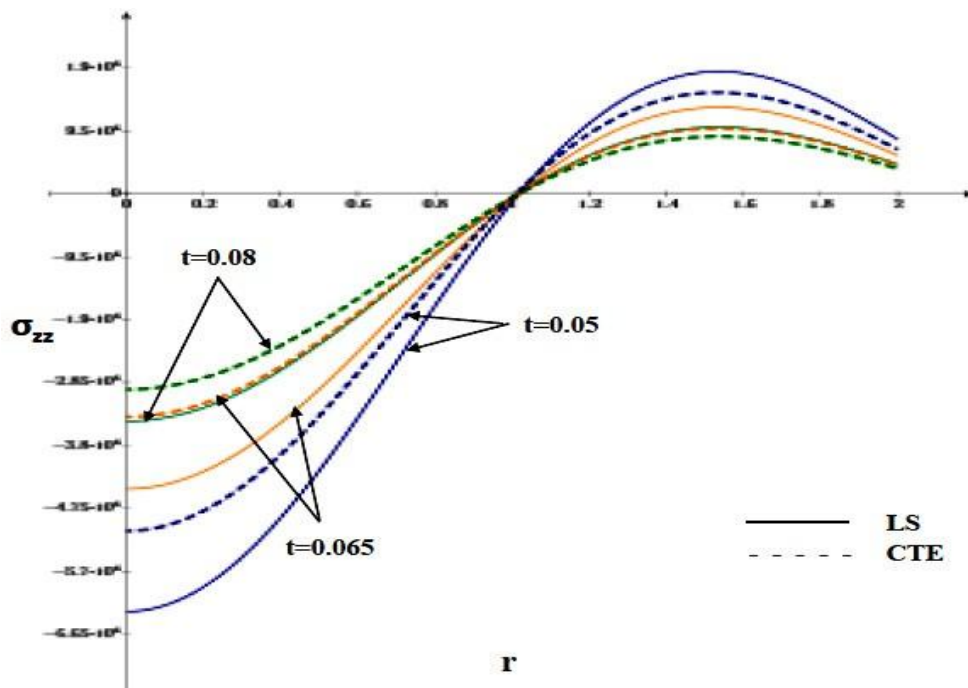


Figure 12. Axial stress σ_{zz} component for distinct values of $t(s)$

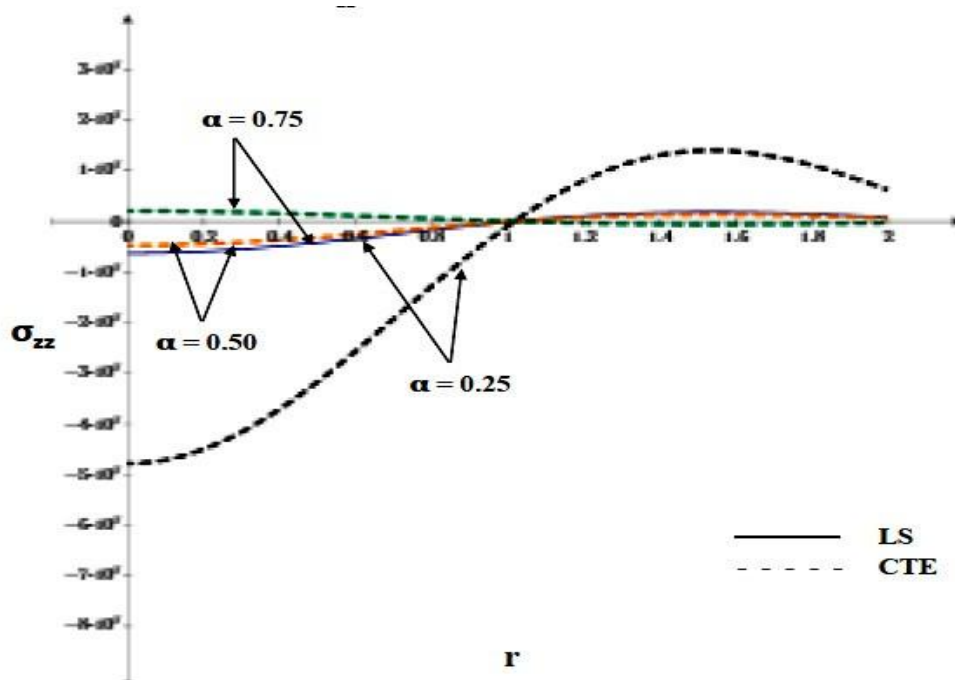


Figure 13. Axial stress σ_{zz} component for distinct values of α

Conclusion

In this chapter, we analytically solved the generalized fractional order thermoelastic problem for a thick circular plate using a direct approach and integral transform techniques.

The important findings of this chapter are as follows:

- The used approach avoids the use of potential function, offering a simplified and efficient route to obtain exact solution in the Laplace domain.
- Graphical results demonstrate how temperature, displacement, stresses, concentration, and chemical potential behave under various time intervals and fractional order parameters.
- A key observation is that the fractional order parameter has no significant effect in the LS model, while it shows noticeable influence in the CTE model. Moreover, both models yield similar outcomes at higher time values, indicating convergence due to the finite speed of thermal wave propagation.
- The findings have significant implications for researchers exploring fractional thermoelasticity, particularly in the design of advanced materials and systems subjected to thermal and diffusion effects.
- Future studies may extend this analysis along the axial direction, potentially revealing further insights into the behaviour of such materials.

Author Contributions All authors reviewed the manuscript.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing Interests The authors have disclosed that they have no potential conflicts of interest and were not aided by any funding in the preparation of this manuscript.

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