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The Cauchy Problems for Fractional Q-Difference Equations with Integral Conditions in Banach Spaces

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Abstract: In In this paper, we investigate the existence of solutions to the fractional q-difference equation of the type ${}^cD_q^\alpha u(t) = Au(t) + f\big(t, u(t)\big),\ t \in J := [0,1]$ with $u(0) = a\int_0^1 u(s)d_qs + b,$ where $0 < \alpha < 1,\ f \in C(J \times X, X)$, and $A(t)$ is a bounded linear operator on a Banach space X The operator ${}^cD_q^\alpha u(t)$ denotes the Caputo fractional q-derivative of order α. Our existence result are obtained by using the Banach fixed point theorem and the Schaefer fixed point theorem.

Keywords: Fixed point, Existence, Caputo fractional q-derivative, Fractional q-difference equations.

1. Introduction

Over the past years, the theory of fractional calculus has received increasing attention from researchers and has become one of the most active areas of research due to its significant importance and wide applications on many subjects. The importance of this theory lies in its ability to contribute to mathematical modeling in various fields such as technical sciences, physics, engineering, biophysics and biomathematics. For more details, see [9–11, 13, 16].

At the beginning of the twentieth century, Jackson was the first to develop quantum calculus, also known as q-difference calculus, by introducing the concept of the q- integral along with several other fundamental notions in this theory. For further details on this topic, see references [8, 12].

In the late 1960s, a new branch known as fractional q-difference calculus emerged as a generalization of the q-difference calculus. This development is attributed to Al-Salam [6] and Agarwal [2]. This branch has received considerable attention in the academic community due to its wide range of applications in modeling mathematical phenomena across various scientific fields.

Recently, Several researchers have studied the fractional q-difference equations in- volving the Caputo fractional q-derivative by using all kinds of fixed point theorems and obtained many interesting results , for example, by Abbas et al [1], Ahmad and al. [5].

In [3], N. Allouch et al studied the existence of solutions to the following fractional

q-difference equations with nonlinear integral conditions:

$$ {}^cD_q^\alpha u(t) = f\big(t, u(t)\big), \quad t \in I = [0, T], \quad 1 < \alpha \le 2, $$

$$ u(0) - u'(0) = \int_0^T g\big(s, u(s)\big)ds, $$

$$u(T) + u'(T) = \int_0^T h\big(s, u(s)\big) ds,$$

where $T > 0,\ q \in]0,1[,\ {}^cD_q^\alpha$ denotes the Caputo fractional q-difference derivative of order $1 < \alpha \leq 2$, and $f, g, h \in C(I \times X, X)$.

In [4], N. Allouch et al applied some standard fixed point theorems and investigated the existence of solutions of fractional q-difference equations of the type:

$$ {}^cD_q^\alpha u(t) = f\big(t, u(t)\big), \quad t \in I = [0, T], \quad 1 < \alpha \leq 1,$$

$$au(0) + bu(T) = c,$$

where $T > 0,\ q \in]0,1[,\ {}^cD_q^\alpha$ denotes the Caputo fractional q-difference derivative of order α, $f \in C(I \times X, X)$ and $a + b \neq 0$.

In this paper, we establish the existence of solutions to the fractional q-difference equations of the type:

$$ {}^cD_q^\alpha u(t) = A(t) + f\big(t, u(t)\big), \quad t \in J := [0,1], \tag{1.1}$$

$$u(0) = a \int_0^1 u(s) d_q s + b, \tag{1.2}$$

where $0 < \alpha < 1$, $f \in C(J \times X, X)$, and $A(t)$ is a bounded linear operator on a Banach space X. The operator ${}^cD_q^\alpha$ denotes the Caputo fractional q-difference derivative of order α. The existence result is based on the fixed point theorem and the Schaefer fixed point theorem.

The paper is structured as follows. In Section 2, we present the notations and definitions required for the study, and we review essential preliminaries from fractional q-calculus. Section 3 contains the principal results: the first derived from the Banach fixed point theorem, and the second from Schaefer's fixed point theorem. Section 4 is devoted to an illustrative example highlighting the applicability of these results.

2. Preliminaries

This section is concerned with presenting basic definitions together with auxiliary results required in the later parts of this paper.

Assume that X is a Banach space. Define $J := [0,1]$ and let $C(J, X)$ denote the Banach space of continuous functions u from J into X with the norm

$$\|u\|_\infty = sup_{t \in J} |u(t)|.$$

Now, we introduce the essential definitions and relevant properties of the fractional q-calculus. For more details, see [8, 12].

We assume that $q \in]0,1[$. For every $a \in \mathbb{R}$, we define

$$[a]_q = \frac{1 - q^a}{1 - q}.$$

Let $a, b \in \mathbb{R}$. The q-analogue of $(a-b)^{(n)}$ is given by:

$$(a-b)^{(n)} = \begin{cases} 1 & if\ n = 0 \\ \prod_{i=0}^{n-1} (a - bq^i) & if\ n \in \mathbb{N}^* \end{cases}$$

For $\beta \in \mathbb{R}$, we have

$$(a-b)^{(\beta)} = a^\beta \prod_{i=0}^{\infty} \left(\frac{a - bq^i}{a - bq^{i+\beta}} \right), \quad a, b \in \mathbb{R}.$$

Note that, if $b = 0$, then $a^{(\beta)} = a^\beta$.

Definition 1 [12] The q-gamma function is defined as follows:

$$\Gamma_q(\beta) = \frac{(1-q)^{(\beta-1)}}{(1-q)^{\beta-1}}, \ \beta > 0.$$

Observe that the q-gamma function verifies $\Gamma_q(\beta+1) = [\beta]_q \Gamma_q(\beta)$.

Definition 2 [12] Let $f: J \to \mathbb{R}$. The q-derivative of order $n \in \mathbb{N}$ is given by:

$$\left(D_q^0 f\right)(t) = f(t), \quad \left(D_q^1 f\right)(t) = \frac{f(t) - f(qt)}{(1-q)t},$$

and

$$\left(D_q^n f\right)(t) = \left(D_q^1 D_q^{n-1} f\right)(t), \ n \in \mathbb{N}^*.$$

Definition 3 [12] Let $J_t = \{tq^n : \ n \in \mathbb{N}\} \cup \{0\}$. The q-integral of a function $f: J_t \to \mathbb{R}$ is defined by:

$$\left(I_q f\right)(t) = \int_0^1 f(s) d_q s = \sum_{n=0}^{\propto} t(1-q) q^n f(tq^n),$$

under the assumption that the series converges.

Note that $\left(D_q I_q f\right)(t) = f(t)$, Furthermore, if f is continuous at 0, then

$$\left(I_q D_q f\right)(t) = f(t) - f(0).$$

Definition 4 [2] Let $f: J \to \mathbb{R}$. the Riemann-Liouville fractional q-integral of order $\alpha \geq 0$ is defined as:

$$\left(I_q^\alpha f\right)(t) = \begin{cases} f(t) & if\ \alpha = 0, \\ \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s) d_q s & if\ \alpha > 0. \end{cases}$$

Observe that when $\alpha = 1$, we have $\left(I_q^1 f\right)(t) = \left(I_q f\right)(t)$.

Lemma 5 [15] For all $\alpha \geq 0$ and $\beta \in]-1,+\infty[$, we have

$$\left(I_q^\alpha (t-a)^{(\beta)}\right)(t) = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)}(t-a)^{(\alpha+\beta)}, \quad 0 < \alpha < t < 1.$$

In particular,

$$\left(I_q^\alpha 1\right)(t) = \frac{1}{\Gamma_q(\alpha+1)} t^{(\alpha)}.$$

Definition 6 [14] The Riemann-Liouville fractional q-derivative of order $\alpha \geq 0$ for a function $f: J \to \mathbb{R}$ is defined as follows:

$$\left(D_q^0 f\right)(t) = f(t) \quad \text{and} \quad \left(D_q^\alpha f\right)(t) = \left(D_q^{[\alpha]} I_q^{[\alpha]-\alpha} f\right)(t), \ t \in J,$$

where $[\alpha]$ is the integer part of α.

Definition 7 [14] Consider a function $f: J \to \mathbb{R}$ and let $\alpha \geq 0$. The Caputo fractional q-derivative of order α is defined as follows:

$$\left(D_q^0 f\right)(t) = f(t) \quad \text{and} \quad \left({}^c D_q^\alpha f\right)(t) = \left(I_q^{[\alpha]-\alpha} D_q^{[\alpha]} f\right)(t), \ t \in J,$$

where $[\alpha]$ is the integer part of α.

Lemma 8 [14] Suppose that $\alpha, \beta \geq 0$, and let $f: J \to \mathbb{R}$ be a given function. Then, the following identities hold:

(i) $\left(I_q^\alpha I_q^\beta f\right)(t) = \left(I_q^{\alpha+\beta} f\right)(t)$,

(ii) $\left(D_q^\alpha I_q^\beta f\right)(t) = f(t)$.

Lemma 9 [14] Assume $\alpha \geq 0$, and let f be a function defined on the interval J. The following identity holds

$$\left(I_q^\alpha \, {}^c D_q^\alpha f\right)(t) = f(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(k+1)} \left(D_q^\alpha f\right)(0).$$

If $\alpha \in]0,1[$, we have

$$\left(I_q^\alpha \, {}^c D_q^\alpha f\right)(t) = f(t) - f(0).$$

Theorem 10 (Banach contraction principle) [7] Suppose that C is a non-empty closed subset of a Banach space X. If $H: C \to C$ is a contraction, then H admits a unique fixed point in C.

Theorem 11 (Schaefer) [17] Let X be a Banach space and let $H: X \to X$ be a completely continuous operator. Assume that the set

$$\mathcal{E} := \{u \in X \mid u = \lambda H(u), \ \lambda \in]0,1[\}$$

is bounded. Then H admits a fixed point in X.

3. Existence

In this section, we investigate the existence of solutions for the fractional qqq-difference problem given by (1.1)-(1.2).

Definition 12 A function $u \in C([0,1],X)$ is called a solution of the fractional q-difference problem (1.1)-(1.2) if u satisfies the equation ${}^{c}D_q^{\alpha}u(t) = Au(t) + f\big(t,u(t)\big)$ on $J = [0,1]$, and the condition $u(0) = a\int_0^1 u(s)d_qs + b$.

To establish the existence of solutions for the fractional problem (1.1)-(1.2), we require the following lemma:

Lemma 13 Let $\alpha \in]0,1[$ and let $h: [0,1] \times X \to X$ be continuous function. The solution of the fractional q-difference problem

$$ {}^{c}D_q^{\alpha}u(t) = A(t) + f\big(t,u(t)\big), \quad t \in J = [0,1], $$

$$ u(0) = a\int_0^1 u(s)d_qs + b, $$

is given by

$$ u(t) = a\int_0^1 u(s)d_qs + b + \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} f\big(s,u(s)\big)d_qs + \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} Au(s)d_qs. $$

The first result is obtained by applying the Banach fixed point theorem.

Theorem 14 Suppose that:

(H_1) There exist $k > 0$ such that

$$ \forall t \in J,\ \forall u,v \in X,\ |f(t,u) - f(t,v)| \leq k|u-v|. $$

If

$$ |\alpha| + \frac{k + \|A\|_{\mathcal{L}(X)}}{\Gamma_q(\alpha+1)} < 1. \tag{3.1} $$

Then, the fractional q-difference problem (1.1)-(1.2) admits a unique solution on [0,1].

Proof 15 The fractional q-difference problem (1.1)-(1.2) can be reformulated in terms of a fixed point problem. For this purpose, we define the operator

$$ F: C([0,1],X) \to C([0,1],X), $$

where

$$F\big(u(t)\big) := a\int_0^1 u(s)d_qs + b + \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} f\big(s,u(s)\big)d_qs \qquad (3.2)$$
$$+\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}Au(s)d_qs.$$

It is evident that the fixed points of the operator F correspond to solutions of the fractional q-difference problem (1.1)-(1.2). We shall use the Banach contraction mapping principle to demonstrate that F defined by (3.2) has a fixed point. We shall demonstrate that F is a contraction.

Let $u, v \in C([0,1], X)$ and $t \in [0,1]$, we obtain

$$|F(u)(t) - F(v)(t)|$$
$$\leq |a|\int_0^1 |u(s)-v(s)|d_qs + \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}\left|f\big(s,u(s)\big) - f\big(s,v(s)\big)\right|d_qs$$
$$+\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}\|A\|_{\mathcal{L}(X)}|u(s)$$
$$-v(s)|d_qs$$
$$\leq |a|sup_{s\in[0,1]}|u(s)-v(s)| + \frac{k}{\Gamma_q(\alpha)}sup_{s\epsilon[0,1]}|u(s)-v(s)|\int_0^t (t-qs)^{(\alpha-1)}d_qs$$
$$+\frac{\|A\|_{\mathcal{L}(X)}}{\Gamma_q(\alpha)}sup_{s\epsilon[0,1]}|u(s)-v(s)|\int_0^t (t-qs)^{(\alpha-1)}d_qs$$
$$\leq |a|sup_{s\in[0,1]}|u(s)-v(s)| + \frac{k}{\Gamma_q(\alpha+1)}sup_{s\epsilon[0,1]}|u(s)-v(s)|$$
$$+\frac{\|A\|_{\mathcal{L}(X)}}{\Gamma_q(\alpha+1)}sup_{s\epsilon[0,1]}|u(s)-v(s)|$$
$$=\left(|a| + \frac{k+\|A\|_{\mathcal{L}(X)}}{\Gamma_q(\alpha+1)}\right)sup_{s\in[0,1]}|u(s)-v(s)|.$$

Hence,

$$\|F(u)-F(v)\|_\infty \leq \left[|a| + \frac{k+\|A\|_{\mathcal{L}(X)}}{\Gamma_q(\alpha+1)}\right]\|u-v\|_\infty.$$

According to (3.2), the operator F satisfies the contraction condition. Hence, by the Banach fixed point theorem, F admits a fixed point, representing the solution of the fractional q-difference problem (1.1)-(1.2).

Theorem 16 Suppose that:

(H_2) The function $f: [0,1] \times X \to X$ is continuous.

(H_3) $\exists M > 0,\ \forall t \in J,\ \forall u \in X,\ \ |f(t,u)| \leq M.$

Hence, the fractional q-difference problem (1.1)-(1.2) admits at least one solution on $[0,1]$.

Proof 17 We apply Schaefer's fixed point theorem to verify that the operator F, defined in (3.2), admits a fixed point. The proof is organized in four steps.

Step 1: F is continuous operator.

Consider a sequence $\{u_n\}$ such that $u_n \to u$ in $C([0,1],X)$. Then for all $t \in [0,1]$:

$$
\begin{aligned}
&|F(u_n)(t) - F(u)(t)| \\
&\qquad \leq |a| \int_0^1 |u_n(s) - u(s)| d_q s \\
&\qquad + \frac{1}{\Gamma_q(\alpha)} \int_0^1 (t - qs)^{(\alpha-1)} \left| f\big(s, u_n(s)\big) - f\big(s, u(s)\big) \right| d_q s \\
&\qquad + \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} \|A\|_{\mathcal{L}(X)} |u_n(s) - u(s)| d_q s \\
&\qquad \leq |a| sup_{s\in[0,1]} |u_n(s) - u(s)| \\
&\qquad + \frac{1}{\Gamma_q(\alpha)} sup_{s\in[0,1]} \left| f\big(s, u_n(s)\big) - f\big(s, u(s)\big) \right| \int_0^t (t - qs)^{(\alpha-1)}\, d_q s \\
&\qquad + \frac{\|A\|_{\mathcal{L}(X)}}{\Gamma_q(\alpha)} sup_{s\in[0,1]} |u_n(s) - u(s)| \int_0^t (t - qs)^{(\alpha-1)} d_q s \\
&\qquad \leq |a| sup_{s\in[0,1]} |u_n(s) - u(s)| + \frac{1}{\Gamma_q(\alpha+1)} sup_{s\in[0,1]} \left| f\big(s, u_n(s)\big) - f\big(s, u(s)\big) \right| \\
&\qquad + \frac{\|A\|_{\mathcal{L}(X)}}{\Gamma_q(\alpha+1)} sup_{s\in[0,1]} |u_n(s) - u(s)| \\
&\qquad = \left(|a| + \frac{\|A\|_{\mathcal{L}(X)}}{\Gamma_q(\alpha+1)} \right) sup_{s\in[0,1]} |u_n(s) - u(s)| \\
&\qquad + \frac{1}{\Gamma_q(\alpha+1)} sup_{s\in[0.1]} \left| f\big(s, u_n(s)\big) - f\big(s, u(s)\big) \right|.
\end{aligned}
$$

Since the function f is a continuous, we have

$$
\begin{aligned}
&\|F(u_n) - F(u)\|_\infty \\
&\qquad \leq \left(|a| + \frac{\|A\|_{\mathcal{L}(X)}}{\Gamma_q(\alpha+1)} \right) \|u_n(\cdot) - u(\cdot)\|_\infty + \frac{1}{\Gamma_q(\alpha+1)} \left\| f\big(\cdot, u_n(\cdot)\big) - f\big(\cdot, u(\cdot)\big) \right\|_\infty \\
&\qquad \xrightarrow[n\to+\infty]{} 0.
\end{aligned}
$$

Step 2: F maps bounded sets into bounded sets in $C([0,1],X)$.

Indeed, it is enough to prove that for every $\eta^* > 0$, there exists a constant $\ell > 0$ such that for all $u \in B_{\eta^*} = \{u \in C([0,1],\mathbb{R}) \colon \ \|u\|_\infty \leq \eta^*\}$, we have $\|F(u)\|_\infty \leq \ell$.

By (H_3) we have for every $t \in [0,1]$:

$$\begin{aligned}
|F(u)(t)| &\leq |a| \int_0^1 |u(s)| d_q s + |b| + \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} \left| f\big(s,u(s)\big) \right| d_q s \\
&\quad + \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} \|A\|_{\mathcal{L}(X)} |u(s)| d_q s \\
&\leq |a|\eta^* + |b| + \frac{M}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} d_q s + \frac{\|A\|_{\mathcal{L}(X)}\eta^*}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} d_q s \\
&\leq |a|\eta^* + |b| + \frac{M}{\Gamma_q(\alpha+1)} + \frac{\|A\|_{\mathcal{L}(X)}\eta^*}{\Gamma_q(\alpha+1)}.
\end{aligned}$$

Hence

$$\|F(u)\|_\infty \leq |a|\eta^* + |b| + \frac{M}{\Gamma_q(\alpha+1)} + \frac{\|A\|_{\mathcal{L}(X)}\eta^*}{\Gamma_q(\alpha+1)} := \ell.$$

Step 3: The operator F maps bounded sets into equicontinuous sets of $C([0,1],X)$.

For $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, and for $u \in B_{\eta^*}$, where B_{η^*} is the bounded subset of $C([0,1],X)$ defined in step 2, we have

$$\begin{aligned}
&|F(u)(t_2) - F(u)(t_1)| \\
&\quad = \left| \frac{1}{\Gamma_q(\alpha)} \int_0^{t_1} \left[(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}\right] f(s,u(s)) d_q s \right. \\
&\quad + \frac{1}{\Gamma_q(\alpha)} \int_0^{t_1} \left[(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}\right] Au(s) d_q s \\
&\quad \left. + \frac{1}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2-qs)^{(\alpha-1)} f(s,u(s)) d_q s + \frac{1}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2-qs)^{(\alpha-1)} Au(s) d_q s \right| \\
&\quad \leq \frac{M}{\Gamma_q(\alpha)} \int_0^{t_1} \left[(t_1-qs)^{(\alpha-1)} - (t_2-qs)^{(\alpha-1)}\right] f\big(s,u(s)\big) d_q s \\
&\quad + \frac{\eta^*\|A\|_{\mathcal{L}(X)}}{\Gamma_q(\alpha)} \int_0^{t_1} \left[(t_1-qs)^{(\alpha-1)} - (t_2-qs)^{(\alpha-1)}\right] d_q s \\
&\quad + \frac{M}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2-qs)^{(\alpha-1)} d_q s + \frac{\eta^*\|A\|_{\mathcal{L}(X)}}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2-qs)^{(\alpha-1)} d_q s \\
&\quad \leq \frac{M}{\Gamma_q(\alpha+1)} \left[(t_2-t_1)^{(\alpha)} + t_1^{(\alpha)} - t_2^{(\alpha)}\right] + \frac{\eta^*\|A\|_{\mathcal{L}(X)}}{\Gamma_q(\alpha+1)} \left[(t_2-t_1)^{(\alpha)} + t_1^{(\alpha)} - t_2^{(\alpha)}\right] \\
&\quad + \frac{M}{\Gamma_q(\alpha+1)} (t_2-t_1)^{(\alpha)} + \frac{\eta^*\|A\|_{\mathcal{L}(X)}}{\Gamma_q(\alpha+1)} (t_2-t_1)^{(\alpha)} \\
&\quad \leq \left[\frac{2M}{\Gamma_q(\alpha+1)} + \frac{2\eta^*\|A\|_{\mathcal{L}(X)}}{\Gamma_q(\alpha+1)}\right] (t_2-t_1)^{(\alpha)} + \left[\frac{M}{\Gamma_q(\alpha+1)} + \frac{\eta^*\|A\|_{\mathcal{L}(X)}}{\Gamma_q(\alpha+1)}\right] (t_2-t_1)^{(\alpha)}.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the preceding inequality tends to zero. As a consequence of step 1 to 3 tougher with the Arzelà-Ascoli theorem, we can conclude that the operator $F: C([0,1],X) \to C([0,1],X)$ is continuous and completely continuous.

Step 4: A priori bounds.

By Schauder's fixed point theorem, it is sufficient to show that the set:

$$\mathcal{E} = \{u \in C([0,1], X) \mid u = \lambda F(u),\ \ \lambda \in]0,1[\}$$

is bounded.

Let $u \in \mathcal{E}$. Then there exists $\lambda \in]0,1[$ such that $u = \lambda F(u)$. Hence, for every $t \in [0,1]$, we have

$$u(t) = \lambda\left[a\int_0^1 u(s)d_qs + b + \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} f\big(s,u(s)\big)d_qs \right.$$
$$\left. + \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} Au(s)d_qs\right].$$

Under hypothesis (H_3), it follows that for every $t \in [0,1]$:

$$\begin{aligned}|F(u)(t)| &\le |a|\int_0^1 |u(s)|d_qs + |b| + \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}\left|f\big(s,u(s)\big)\right|d_qs \\ &+ \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}\,\|A\|_{\mathcal{L}(X)}|u(s)|d_qs \\ &\le |a|\ sup_{s\in[0,1]}|u(s)| + |b| + \frac{M}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}d_qs \\ &+ \frac{\|A\|_{\mathcal{L}(X)}}{\Gamma_q(\alpha)}\ sup_{s\in[0,1]}|u(s)|\int_0^t (t-qs)^{(\alpha-1)}\,d_qs \\ &\le |a|\ sup_{s\in[0,1]}|u(s)| + |b| + \frac{M}{\Gamma_q(\alpha+1)} + \frac{\|A\|_{\mathcal{L}(X)}}{\Gamma_q(\alpha+1)}sup_{s\in[0,1]}|u(s)|.\end{aligned}$$

Then, for every $t \in [0,1]$, we have

$$\|F(u)\|_\infty \le |a|\ sup_{s\in[0,1]}|u(s)| + |b| + \frac{M}{\Gamma_q(\alpha+1)} + \frac{\|A\|_{\mathcal{L}(X)}}{\Gamma_q(\alpha+1)}sup_{s\in[0,1]}|u(s)| \ := R.$$

This shows that the set $\mathcal{E}$ is bounded.

Consequently, by Schaefer fixed point theorem, we conclude that F admits a fixed point which is a solution of the problem (1.1)-(1.2).

4. Application

In this section, we provide an illustrative example to demonstrate the applicability of our result.

Consider the following fractional q-difference problem with an integral boundary condition:

$${}^cD^{\frac{1}{2}}_{\frac{1}{3}}u(t) = \frac{t}{100}u(t) + \frac{u(t)}{(1+cos(u(t)))}, \quad t \in [0,1], \tag{4.1}$$

$$u(0) = \frac{2}{3}\int_0^1 u(s)d_{\frac{1}{3}}s + \frac{1}{2}. \tag{4.2}$$

Let

$$f(t,u) = \frac{u}{\left(1 + cos(u)\right)}, \quad (t,u) \in [0,1] \times]0,+\infty[$$

and

$$A(t) = \frac{t}{100}.$$

For all $u, v \in]0,+\infty[$ and $t \in [0,1]$. We have

$$|f(t,u) - f(t,v)| \leq \left|\frac{u-v}{(1+\cos(u))(1+\cos(v))}\right| \leq \frac{1}{4}|u-v|.$$

We see that the condition $|a| + \frac{k+\|A\|}{\Gamma_q(\alpha+1)} \approx 0.93 < 1$ holds with $\alpha = \frac{1}{2}, k = \frac{1}{4},\ q = \frac{1}{3},\ \|A\| = \frac{1}{100}$ and $\Gamma_{\frac{1}{3}}\left(\frac{3}{2}\right) \approx 0.9376$. Consequently, by Theorem 14, the fractional q-difference problem (4.1)-(4.2) admits a unique solution on the interval $[0,1]$.

5. Conclusion

In this study, we establish the existence of solutions to Cauchy problems for fractional *q*-difference equations with integral conditions in Banach spaces. The analysis is conducted employing the Banach fixed point theorem and the Schaefer fixed point theorem. Furthermore, an illustrative example is provided to demonstrate the applicability and effectiveness of the obtained results.

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