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# The Nonlinear Schrödinger Equation Derived from the higher order Kaup–Kupershmidt (KK) Equation Using Multiple Scales Method

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**Abstract:**
The mathematical models of problems that arise in many branches of science are nonlinear equations of evolution (NLEE). For this reason, nonlinear equations of evolution have served as a language in formulating many engineering and scientific problems. Although the origin of nonlinear evolution equations dates back to ancient times, significant developments have been made in these equations up to the present day. The main reason for this situation is that nonlinear equations of evolution involve the problem of nonlinear wave propagation. Therefore, many different and effective techniques have been developed regarding nonlinear evolution equations and solution methods. Studies conducted in recent years show that evolution equations are becoming increasingly important in applied mathematics. This study is about the multiple scales methods, known as the perturbation method, for nonlinear equations of evolution (NLEE). In this report, the multiple scales method was applied for the analysis of the (1 + 1) dimensional higher-order nonlinear Kaup–Kupershmidt (KK) equations, and the nonlinear Schrödinger (NLS) equation was obtained. Also, the approximate solution of the (1 + 1) dimensional higher-order nonlinear Kaup–Kupershmidt (KK) equation is obtained.

**Keywords**: Multiple scales method, KdV equation, NLS type equations, Kaup–Kupershmidt (KK) equation.

## 1. Introduction

Nonlinear evolution equations (NLEE) [1, 2, 3] play a significant role in fields such as optics, fluid physics, plasma physics, statistical physics, biology, chemistry, communication, and engineering technology. Therefore, the study of (NLEE) has become a critical research topic.

The current research on the nonlinear evolution equations (NLEE) mainly includes the existence of solutions [4], stability of solutions [5], uniqueness of solutions [6], numerical solutions [7], and exact solutions [8]. Based on the existing practical problems in the real world, obtaining analytical solutions for such (NLEE) is an urgent practical problem that needs to be solved. Since data of their exact solutions facilitates the confirmation of numerical solvers and supports in decision-making analysis of solutions, the analytical study of these (NLEE) is significant. This not only helps to better understand the solutions but also helps us to understand the phenomenon they describe [9, 10, 11, 12, 13]. The studies of engineering and science demand visualization of the mathematical model for real-life phenomena and the development of a formula. Numerous real-world problems are modelled using higher-order (NLEE) equations, either as ordinary or partial differential equations (PDEs). For instance, higher-order differential equations are used to examine the effects of unsafe contact with an infected corpse by the NiV virus and to study the dynamics of an accelerated mass-spring system [14, 15]. They are also employed to investigate chaos, bifurcations, signal processing, heat distribution, secure communications, plasma physics, and the behaviour of surface water waves in fluid dynamics, and describe the propagation of electrostatic waves in plasmas [16, 17, 18, 19].

Known for its distinguished role in the development of nonlinear physics, the Korteweg-de Vries (KdV) equation has been derived in a variety of physical contexts. For example, it can be thought of as modelling the unilateral propagation of long-wavelength gravitational waves of small amplitude in a shallow channel. In any case, the Korteweg-de Vries equation is obtained with a certain degree of approximation and, therefore, cannot be considered to represent physical reality with perfect accuracy. Therefore, an important question arises about what would happen to the solutions of Korteweg-de Vries' equation when perturbations from neglected terms in the derivation come into play. For instance, would the perturbed Korteweg-de Vries equation still have a well-localized single-wave solution? The answer, sure, would depend on the nature and physical origin of the perturbation. Currently, studies are underway on the nonlinear fifth-order KdV-type equations as they can describe real properties in various scientific applications and engineering fields, and have practical and physical significance.

The nonlinear Schrödinger (NLS) equation is an example of a universal nonlinear model because it describes a wide variety of physical systems. As a result, the equation may be used to describe a wide range of nonlinear physical events [20, 21, 22]. It is renowned that a multiscale analysis of the KdV equation gives rise to the NLS equation for modulated amplitude [23, 24, 25, 26, 27]. In [23], Zakharov and Kuznetsov demonstrated a much deeper correspondence between these integrable equations, not only at the equation level but also at the linear spectral problem level, by showing that multiscale analysis of the Schrödinger spectral problem yields the Zakharov-Shabat problem for the NLS equation. Özer and Dag˘ demonstrated a similar link between the NLS and integrable fifth-order nonlinear evolution equations [28]. Additionally, similar results were obtained using multiscale analysis in different equations [29, 30, 31, 32].

In this paper, we apply a multiple scale method following Zakharov and Kuznetsov [23] related to the connection of the KdV and NLS equations. This is an important derivation because the KdV flow equations follow from the NLS equation. The strength of this method is that for each degree of coefficients in epsilon, the equations contain no secular terms. Therefore, there is no freedom in choosing the coefficients, and the expansion is uniquely determined. The derivation of this hierarchy is not a simple case of algorithms, but basically relies on two facts. First, different time flows must commute; i.e.: $u_{t_i t_j} = u_{t_j t_i}$. Second, in each order in epsilon, the coefficient equations contain secular terms. Eliminating the secular terms requires $u_{t_i}$ to have a certain high symmetry (flow) of the hierarchy, and this way all coefficients of expansion are fixed and no arbitrariness is left.

After the introduction, in chapter 2, we briefly expressed the fifth-order Korteweg-de Vries Equation (KdV5) flow equations, the seventh-order Korteweg-de Vries Equation (KdV7) flow equations, and the multiple scales method, respectively. In Chapter 3, we applied the method given in Chapter 2 to the (1+1) dimensional fifth-order and seventh-order Kaup–Kupershmidt (KK) equation. The last part consists of the conclusion part.

## 2. Background Materials

In this section, we present some background material on the best-known fifth-order, seventh-order KdV equations and the multi-scale method.

### 2.1 The fifth-order Korteweg-de Vries Equation (fKdV) Flow Equations

The best-known fifth-order KdV equations look like this

$$u_t = \omega u_{xxxxx} + \alpha u_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x. \quad (1)$$

where $\alpha$, $\beta$, $\gamma$ and $\omega$ are arbitrary nonzero, and real parameters, and $u = u(x, t)$ is a smooth enough function. Since the parameters, $\alpha$, $\beta$, $\gamma$ and $\omega$ are arbitrary and take different values, this will greatly change the properties of the fKdV equation (1). Changing the actual values of the parameters allows you to generate many different variations of the fKdV equation. The fKdV equations, which are widely used in nonlinear optics and quantum physics, are an important mathematical model. Characteristic examples are commonly utilized in many domains, including plasma physics, quantum field theory, solid-state physics, and fluid physics [33, 42].

Some important special cases of Eq. (1) are:

Kaup-Kupershmidt (KK) equation [37, 38, 39, 40, 41]

$$u_t = u_{xxxxx} + 10uu_{xxx} + 25u_x u_{xx} + 20u^2 u_x, \quad (2)$$

Sawada-Kotera (SK) equation [34, 43]

$$u_t = u_{xxxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x, \quad (3)$$

Caudrey-Dodd-Gibbon (CDG) equation

$$u_t = u_{xxxxx} + 30u_{xxx} + 30u_x u_{xx} + 180u^2 u_x, \quad (4)$$

Lax equation [33]

$$u_t = u_{xxxxx} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x, \tag{5}$$

Ito equation [35, 36]

$$u_t = u_{xxxxx} + 3u_{xxx} + 6u_xu_{xx} + 2u^2u_x, \tag{6}$$

As the values of *α, β, and γ* vary, the characteristics of the equation (1) change drastically. For example, the KK equation with α= 10, *β* = 25, and *γ* = 20, which is known to be integrable [39], and has bilinear representations [39, 41], but the obvious form of the *N*-soliton solutions is not known. Another example is the SK equation, where α= *β* = *γ* = 5, and the Lax equation with α= 10, *β* = 20, and *γ* = 30, are both fully integrable. These two equations have *N*-soliton solutions and an endless set of conserved densities. One last equation in this class is the Ito equation, with α= 3, *β* = 6, and *γ* = 2, which cannot be fully integrated but has a limited number of special conserved densities [36].

## 2.2 The seventh-order Korteweg-de Vries Equation (sKdV) Flow Equations

The best-known fifth-order KdV equations look like this

$$u_t + au^3u_x + bu^3 + cuu_xu_{2x} + du^2u_{3x} + eu_{2x}u_{3x} + fu_xu_{4x} + guu_{5x} + u_{7x} = 0 \tag{7}$$

where *a*, *b*, *c, d, e, f* and *g* are constant parameters also, these parameters cannot be zero, and $u_{ix} = (\partial^i/\partial_x^i)u$. Pomeau et al. [44] have introduced the seventh-order KdV equation for discussing the structural stability of KdV equation under a singular perturbation.

When *a* = 252, *b* = 63, *c* = 378, *d* = 126, *e* = 63, *f* = 42 and *g* = 21, we have:

$$\begin{aligned} &u_t + 252u^3u_x + 63u_x^3 + 378u_xu_{2x} + 126u^2u_{3x} \\ &+63u_{2x}u_{3x} + 42u_xu_{4x} + 21u_{5x} + u_{7x} = 0 \end{aligned} \tag{8}$$

This equation is the well-known seventh-order Sawada–Kotera–Ito equation [45, 46, 47].

When *a* = 140, *b* = 70, *c* = 280, *d* = 70, *e* = 70, *f* = 42 and *g* = 14, we have:

$$\begin{aligned} &u_t + 140u^3u_x + 70u_x^3 + 280u_xu_{2x} + 70u^2u_{3x} \\ &+70u_{2x}u_{3x} + 42u_xu_{4x} + 14u_{5x} + u_{7x} = 0 \end{aligned} \tag{9}$$

This equation is the well-known seventh-order Lax equation [45, 46, 47]. When a = 2016, b = 630, c = 2268, d = 504, e = 252, *f* = 147 and g = 42, we have:

$$\begin{aligned} &u_t + 2016u^3u_x + 630u_x^3 + 2268u_xu_{2x} + 504u^2u_{3x} \\ &+252u_{2x}u_{3x} + 147u_xu_{4x} + 42u_{5x} + u_{7x} = 0 \end{aligned} \tag{10}$$

This equation is the well-known seventh-order Kaup–Kupershmidt equation [45, 46, 47].

### 2.3 The Multiple Scales Method

The multiple scales method is a perturbation method. In the multiple scales method first recommended by Zakharov and Kuznetsov [23], Zakharov and Kuznetsov used this method to decrease the KdV equation to the NLS equation and apply it to a class of nonlinear evolution equations. Using this method, they showed that integrable systems can be decreased to integrable systems. If the system we have taken at the beginning is not an integrable system, it has been seen that the reduced system as a result of the application of the method is either integrable or non-integrable. However, if the method is applied to a suitable integrable system, it is seen that the system obtained as a result of the analysis is always an integrable system. This is the master purpose of applying the multi-scale expansion method to integrable systems. In this section, multiple scales method of nonlinear evolution equations is discussed. By applying the Zakharov and Kuznetsov (1986) technique, the steps of the multi-scale method in obtaining NLS type equations from KdV equations are shown in order.

Let consider the general evolution equation in the following form

$$u_t = K(u, u_x, u_y \ldots) \tag{11}$$

Where $K[u]$ is a function of $u$ and its derivatives with respect to the $x$-spatial variables. The well known of this type equations is KdV equation. $L\lfloor \partial_x, \partial_y \rfloor u$ is the linear part of $K[u]$. So, using $K[u]$, we can reach the dispersion relation for Eq. (11). Substituting the wave solution space

$$\begin{aligned} u_k &= Ae^{i(kx+ry-\omega(k,r)t)} \\ &\equiv Ae^{i\theta} \end{aligned} \tag{12}$$

into the linear part of Eq. (11)

$$u_t = L\lfloor \partial_x, \partial_y \rfloor u \tag{13}$$

we get the dispersion relation

$$\omega(k,r) = iL[ik, ir] \tag{14}$$

Then, the dispersion relation (14) is substituted in Eq. (11). We assume the following series expansions for the solution of Eq. (11):

$$u(x,y,t) = \sum_{n=1}^{\infty} \varepsilon^n U_n(x,t,\xi,\tau)$$

Based on this solution, we also define slow space $\xi$ and multiple time variable $\tau$ with respect to the scaling parameter $\varepsilon > 0$, respectively, as follows.

$$\begin{aligned} \xi &= \varepsilon\left(x - \frac{d\omega(k,r)}{dk}t\right) \\ \tau &= -\tfrac{1}{2}\varepsilon^2\left(\frac{d^2\omega(k,r)}{dk^2}\right)t \end{aligned} \tag{15}$$

A nonlinear equation modulates the amplitude of this plane wave solution in such a way that may consider it dependent upon the slow variables. If we choose the slow variables' different forms, we can derive higher-order NLS equations. The multiple scales analysis starts with the assumption:

$$u(x,y,t) = U(x,y,t,\xi,\tau) \tag{16}$$

and solution of $U$ is in the form

$$U(x,y,t,\xi,\tau) = (\varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \ldots) \tag{17}$$

In this case, considering the transformation (16) and solution (17), using (14) and (15), the terms included derivative in Eq.(11) are obtained. Substituting these terms with (16) and (17) into the Eq.(11), we get a polynomial in zero. We obtain a series of algebraic equations by equalizing each coefficient of this polynomial to zero. Using wave solution space (12) and dispersion relation (14), these equations may be solved by iteration and reduce. Thus, we can obtain NLS type equations from Eq. (11). Furthermore, this approach allows us to obtain numerical solutions to KdV-type problems.

## 3. Applications

### 3.1 The fifth-order Kaup-Kupershmidt (KK) equation

Following the method given in Section 2.3 we use a multiple scales method to derive the NLS equations from the fifth-order Kaup–Kupershmidt (KK) equation (2). To find the dispersion relation for (2), we consider the linear part of (2) in the form

$$u_t = u_{xxxxx}, \tag{18}$$

and linear differential equation (18) satisfies the solution

$$u(x,t) = e^{i\theta}, \quad \theta = kx - w(k)t \tag{19}$$

Substituting the solution (19) into the linear differential equation (18), we get and from this we reach the

$$w(k) = -k^5 \tag{20}$$

dispersion relation. Thus, the solution of the linear differential equation (18) is as follows:

$$u(x,t) = e^{i(kx - w(k)t)}, \tag{21}$$

Let the solution of Eq. (2) be in the form

$$u(x,t) = U(x,t,\xi,\tau) \quad , \quad U(x,t,\xi,\tau) = \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \ldots \tag{22}$$

ε scale parameter and slow variables

$$\begin{aligned} \xi &= \varepsilon(x - \tfrac{dw(k)}{dk}t) \\ &= \varepsilon(x - 5k^4 t) \\ \tau &= -\tfrac{1}{2}\varepsilon^2(\tfrac{d^2w(k)}{dk^2}t) \\ &= -10\varepsilon^2 k^3 t \end{aligned} \tag{23}$$

Then we assume the following series expansions for solutions:

$$u(x,t) = U(x,t,\xi,\tau) \quad , \quad U(x,t,\xi,\tau) = \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \ldots \tag{24}$$

In this case, considering the transformation and solution in (22), using (20) and (23), the terms included in the derivative in Eq. (2) are obtained. Substituting these terms and (22) into Eq.(2), we get a polynomial in. Equalizing each coefficient of this polynomial to zero, we get a set of algebraic equations. If we let $\varepsilon \to 0$, we find the following:

$$\varepsilon: \quad u_{1t} — u_{1xxxxx} = 0, \tag{25}$$

$$\varepsilon^2: \quad u_{2t} + u_{2xxxxx} - 5k^4 u_1 + 5u_{1\ xxxx} + 10u_1 u_{1xxx} + 25u_{1x}u_{1xx} = 0, \tag{26}$$

$$\begin{aligned} \varepsilon_3: \; & u_{3t} + u_{3xxxxx} - 5k4u_{2\xi} — 10k3u_{1\tau} + 5u_{2\xi xxxx} + 10u_{1\xi\xi xxx} \\ & +10u_1(u_{2xxx} + 3u_{1\xi xx}) + 10u_2 u_{1xxx} + 25u_{1x}(u_{2xx} + 2u_{1\xi x}) \\ & +25u_{1xx}(u_{2x} + 2u_{1\xi})_1 + 20u^2 u_{1x} = 0 \end{aligned} \tag{27}$$

Then, we can find the solution of (25) as follows

$$u_1\,(x, t, \xi\,, \tau) = v_1\,(\xi, \tau)\, e^{i(kx—k^5\,t)} + c.c. \tag{28}$$

where *c.c.* is complex conjugate of $v_1$. Substituting the solution (28) into (26), the solution of (26) is in the form

$$u_2(x,t,\xi,\tau) = v_2(\xi,\tau)e^{2i(kx-k^5t)} + c.c + f_0(\xi,\tau) \tag{29}$$

where $f_0$ is an integration constant. Thus, we get

$$v_2(\xi,\tau) = \frac{7v_1^2(\xi,\tau)}{6k^2} \quad , \quad v_{-2}(\xi,\tau) = \frac{7v_{-1}^2(\xi,\tau)}{6k^2} \tag{30}$$

where $v_{-1}$ is the complex conjugate of $v_1$ and $v_{-2}$ is the complex conjugate of $v_2$. Substituting solutions (28), (29) and (30) into the (27), we find the solution of (27) in the form

$$u_3(x,t,\xi,\tau) = v_3(\xi,\tau)e^{3i(kx-k^5t)} + c.c + f_1(\xi,\tau)e^{2i(kx-k^5t)} + f_2(\xi,\tau)e^{-2i(kx-k^5t)} \quad (31)$$

where $f_1$ and $f_2$ is integration constant and $v_{-3}$ is the complex conjugate of $v_3$. Then we get

$$v_3(\xi,\tau) = \frac{13v_1^3(\xi,\tau)}{12k^4} \quad , \quad v_{-3}(\xi,\tau) = \frac{13v_{-1}^3(\xi,\tau)}{12k^4} \quad (32)$$

$$f_0(\xi,\tau) = \frac{6v_1(\xi,\tau)v_{-1}(\xi,\tau)}{k^2}$$
$$f_1(\xi,\tau) = \frac{-7iv_{-1}(\xi,\tau)v_{-1\xi(\xi,\tau)}}{3k^3}$$
$$f_2(\xi,\tau) = \frac{7iv_1(\xi,\tau)v_{1\xi}(\xi,\tau)}{3k^3}$$

and

$$iv_{1\tau} = v_{1\xi\xi} - \frac{2}{k^2}v_1v_{-1}$$
$$iv_{1\tau} = v_{1\xi\xi} - \frac{2}{k^2}v_1|v_1|^2 \quad (33)$$

Describing as $q = \frac{v_1}{k}$ and $q_{-1} = \frac{v_{-1}}{k}$ (33) equation, we get the NLS type equations in the form

$$iq_\tau = q_{\xi\xi} - 2q|q^2| \quad (34)$$

Also, the numerical solution of the (1 + 1) fifth-dimensional Kaup–Kupershmidt (KK) equation (2) is found as

$$\begin{aligned} u(x,t) &= \varepsilon k(q(\xi,\tau)e^{i\theta} + q_{-1}(\xi,\tau)e^{-i\theta}) \\ &+\varepsilon^2(3q(\xi,\tau)q_{-1}(\xi,\tau) + \frac{6}{7}q^2(\xi,\tau)e^{2i\theta)} + \frac{6}{7}q_{-1}^2(\xi,\tau)e^{-2i\theta}) \\ &+k\varepsilon^3(-\frac{7}{3}iq_{-1}(\xi,\tau)q_{-1}(\xi,\tau)e^{-2i\theta} + \frac{7}{3}iq(\xi,\tau)q_\xi(\xi,\tau)e^{-2i\theta}) \\ &-k\varepsilon^3\frac{13}{12}(q^3(\xi,\tau)e^{2i\theta)} + q_{-1}^3(\xi,\tau)e^{-2i\theta}) \end{aligned} \quad (35)$$

where q is solution of NLS equation.

### 3.2 The seventh-order Kaup-Kupershmidt (KK) equation

Following the method given in Section 2.3 we use a multiple scales method to derive the NLS equations from the seventh-order Kaup–Kupershmidt (KK) equation (10). To find dispersion relation for (10), we consider the linear part of (10) in the form

$u_t = u_{xxxxxxx}$, (36)

and linear differential equation (18) satisfies the solution

$$u(x,t) = e^{i\theta}, \quad \theta = kx - w(k)t \quad (37)$$

Substituting the solution (19) into the linear differential equation (18), we get and from this we reach the

$$w(k) = k^7 \tag{38}$$

dispersion relation. Thus, the solution of the linear differential equation (18) is as follows:

$$u(x, t) = e^{i(kx—w(k)t)}, \tag{39}$$

Let the solution of Eq. (10) be in the form

$$u(x,t) = U(x,t,\xi,\tau) \quad , \quad U(x,t,\xi,\tau) = \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \dots \tag{40}$$

ε scale parameter and slow variables

$$\xi = \varepsilon\left(x - \frac{dw(k,r)}{dk}t\right)$$
$$\tau = -\tfrac{1}{2}\varepsilon^2\left(\frac{d^2w(k,r)}{dk^2}\right)t \tag{41}$$

Then we assume the following series expansions for solutions:

$$u(x,t) = U(x,t,\xi,\tau) \quad , \quad U(x,t,\xi,\tau) = \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \dots \tag{42}$$

In this case, considering the transformation and solution in (22), using (20) and (23), the terms included in the derivative in Eq. (2) are obtained. Substituting these terms and (22) into Eq.(10), we get a polynomial in. Equalizing each coefficient of this polynomial to zero, we get a set of algebraic equations. If we let $\varepsilon \to 0$, we find the following:

$$\varepsilon: \quad u_{1t} + u_{1xxxxxxx} = 0, \tag{43}$$

$$\varepsilon^2: \ u_{2t} - 7k^6 u_{1\xi} - u_{2xxxxxxx} - 7u_{1xxxxxx\xi} - 42u_1 u_{1xxxxx} - 147u_{1x}u_{1xxxx} - 252u_{1xx}u_{1xxx} = 0 \tag{44}$$

$$\begin{aligned}\varepsilon^3: \ & u_{3t} - 7k^6 u_{2\xi} - 21k^5 u_{1\tau} - u_{3xxxxxxx} - 7u_{2xxxxxx\xi} \\ & -21u_{1xxxxx\xi\xi} - 42u_1(u_{2xxxxx} + 5u_{1xxxx\xi}) - 42u_2 u_{1xxxxx} \\ & -147u_{1x}(u_{2xxxx} + 4u_{1xxx\xi}) - 147(u_{2x} + u_{1\xi})u_{1xxxx} \\ & -252u_{1xx}(u_{2xxx} + 3u_{1xx\xi}) - 252(u_{2xx} + 2u_{1x\xi})u_{1xxx} \\ & -504u_1^2 u_{1xxx} - 2268u_1 u_{1x} u_{1xx} - 630u_{1x}^3 = 0\end{aligned} \tag{45}$$

Then, we can find the solution of (43) as follows

$$u_1(x, t, \xi, \tau) = v_1(\xi, \tau)\, e^{i(kx—k^7 t)} + c.c. \tag{46}$$

where *c.c.* is complex conjugate of $v_1$. Substituting the solution (46) into (44), the solution of (44) is in the form

$$u_2(x,t,\xi,\tau) = v_2(\xi,\tau)e^{2i(kx-k^7t)} + c.c + f_0(\xi,\tau) \tag{47}$$

where $f_0$ is integration constant. Thus, we get

$$v_2(\xi,\tau) = \frac{7v_1^2(\xi,\tau)}{2k^2} \quad , \quad v_{-2}(\xi,\tau) = \frac{7v_{-1}^2(\xi,\tau)}{2k^2}$$
(48)

where $v_{-1}$ is the complex conjugate of $v_1$ and $v_{-2}$ is the complex conjugate of $v_2$. Substituting solutions (46), (47) and (48) into the (45), we find the solution of (45) in the form

$$u_3(x,t,\xi,\tau) = v_3(\xi,\tau)e^{3i(kx-k^7t)} + c.c + f_1(\xi,\tau)e^{2i(kx-k^7t)} + f_2(\xi,\tau)e^{-2i(kx-k^7t)} \tag{49}$$

where $f_1$ and $f_2$ is integration constant and $v_{-3}$ is the complex conjugate of $v_3$. Then we get

$$v_3(\xi,\tau) = \frac{39v_1^3(\xi,\tau)}{4k^4} \quad , \quad v_{-3}(\xi,\tau) = \frac{39v_{-1}^3(\xi,\tau)}{4k^4} \tag{50}$$

$$f_0(\xi,\tau) = \frac{-6v_1(\xi,\tau)v_{-1}(\xi,\tau)}{k}$$
$$f_2(\xi,\tau) = \frac{-7iv_{-1}(\xi,\tau)v_{-1\xi(\xi,\tau)}}{k^3}$$
$$f_3(\xi,\tau) = \frac{7iv_1(\xi,\tau)v_{1\xi}(\xi,\tau)}{k^3}$$

and

$$iv_{1\tau} = v_{1\xi\xi} - \frac{2}{k^2}v_1v_{-1}$$
$$iv_{1\tau} = v_{1\xi\xi} - \frac{2}{k^2}v_1|v_1|^2 \tag{51}$$

Describing as $q = \frac{v_1}{k}$ and $q_{-1} = \frac{v_{-1}}{k}$ (51) equation, we get the NLS type equations in the form

$$iq_\tau = q_{\xi\xi} - 2q|q^2| \tag{52}$$

Also, numerical solution of the (1+1) seventh-dimensional Kaup–Kupershmidt equation (KK) equation (8) is found as

$$\begin{aligned} u(x,t) &= \varepsilon k(q(\xi,\tau)e^{i\theta} + q_{-1}(\xi,\tau)e^{-i\theta}) \\ &+\varepsilon^2(-6q(\xi,\tau)q_{-1}(\xi,\tau) + 7q^2(\xi,\tau)e^{2i\theta} + 7q_{-1}^2(\xi,\tau)e^{-2i\theta}) \\ &+k\varepsilon^3(-7iq_{-1}(\xi,\tau)q_{-1}(\xi,\tau)e^{-2i\theta} + 7iq(\xi,\tau)q_{-1}(\xi,\tau)e^{-2i\theta}) \\ &-k\varepsilon^3\frac{39}{4}(q^3(\xi,\tau)e^{2i\theta)} + q_{-1}^3(\xi,\tau)e^{-2i\theta}) \end{aligned} \tag{53}$$

## 4. Conclusion

The multiple scales method, which is a perturbation method, is used to find approximate solutions of nonlinear evolution equations. This method allows us to find the solution of the given nonlinear equation depending on the solution of the linear part by using multiple scales, which are defined as the slow variables and depend on a parameter, in time and space variables. First, Zakharov and Kuznetsov showed that integrable systems can be reduced to other integrable systems using this method. If the initially taken system is not integrable, it is seen that the reduced system is either integrable or non-integrable as a result of the application of the method. However, if the method is applied to an inte- grable system, it is seen that the system obtained as a result of the analysis is always integrable. This is the main purpose of applying multiple-scale methods to integrable systems. This study examined how only NLS equation is derived from higher-order nonlinear Kaup–Kupershmidt (KK) equations using the mul- tiple scales method. We hope this conclusion serves as a valuable resource for researchers, scientists, and engineers interested in nonlinear wave theory, inspir- ing a deeper desire to explore further and reveal the mysteries still held within the intricate language of the NLS equation. At the same time, we think that the obtained results will form the basis of numerical calculations. Also, the method can be applied to many dierent NLEE equations.

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## 6. Data Availability

All data generated or analysed during this study are included in this published article.

## 7. Conflict of interest

The authors declare that they have no conflict of interest regarding the publication of this paper.

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