

## A Mathematical Study of Three Species Commensalism

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### Article History:

**Received:** 26-08-2025

**Revised:** 19-09-2025

**Accepted:** 06-10-2025

### Abstract:

In this paper, we discuss the stability of Three Species (S1, S2, and S3) Commensalism with unlimited resources for the second species. Commensalism is a symbiotic interaction between two or more populations that live together, and in which only one of the populations (commensal) is benefited while the other (host) is not affected. The mathematical model equations constitute a set of three first-order non-linear simultaneous differential equations in the strengths  $N_1$ ,  $N_2$ , and  $N_3$  of S1, S2, and S3 respectively. Criteria for the asymptotic stability of all the six equilibrium states are established. The system would be stable, if all the characteristic roots are negative, in case they are real, and have negative real parts, in case they are complex. Trajectories of the perturbations over the equilibrium points are illustrated. Criteria for the global stability of a normal steady state are derived by constructing suitable a Lyapunov's function. Further, the numerical solutions for the growth rate equations are computed using Runga-Kutta fourth-order method.

Keywords: Commensal, Equilibrium state, Host, Liapunov's function, Stable, Trajectories, Unstable.

2010 Mathematics Subject Classification: 92D25; 92D40

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### Introduction

Mathematical models serve as vital tools in biological research, enabling scientists to analyze complex interactions through iterative data collection and theoretical simulations. When properly constructed, these models reveal relationships between physical variables and underlying processes, guiding experimental design and data interpretation. Given the complexity of real-life systems, mathematical formulations often replicate experimental outcomes without fully representing the actual mechanisms. Despite this, such models are invaluable in predicting system behavior and exploring interactions among different components. Empirical adjustments allow researchers to refine models, yielding insights applicable to real-world scenarios. Several researchers have contributed to the field of biological modeling, Ma [6], Moghadas [7], Murray [8], and Sze-Bi Hsu [30] established foundational frameworks. In Competitive Ecosystems: Srinivas [29] analyzed two- and three-species systems with limited/unlimited resources. Prey-Predator Dynamics: Narayan [9] studied models incorporating prey cover and alternate food sources for predators.

Commensalism Models: Kumar [5] explored mathematical representations of commensal relationships. Syn-Ecosystems: Prasad [10–27] investigated continuous and discrete models for two-, three-, and four-species systems. Ecology is the study of living organisms in relation to their environments, examining how they interact with their habitats and each other. As a branch of evolutionary biology, it seeks to understand the mechanisms regulating species in nature, including population dynamics, species distribution, and ecological relationships such as predator-prey interactions and competition.

The field of ecology is broadly categorized into autecology (the study of individual species populations) and synecology (the study of interactions among multiple species communities). Synecological research has led to the development of the ecosystem concept, which integrates living organisms-plants, animals, and micro-organisms-with their physical surroundings. This foundational idea emerged from the collective work of generations of biologists, ecologists, and botanists. Theoretical ecology has been significantly advanced by researchers such as Gillman [3] and Kot [4], with contributions from both ecologists and mathematicians. Mathematical ecology, in particular, is divided into autecology and synecology, as explored in the works of Anna Sher [1], Arumugam [2], and Sharma [28].

### Notation

$N_i(t)$  : The population strength of  $S_i$  at time  $t$ ,  $i = 1, 2, 3$

$t$  : Time instant

$a_i$  : Natural growth rate of  $S_i$ ,  $i = 1, 2, 3$

$a_{ii}$  : Self inhibition coefficients of  $S_i$ ,  $i = 1, 3$

$a_{12}$  : Interaction coefficient of  $S_1$  due to  $S_2$

$a_{21}, a_{23}$  : Interaction coefficients of  $S_2$  due to  $S_1$  and  $S_3$

$a_{31}$  : Intersection coefficients of  $S_3$  due to  $S_1$

$k_i = \frac{a_i}{a_{ii}}$  : Carrying capacities of  $S_i$ ,  $i = 1, 3$

Further the variables  $N_1, N_2, N_3$  are non-negative and the model parameters  $a_1, a_2, a_3, a_{21}, a_{11}, a_{12}, a_{31}, a_{33}, a_{23}, k_1, k_3$  are assumed to be non-negative constants.

## 2. Basic Equations

The model equations for syn ecosystem is given by the following system of first order non-linear ordinary differential equations.

$$\frac{dN_1}{dt} = a_1 N_1 - a_{11} N_1^2 + a_{12} N_1 N_2 \quad (1)$$

$$\frac{dN_2}{dt} = a_2 N_2 + a_{21} N_1 N_2 + a_{23} N_2 N_3 \quad (2)$$

$$\frac{dN_3}{dt} = a_3 N_3 - a_{33} N_3^2 + a_{31} N_1 N_3 \quad (3)$$

### 3. Equilibrium States:

At  $\frac{dN_i}{dt} = 0, i = 1, 2, 3$ , the equations (1), (2) & (3) have six equilibrium states given by

(i) Fully washed-out state

$$E_1 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0$$

(ii) States in which two of the three species are washed out and the third is not.

$$E_2 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = k_3$$

$$E_3 : \bar{N}_1 = k_1, \bar{N}_2 = 0, \bar{N}_3 = 0$$

(iii) Only one of the three species is washed out while the other two are not.

$$E_4 : \bar{N}_1 = -\frac{a_2}{a_{21}}, \bar{N}_2 = -\alpha_2, \bar{N}_3 = 0; \text{ where } \alpha_2 = \frac{1}{a_{12}} \left( a_1 + \frac{a_{11} a_2}{a_{21}} \right) > 0$$

$$E_5 : \bar{N}_1 = k_1, \bar{N}_2 = 0, \bar{N}_3 = k_3 + \frac{a_{31} k_1}{a_{33}}$$

(iv) The normal steady state

$$E_6 : \bar{N}_1 = -\frac{\alpha_1}{\alpha}, \bar{N}_2 = -\left( \frac{a_1}{a_{12}} + \frac{a_{11} \alpha_1}{a_{12} \alpha} \right), \bar{N}_3 = \frac{\alpha_3}{\alpha}$$

Where  $\alpha = (a_{31} a_{23} + a_{21} a_{33}) > 0; \alpha_1 = (a_3 a_{23} + a_2 a_{33}) > 0; \alpha_3 = a_3 a_{21} - a_2 a_{31}$

### 4. Stability Analysis of the Equilibrium States:

Let  $N = (N_1, N_2, N_3) = \bar{N} + U$

where  $U = (u_1, u_2, u_3)^T$  is very small perturbation upon the equilibrium point  $\bar{N} = (\bar{N}_1, \bar{N}_2, \bar{N}_3)$ .

The basic equations (1), (2) & (3) are quasi-linearized to obtain the equations for the perturbed state as

$$\frac{dU}{dt} = AU \quad (4)$$

$$\text{where } A = \begin{bmatrix} a_1 - 2a_{11}\bar{N}_1 + a_{12}\bar{N}_2 & a_{12}\bar{N}_1 & 0 \\ a_{21}\bar{N}_2 & a_2 + a_{21}\bar{N}_1 + a_{23}\bar{N}_3 & a_{23}\bar{N}_2 \\ a_{31}\bar{N}_3 & 0 & a_3 - 2a_{33}\bar{N}_3 + a_{31}\bar{N}_1 \end{bmatrix}$$

The characteristic equation for the system is  $|A - \lambda I| = 0$  (5)

The equilibrium state is stable if all the roots of the equation (5) are negative in case they are real or have negative real parts in case they are complex.

**4.1 Fully washed out state:**  $E_1 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0.$

$$\text{In this state, we have } A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

The characteristic equation is  $|A - \lambda I| = 0$  (6)

The characteristic roots of (6) are  $a_1, a_2, a_3$ . Since all the three roots are positive. Hence the fully washed out state is **unstable** and the solutions of the equations (4) are

$$u_1 = u_{10}e^{a_1 t}, u_2 = u_{20}e^{a_2 t}, u_3 = u_{30}e^{a_3 t} \quad (7)$$

**Trajectories of perturbations:**

The trajectories in the  $u_1 - u_2, u_2 - u_3$  and  $u_3 - u_1$  planes are given by

$$(x_1)^{\frac{1}{a_1}} = (x_2)^{\frac{1}{a_2}} = (x_3)^{\frac{1}{a_3}}$$

$$\text{where } x_1 = \begin{pmatrix} u_1 \\ u_{10} \end{pmatrix}; x_2 = \begin{pmatrix} u_2 \\ u_{20} \end{pmatrix}; x_3 = \begin{pmatrix} u_3 \\ u_{30} \end{pmatrix}$$

**4.2 Equilibrium state:**  $E_2 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = k_3$

$$\text{At this state, we get } A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 + a_{23}k_3 & 0 \\ a_{31}k_3 & 0 & -a_3 \end{bmatrix}$$

In this state the characteristic equation is

$$(\lambda - a_1)[\lambda - (a_2 + a_{23}k_3)](\lambda + a_3) = 0 \quad (8)$$

The characteristic roots of (8) are  $a_1, a_2 + a_{23}k_3, -a_3$ . Since two of the three roots are positive. Hence the state is **unstable** and the solutions are

$$u_1 = u_{10}e^{a_1 t}, u_2 = u_{20}e^{(a_2+a_{23}k_3)t}, u_3 = \left( u_{30} - \frac{a_{31}k_3 u_{10}}{a_1 + a_3} \right) e^{-a_3 t} + \frac{a_{31}k_3 u_{10}}{a_1 + a_3} e^{a_1 t} \quad (9)$$

### Trajectories of perturbations:

The trajectories are

$$\left( \frac{u_1}{u_{10}} \right)^{(a_2+a_{23}k_3)} = \left( \frac{u_2}{u_{20}} \right)^{a_1}, x_3 = \left[ 1 - \frac{a_{31}k_3 u_{10}}{u_{30}(a_1 + a_3)} \right] x_2^{-\frac{a_3}{a_2+a_{23}k_3}} + \frac{a_{31}k_3 u_{10}}{u_{30}(a_1 + a_3)} x_2^{\frac{a_1}{a_2+a_{23}k_3}}$$

$$\text{and } x_3 = \left[ 1 - \frac{a_{31}k_3 u_{10}}{u_{30}(a_1 + a_3)} \right] x_1^{-\frac{a_3}{a_1}} + \frac{a_{31}k_3 u_{10}}{u_{30}(a_1 + a_3)} x_1$$

### 4.3 Equilibrium state $E_3 : \bar{N}_1 = k_1, \bar{N}_2 = 0, \bar{N}_3 = 0$ .

$$\text{Here, } A = \begin{bmatrix} -a_1 & a_{12}k_1 & 0 \\ 0 & a_2 + a_{21}k_1 & 0 \\ 0 & 0 & a_3 + a_{31}k_1 \end{bmatrix}$$

The characteristic equation is

$$(\lambda + a_1)[\lambda - (a_2 + a_{21}k_1)][\lambda - (a_3 + a_{31}k_1)] = 0 \quad (9)$$

The characteristic roots are  $-a_1, a_2 + a_{21}k_1, a_3 + a_{31}k_1$ . Since two of the three roots are positive. Hence the state is **unstable** and the solutions of the equations are given by

$$u_1 = \left[ u_{10} - \frac{a_{12}k_1 u_{20}}{a_1 + a_2 + a_{21}k_1} \right] e^{-a_1 t} + \frac{a_{12}k_1 u_{20}}{a_1 + a_2 + a_{21}k_1} e^{(a_2+a_{21}k_1)t}$$

$$u_2 = u_{20}e^{(a_2+a_{21}k_1)t}, u_3 = u_{30}e^{(a_3+a_{31}k_1)t} \quad (10)$$

### Trajectories of perturbations:

The trajectories in  $u_1 - u_2, u_2 - u_3, u_3 - u_1$  planes are

$$x_1 = \left[ 1 - \frac{a_{12}k_1 u_{20}}{u_{10}(a_1 + a_2 + a_{21}k_1)} \right] x_2^{-\frac{a_1}{a_2+a_{21}k_1}} + \frac{a_{12}k_1 u_{20}}{u_{10}(a_1 + a_2 + a_{21}k_1)} x_2, (x_2)^{a_3+a_{31}k_1} = (x_3)^{a_2+a_{21}k_1}$$

$$x_1 = \left[ 1 - \frac{a_{12}k_1 u_{20}}{u_{10}(a_1 + a_2 + a_{21}k_1)} \right] x_3^{-\frac{a_1}{a_3+a_{31}k_1}} + \frac{a_{12}k_1 u_{20}}{u_{10}(a_1 + a_2 + a_{21}k_1)} x_3^{\frac{a_2+a_{21}k_1}{a_3+a_{31}k_1}}$$

### 4.4 Equilibrium state $E_4 : \bar{N}_1 = -\frac{a_2}{a_{21}}, \bar{N}_2 = -\alpha_2, \bar{N}_3 = 0$

$$A = \begin{bmatrix} \frac{a_2 a_{11}}{a_{21}} & -\frac{a_2 a_{12}}{a_{21}} & 0 \\ -a_{21} \alpha_2 & 0 & 0 \\ 0 & 0 & \beta_3 \end{bmatrix}; \text{ where } \beta_3 = a_3 - \frac{a_2 a_{31}}{a_{21}}$$

In this state the characteristic equation is

$$(\lambda - \beta_3) \left( \lambda^2 - \frac{a_2 a_{11}}{a_{21}} \lambda - a_2 \alpha_2 a_{21} \right) = 0 \quad (11)$$

One of the three roots is  $\beta_3$ .

According to Routh-Hurwitz's criteria, all the coefficients of the quadratic polynomial R.H.S. of the equation (11) have not same sign. Hence the state is **unstable**.

Let  $\lambda_1, \lambda_2$  be the zeros of the quadratic polynomial R.H.S. of the above equation (11).

There fore, the solutions are

$$u_1 = \left[ \frac{a_{21} \alpha_2 \lambda_1 u_{10} + \lambda_1 \lambda_2 u_{20}}{a_{21} \alpha_2 (\lambda_1 - \lambda_2)} \right] e^{\lambda_1 t} + \left[ \frac{a_{21} \alpha_2 \lambda_2 u_{10} + \lambda_1 \lambda_2 u_{20}}{a_{21} \alpha_2 (\lambda_2 - \lambda_1)} \right] e^{\lambda_2 t}$$

$$u_2 = \left( \frac{a_{21} \alpha_2 u_{10} + \lambda_2 u_{20}}{\lambda_2 - \lambda_1} \right) e^{\lambda_1 t} + \left( \frac{a_{21} \alpha_2 u_{10} + \lambda_1 u_{20}}{\lambda_1 - \lambda_2} \right) e^{\lambda_2 t}, u_3 = u_{30} e^{\beta_3 t} \quad (12)$$

### Trajectories of perturbations:

The trajectories in  $u_3 - u_1, u_3 - u_2$  planes are

$$x_1 = \left[ \frac{a_{21} \alpha_2 \lambda_1 u_{10} + \lambda_1 \lambda_2 u_{20}}{a_{21} \alpha_2 u_{10} (\lambda_1 - \lambda_2)} \right] x_3^{\frac{\lambda_1}{\beta_3}} + \left[ \frac{a_{21} \alpha_2 \lambda_2 u_{10} + \lambda_1 \lambda_2 u_{20}}{a_{21} \alpha_2 u_{10} (\lambda_2 - \lambda_1)} \right] x_3^{\frac{\lambda_2}{\beta_3}}$$

$$x_2 = \left[ \frac{a_{21} \alpha_2 u_{10} + \lambda_2 u_{20}}{u_{20} (\lambda_2 - \lambda_1)} \right] x_3^{\frac{\lambda_1}{\beta_3}} + \left[ \frac{a_{21} \alpha_2 u_{10} + \lambda_1 u_{20}}{u_{20} (\lambda_1 - \lambda_2)} \right] x_3^{\frac{\lambda_2}{\beta_3}}$$

**4.5 Equilibrium state**  $E_5 : \bar{N}_1 = k_1, \bar{N}_2 = 0, \bar{N}_3 = k_3 + \frac{a_{31} k_1}{a_{33}}$ .

At this point, we have  $A = \begin{bmatrix} -a_1 & a_{12} k_1 & 0 \\ 0 & \alpha_7 & 0 \\ a_{31} \left( k_3 + \frac{a_{31} k_1}{a_{33}} \right) & 0 & -(a_3 + a_{31} k_1) \end{bmatrix}$

The characteristic equation is given by

$$(\lambda + a_1)(\lambda - \alpha_7)[\lambda + (a_3 + a_{31}k_1)] = 0 \quad (13)$$

$-a_1, \alpha_7, -(a_3 + a_{31}k_1)$  are the characteristic roots of (13). Since one of the three roots is positive. Hence the state is **unstable** and the solutions of (4) are

$$u_1 = \left[ u_{10} - \frac{a_{12}k_1u_{20}}{\alpha_7 + a_1} \right] e^{-a_1t} + \frac{a_{12}k_1u_{20}}{\alpha_7 + a_1} e^{\alpha_7t}, u_2 = u_{20}e^{\alpha_7t}$$

$$u_3 = \left[ u_{30} - (A_{10} + A_{20}) \right] e^{-(a_3 + a_{31}k_1)t} + A_{10}e^{-a_1t} + A_{20}e^{\alpha_7t} \quad (14)$$

where  $A_{10} = \frac{\alpha_8 \left[ u_{10} - \frac{a_{12}k_1u_{20}}{\alpha_7 + a_1} \right]}{a_3 + a_{31}k_1 - a_1}, A_{20} = \frac{\alpha_8 \left[ \frac{a_{12}k_1u_{20}}{\alpha_7 + a_1} \right]}{\alpha_7 + a_3 + a_{31}k_1} > 0$ , with  $(a_3 + a_{31}k_1) \neq a_1$

$$\alpha_7 = a_2 + a_{21}k_1 + a_{23}k_3 + \frac{a_{23}a_{31}k_1}{a_{33}} > 0, \alpha_8 = a_{31}k_3 + \frac{a_{31}^2k_1}{a_{33}} > 0$$

The trajectories are given by

$$x_1 = \left[ 1 - \frac{a_{12}k_1u_{20}}{u_{10}(\alpha_7 + a_1)} \right] x_2^{\frac{-a_1}{\alpha_7}} + \frac{a_{12}k_1u_{20}x_2}{u_{10}(\alpha_7 + a_1)}$$

$$x_3 = \left[ 1 - \frac{A_{10} + A_{20}}{u_{30}} \right] x_2^{\frac{-(a_3 + a_{31}k_1)}{\alpha_7}} + \frac{A_{10}}{u_{30}} x_2^{\frac{-a_1}{\alpha_7}} + \frac{A_{20}x_2}{u_{30}}$$

**4.6 Equilibrium state**  $E_6 : \bar{N}_1 = -\frac{\alpha_1}{\alpha}, \bar{N}_2 = -\left( \frac{a_1}{a_{12}} + \frac{a_{11}\alpha_1}{a_{12}\alpha} \right), \bar{N}_3 = \frac{\alpha_3}{\alpha}$

In this normal state, we have

$$A = \begin{bmatrix} \frac{a_{11}\alpha_1}{\alpha} & -\frac{a_{12}\alpha_1}{\alpha} & 0 \\ -a_{21} \left( \frac{a_1}{a_{12}} + \frac{a_{11}\alpha_1}{a_{12}\alpha} \right) & 0 & -a_{23} \left( \frac{a_1}{a_{12}} + \frac{a_{11}\alpha_1}{a_{12}\alpha} \right) \\ \frac{a_{31}\alpha_1}{\alpha} & 0 & -a_{33}\alpha_3 \end{bmatrix}$$

The characteristic equation is  $\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$  (15)

Where  $b_1 = a_3a_{33}a_{21} - \left( a_2a_{33}a_{31} + \frac{a_{11}\alpha_1}{\alpha} \right); b_2 = \frac{a_{21}\alpha_1}{\alpha} \left( a_1 + \frac{a_{11}\alpha_1}{\alpha} \right) > 0$

$$b_3 = \frac{\alpha_1}{\alpha} \left( a_1 + \frac{a_{11}\alpha_1}{\alpha} \right) \left( \frac{a_{23}a_{31}\alpha_3}{\alpha} - a_{21}a_{33}\alpha_3 \right)$$

According to Routh-Hurwitz's criteria, the necessary and sufficient conditions for local stability of co-existent points are  $b_1 > 0, b_3 > 0$  and  $b_3(b_1b_2 - b_3) > 0$

It is evident that,

$$b_1 > 0, \text{ when } a_3a_{33}a_{21} > \left( a_2a_{33}a_{31} + \frac{a_{11}\alpha_1}{\alpha} \right)$$

$$b_3 > 0, \text{ when } \left( \frac{a_{21}a_{23}a_{31}a_3}{\alpha} + a_2a_{21}a_{33}a_{31} \right) > \left( \frac{a_2a_{23}a_{31}^2}{\alpha} + a_{21}^2a_3a_{33} \right)$$

$$(b_1b_2 - b_3) > 0, \text{ when } \left( \frac{a_2a_{23}a_{31}^2}{\alpha} + 2a_2a_{21}^2a_{33} \right) > \left( \frac{a_3a_{21}a_{23}}{\alpha} + a_{21}^2a_3a_{33} \right)$$

$$\text{i.e., } b_1 > 0, b_3 > 0 \text{ and } b_3(b_1b_2 - b_3) > 0$$

Hence, the co-existent state is locally asymptotically stable.

Let  $\lambda_1, \lambda_2, \lambda_3$  be three roots of (15).

The solution of the perturbation equations is

$$\begin{aligned} u_1 &= B_{10}e^{\lambda_1 t} + B_{20}e^{\lambda_2 t} + B_{30}e^{\lambda_3 t} \\ u_2 &= X_2B_{10}e^{\lambda_1 t} + Y_2B_{20}e^{\lambda_2 t} + Z_2B_{30}e^{\lambda_3 t} \\ u_3 &= X_3B_{10}e^{\lambda_1 t} + Y_3B_{20}e^{\lambda_2 t} + Z_3B_{30}e^{\lambda_3 t} \end{aligned} \tag{15}$$

Where

$$B_{10} = \frac{(u_{10}Y_2Z_3 + u_{20}Y_3 + u_{30}Z_2) - (u_{10}Y_3Z_2 + u_{20}Z_3 + u_{30}Y_2)}{(Y_2Z_3 + X_2Y_3 + X_3Z_2) - (Y_3Z_2 + X_2Z_3 + X_3Y_2)}$$

$$B_{20} = \frac{(u_{10}X_3Z_2 + u_{20}Z_3 + u_{30}X_2) - (u_{10}X_2Z_3 + u_{20}X_3 + u_{30}Z_2)}{(Y_2Z_3 + X_2Y_3 + X_3Z_2) - (Y_3Z_2 + X_2Z_3 + X_3Y_2)}$$

$$B_{30} = \frac{(u_{10}X_2Y_3 + u_{20}X_3 + u_{30}Y_2) - (u_{10}X_3Y_2 + u_{20}Y_3 + u_{30}X_2)}{(Y_2Z_3 + X_2Y_3 + X_3Z_2) - (Y_3Z_2 + X_2Z_3 + X_3Y_2)}$$

$$X_2 = \frac{a_{11}\alpha_1 - \alpha\lambda_1}{a_{12}\alpha_1}, Y_2 = \frac{a_{11}\alpha_1 - \alpha\lambda_2}{a_{12}\alpha_1}, Z_2 = \frac{a_{11}\alpha_1 - \alpha\lambda_3}{a_{12}\alpha_1}$$

$$X_3 = \frac{\alpha\lambda_1(a_{11}\alpha_1 - \alpha\lambda_1)}{a_{11}a_{23}\alpha_1(a_1\alpha + a_{11}\alpha_1)} - \frac{a_{21}}{a_{23}}, Y_3 = \frac{\alpha\lambda_2(a_{11}\alpha_1 - \alpha\lambda_2)}{a_{11}a_{23}\alpha_1(a_1\alpha + a_{11}\alpha_1)} - \frac{a_{21}}{a_{23}}$$

$$Z_3 = \frac{\alpha\lambda_3(a_{11}\alpha_1 - \alpha\lambda_3)}{a_{11}a_{23}\alpha_1(a_1\alpha + a_{11}\alpha_1)} - \frac{a_{21}}{a_{23}}$$

### 5. Lyapunov's function for global stability:

In section 4, we discussed the local stability of all six equilibrium states. From which only the normal steady state is **stable** and rest of them are **unstable**. We now examine the global stability of dynamical system (1), (2) & (3) at this state by suitable Liapunov's function.

**Theorem:** The normal steady state is globally asymptotically stable.

**Proof:** Let us consider the following Liapunov's function

$$L(N_1, N_2, N_3) = N_1 - \bar{N}_1 - \bar{N}_1 \ln\left(\frac{N_1}{\bar{N}_1}\right) + l_1 \left[ N_2 - \bar{N}_2 - \bar{N}_2 \ln\left(\frac{N_2}{\bar{N}_2}\right) \right] + l_2 \left[ N_3 - \bar{N}_3 - \bar{N}_3 \ln\left(\frac{N_3}{\bar{N}_3}\right) \right] \quad (16)$$

where  $l_1$  &  $l_2$  are suitable positive constants to be determined as in the subsequent steps.

The line derivative of L, we get

$$\begin{aligned} \frac{dL}{dt} &= \left(\frac{N_1 - \bar{N}_1}{N_1}\right) \frac{dN_1}{dt} + l_1 \left(\frac{N_2 - \bar{N}_2}{N_2}\right) \frac{dN_2}{dt} + l_2 \left(\frac{N_3 - \bar{N}_3}{N_3}\right) \frac{dN_3}{dt} \\ &= (N_1 - \bar{N}_1)(a_1 - a_{11}N_1 + a_{12}N_2) + l_1(N_2 - \bar{N}_2)(a_2 + a_{21}N_1 + a_{23}N_3) + l_2(N_3 - \bar{N}_3)(a_3 - a_{33}N_3 + a_{31}N_1) \\ &= -\left[\sqrt{a_{11}}(N_1 - \bar{N}_1) + \sqrt{l_2 a_{33}}(N_3 - \bar{N}_3)\right]^2 + (2\sqrt{l_2 a_{11} a_{33}} + l_2 a_{31})(N_1 - \bar{N}_1)(N_3 - \bar{N}_3) + (a_{12} - l_1 a_{21})(N_1 - \bar{N}_1)(N_2 - \bar{N}_2) \end{aligned} \quad (17)$$

The positive constants  $l_1$  and  $l_2$  as so chosen that, the coefficients of  $(N_1 - \bar{N}_1)(N_2 - \bar{N}_2)$  and  $(N_1 - \bar{N}_1)(N_3 - \bar{N}_3)$  in (17) vanish.

Then we have  $l_1 = \frac{a_{12}}{a_{21}} > 0$  and  $l_2 = \frac{4a_{11}a_{33}}{a_{31}^2} > 0$ , with this choice of the constants  $l_1$  and  $l_2$

$$\frac{dL}{dt} < 0, \text{ when } N_2 + \bar{N}_3 < N_3 + \bar{N}_2$$

Hence, the steady state is globally asymptotically stable.

### 6. Numerical approach

The numerical solutions of the growth rate equations (1), (2) and (3) computed employing the fourth order Runge-Kutta method for specific values of the various parameters that characterize the model and the initial conditions. The results are illustrated in Figures 1 to 4.

Case 1:  $a_1=1.1, a_2=0.25, a_3=0.41, a_{11}=1.28, a_{12}=1.01, a_{23}=0.1, a_{21}=0.29, a_{31}=0.01, a_{33}=0.07$

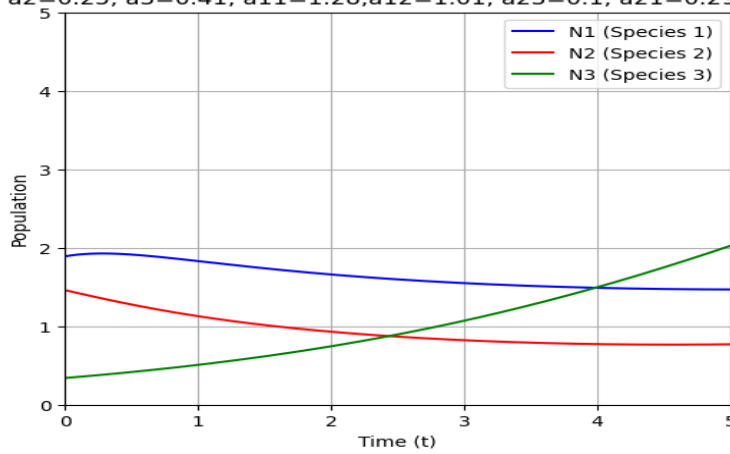


Figure-1

Case 2:  $a_1=1.05, a_2=0.51, a_3=0.3, a_{11}=1.31, a_{12}=1.1, a_{21}=0.81, a_{23}=0.6, a_{31}=0.04, a_{33}=0.42$

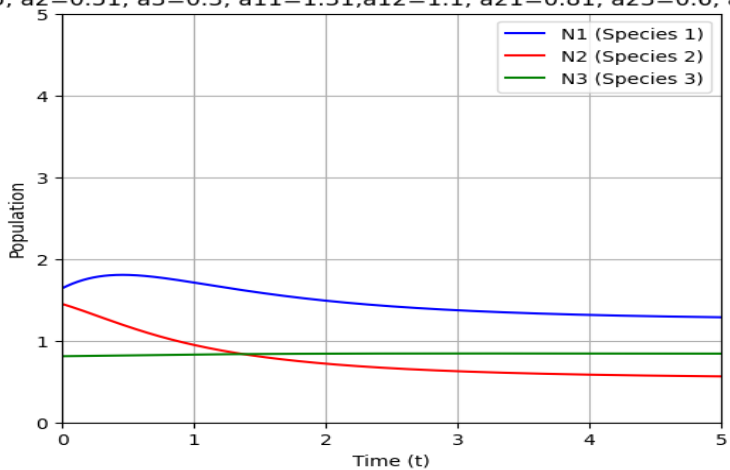


Figure-2

Case 3:  $a_1=1.69, a_2=1.18, a_3=0.38, a_{11}=1.08, a_{12}=0.79, a_{21}=0.52, a_{23}=0.23, a_{31}=0.02, a_{33}=0.11$

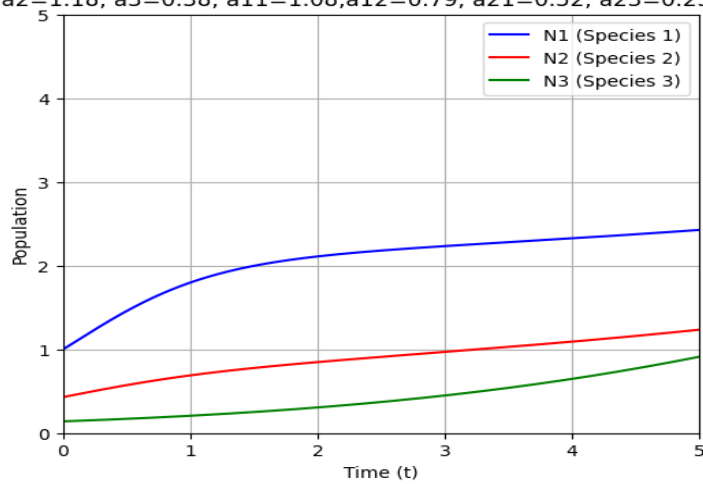


Figure-3

Case 4:  $a_1=1.53, a_2=0.89, a_3=0.35, a_{11}=1.46, a_{12}=1.21, a_{23}=0.31, a_{21}=0.75, a_{31}=0.05, a_{33}=0.11$

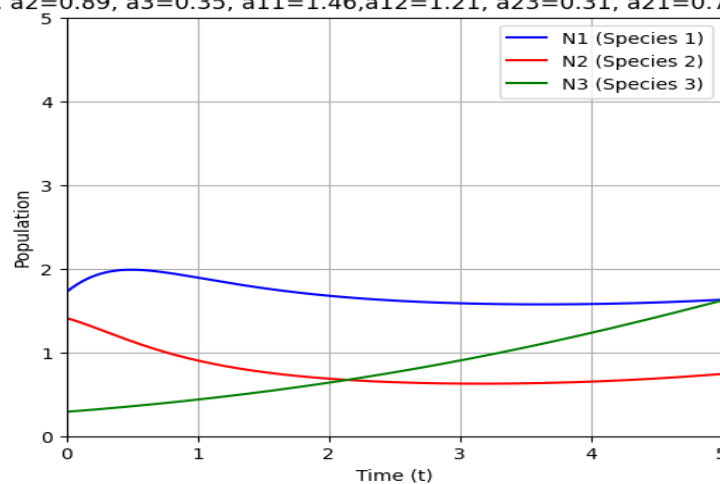


Figure-4

### 7. Observations of the above graphs:

**Case 1:** In this case initially the third species dominates by the second species up to the time  $t^* = 2.41$  and the first species up to the time  $t^* = 4$  and the dominances are reversed. The initial conditions of  $S_1, S_2, S_3$  are in decreasing order. Further we notice that the first species has the greatest natural growth rate and the third species increase gradually. This is illustrated in Figure 1.

**Case 2:** In this case the third species has the least natural birth rate. The first and second species decrease initially while the third species is a weak competitor with no appreciable growth even from the start. The second species has strong mutual interactions, suppressed by  $S_1$  but boosted by  $S_3$ . Further the second species dominates over the third species up to the time instant  $t^* = 1.4$  after which the dominance is reversed as shown in Figure 2.

**Case 3:** In this case the first species dominates over the second and third species. The natural birth rates of the first, second and third species are in decreasing order.  $S_3$  has a very weak self-limitation, strong support from  $S_1$  grows continuously, may reach higher equilibrium. This is shown in Figure 3.

**Case 4:** In this case the second species dominates over the third species up to the time instant  $t^* = 2.2$  after which the dominance is reversed. Further, the third species has the least natural birth rate.  $S_1$  has a strong self-limitation but high interaction with  $S_2$ . (Figure 4).

### 8. Conclusion

In this paper, we discussed the stability analysis of three species ecological commensalism. The model equations constitute a set of three first order non-linear coupled differential equations. All possible equilibrium states of the model are identified and the local stability is discussed. It is observed that, in all six equilibrium states, only the normal state is locally stable. Further, the global stability of the system is established with the aid of suitably constructed Liapunov's function and the growth rates of the species are numerically estimated using Runge-Kutta fourth order method.

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