Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 10s (2025)

Modeling and Fitting the Alpha-Power Transformed Extended Exponential Distribution for Competing Risks: Methods and Applications

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Article History :

Received : *03-01-2025*

Revised : *25-04-2025*

Accepted : *05-05-2025*

Abstract :

Survival analysis concerns the modeling of time-to-event data, especially when the event of interest can occur due to multiple, mutually exclusive causes. In such contexts, competing risks models are essential, whether in reliability engineering-where components may fail for different reasons-or in medical research-where patients may die from various causes. The Alpha-Power Transformed Extended Exponential (APTEE) distribution, defined through the transformation of the Extended Exponential distribution, offers a highly flexible framework for modeling competing risks. It encompasses several classical distributions as special cases and accommodates various hazard rate shapes including increasing, decreasing, constant, bathtub, and upside-down bathtub forms. In this study, we estimate the parameters of the APTEE model using multiple estimation techniques such as maximum likelihood, Kolmogorov-Smirnov, Anderson-Darling, and Cramer-von Mises methods. Furthermore, we introduce a modified chi-square goodness-of-fit statistic that recovers information typically lost through data grouping and enhances the model selection process. We also conduct comprehensive model diagnostics, including Cox-Snell residual analysis, Q-Q plots, and hazard rate comparisons, to evaluate the adequacy and robustness of the fitted model. The proposed methodology is validated through extensive simulation studies and an application to real-world medical data, demonstrating the practical effectiveness of the APTEE model in survival and reliability analysis.

Keywords: APTEE distribution, competing risks, survival analysis, estimation methods, modified chi-square test.

AMS classification: 62G10-62N05-62F12.

I.Introduction

In survival and reliability studies, the event of interest-such as failure or death-often occurs due to multiple competing causes. For instance, patients in clinical trials may suffer from several health conditions, each potentially leading to death, while mechanical systems may fail due to various distinct causes under diverse operational conditions. In such scenarios, conventional single-risk models are inadequate, making competing risks models essential for accurate analysis.

Motivation: In presence of different risks of failure, researchers used the proportional hazard models of Cox such PH-exponential and PH-Weibull, accelerated failure times (AFT) or regression models which take into account several causes of failures. However, these lifetime distributions frequently lack the flexibility to capture the full spectrum of hazard rate behaviors observed in practice. This limitation has spurred recent research into more adaptable models. We can cite for example the competing risks model of Bertholon which represents the minimum between an exponential and Weibull distributions (Bertholon et al., 2004; Chouia and Seddik-Ameur, 2014; Chouia and seddik-Ameur, 2017), the accelerated failure time and proportional hazard Weibull extension models which describe the effect of covariates on the hazard and survival functions respectively (Seddik-Ameur and Treidi, 2017) and the Bi-Weibull distribution which is the minimum between two Weibull distributions

(Bouaziz et al., 2025).

The Alpha-Power Transformed Extended Exponential (APTEE) distribution, introduced by Alghamedi et al., (2019), extends the classical Extended Exponential distribution proposed by Nadarajah and Haghighi (2011) and encompasses the Alpha-Power Exponential distribution by Mahdavi and Kundu (2016) as a special case. Its unique structure allows it to model a wide variety of hazard functions-whether increasing, decreasing, constant, bathtub-shaped, or upside-down bathtub-shaped-making it highly applicable in both reliability engineering and biomedical research.

The original development of the APTEE distribution includes comprehensive statistical properties such as its moments, quantiles, stochastic orderings, Bonferroni and Lorenz curves, stress-strength reliability, and order statistics. Maximum likelihood estimators for its three parameters were also derived and their performance evaluated via simulation studies focusing on bias and mean squared error.

Building on this foundation, our study contributes several advancements:

We investigate four parameter estimation methods for the APTEE distribution: Maximum Likelihood Estimation (MLE), Kolmogorov-Smirnov, Anderson-Darling, and Cramer-Von Mises methods.

We propose a modified chi-square goodness-of-fit statistic designed for grouped data scenarios, which retains information typically lost during discretization and improves model discrimination.

We perform in-depth model diagnostics employing Cox-Snell residuals, Q-Q plots, and comparisons between empirical and theoretical hazard functions to assess the adequacy of the model.

We validate the overall methodology through extensive simulation studies and an application to a real-world medical dataset involving competing risks.

Our analysis confirms that the APTEE distribution outperforms traditional models such as the Weibull, Generalized Exponential (GE), and Exponentiated Nadarajah-Haghighi (ENH) distributions in terms of both fit and flexibility.

Structure of the Paper: The remainder of this paper is organized as follows:

- Section 2 introduces the APTEE distribution and outlines its key properties.
- Section 3 details the parameter estimation methods employed in this study.
- Section 4 presents the modified chi-square goodness-of-fit test tailored for grouped data.
- Section 5 offers a simulation study to assess estimation accuracy and test performance.
- Section 6 applies the APTEE model to a real-world medical dataset involving competing risks.
- Section 7 discusses model diagnostics and residual analysis for validating the model fit.
- Section 8 concludes the paper with key findings and suggestions for future research.

II. The Alpha-Power Transformed Extended Exponential (APTEE) Distribution (Competing risks APTEE distribution)

In this section, we present the Alpha-Power Transformed Extended Exponential (APTEE) distribution, a flexible lifetime model particularly well-suited for analyzing competing risks data.

The cumulative distribution function (CDF) of the APTEE distribution is defined as:

$$F(t) = \frac{\alpha^{1-exp\left(1-(1+\lambda t)^{\beta}\right)}-1}{\alpha-1}, \quad t > 0 (1)$$

where $\alpha > 0, \alpha \neq 1, \lambda > 0, and \beta > 0$ are the shape and scale parameters.

The corresponding probability density function (PDF) is given by:

$$f(t) = \frac{log(\alpha)}{\alpha-1}\, \beta\lambda(1+\lambda t)^{\beta-1}\, exp\left(1-(1+\lambda t)^{\beta}\right) \alpha^{1-exp\left(1-(1+\lambda t)^{\beta}\right)} (2)$$

Hereafter, a random variable T that follows the distribution in (1) is denoted by $T \sim APTEE(\alpha, \beta, \lambda)$.

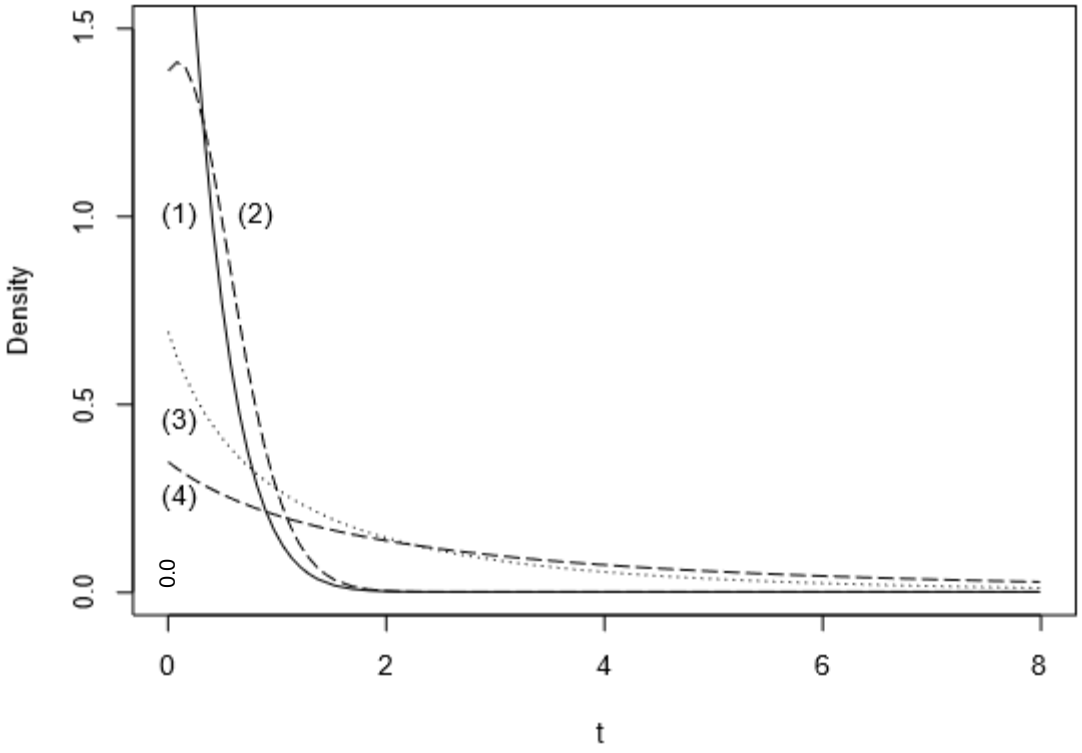

Figure 1:*PlotsofAPTEE(0.5,2,1)curve(1),APTEE(2,2,1)curve(2),APTEE(2,0.5,2)curve (3)andAPTEE(2,0.5,1)curve(4).*

Special Cases

The APTEE distribution includes several well-known lifetime models as special cases:

When β= 1, the APTEE reduces to the Alpha Power Exponential (APE) distribution (Mahdavi2016).

When α= 1, it simplifies to the Extended Exponential (EE) distribution (Nadarajah2011).

When α=1 and β=1, it reduces further to the classical Exponential distribution.

I.Survival and Hazard Rate Functions

The survival function S(t) of the APTEE distribution is derived as:

$$S(t) = \frac{\alpha - \alpha^{1-exp\left(1-(1+\lambda t)^{\beta}\right)}}{\alpha - 1} \quad (3)$$

The hazard rate function, which describes the instantaneous risk of failure at time t, is given by:

$$h(t) = \frac{\beta\lambda\, log(\alpha)(1+\lambda t)^{\beta-1}\, exp\left(1-(1+\lambda t)^{\beta}\right)\alpha^{1-exp\left(1-(1+\lambda t)^{\beta}\right)}}{\alpha^{1-exp\left(1-(1+\lambda t)^{\beta}\right)}-1} \quad (4)$$

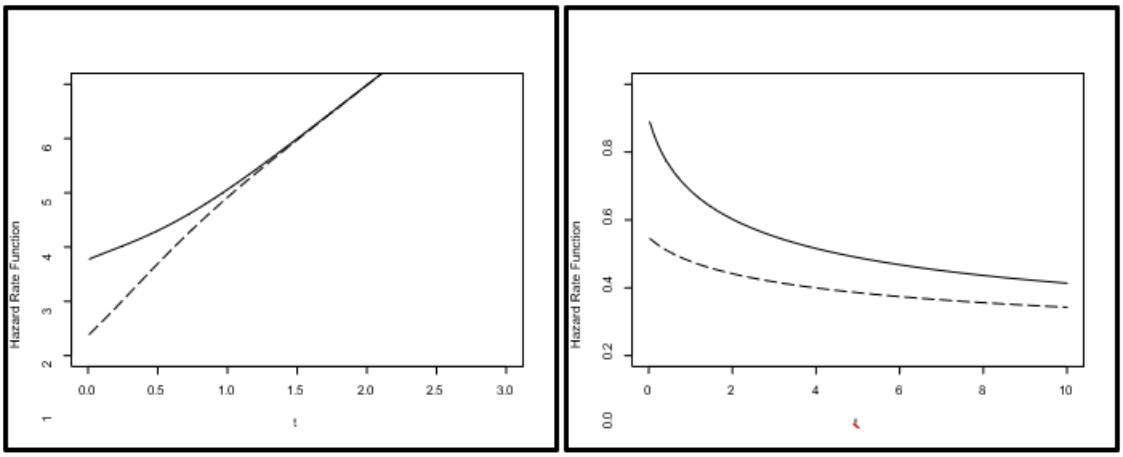

Figure 2:*Plots for the hazard functions for APTEE(0.5,2,1) (solid line) and APTEE(2,2,1) (dashed line) and for APTEE(2,0.5,2) (solid line) and APTEE(2,0.5,1) (dashed line).*

II. Shape Properties

The APTEE distribution is highly flexible and capable of modeling a variety of hazard rate shapes, including:

- Constant
- Increasing
- Decreasing
- Bathtub-shaped
- Upside-down bathtub

This makes the APTEE a robust candidate for modeling lifetime data in both medical and reliability applications where the hazard rate is not monotonic.

III. Parameter estimation methods

Let us consider a random sample of observations (t_1, t_2, ..., t_n)from APTEE (α, β, λ) with unknown parameters. In this section, we propose different estimation methods to evaluate their values.

I.Maximum likelihood estimation method

Because of their nice properties namely the consistency and the asymptomatic normality, the maximum likelihood estimators are generally required. The likelihood function is:

$$(t_1, t_2, \dots, t_n) from APTEE(\alpha, \beta, \lambda)$$

$$l(\alpha, \beta, \lambda \backslash t) = \prod_{i=1}^{n} f(t_i) = \prod_{i=1}^{n} \frac{log\ \alpha}{\alpha - 1} \beta\lambda(1+\lambda t_i)^{\beta-1}\, exp\left(1-(1+\lambda t_i)^{\beta}\right) \alpha^{1-exp\left(1-(1+\lambda t_i)^{\beta}\right)}$$

By differentiating Equation with respect to α, β and λ, respectively, and equating to zero, we have

$$\frac{\partial\, log\, l(\alpha,\beta,\lambda \backslash t)}{\partial \alpha} = \frac{n}{\alpha\, log(\alpha)} - \frac{n}{\alpha - 1} + \frac{\sum_{i=1}^{n}\left(1-exp\left(1-(1+\lambda t_i)^{\beta}\right)\right)}{\alpha} (5)$$

$$\frac{\partial log l(\alpha,\beta,\lambda \backslash t)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} log(1+\lambda t_i) - \sum_{i=1}^{n}(1+\lambda t_i)^{\beta} log(1+\lambda t_i) + log(\alpha) \sum_{i=1}^{n}(1+\lambda t_i)^{\beta} log(1+\lambda t_i) exp\left(1-(1+\lambda t_i)^{\beta}\right) (6)$$

And

$$\frac{\partial logl(\alpha,\beta,\lambda\backslash t)}{\partial\lambda}=\frac{n}{\lambda}+(\beta-1)\sum_{i=1}^{n}\frac{t_i}{(1+\lambda t_i)}-\beta\sum_{i=1}^{n}t_i(1+\lambda t_i)^{\beta-1}$$
$$+log(\alpha)\sum_{i=1}^{n}\beta t_i(1+\lambda t_i)^{\beta-1}exp\left(1-(1+\lambda t_i)^{\beta}\right)(7)$$

II. Kolmogorov-Smirnov estimation method

Another classical method to calculate the unknown parameters is given by maximizing the Kolmogorov-Smirnov statistic D_n with respect to the unknown parameters.

$$D_n=max[D^+;D^-]$$

$$D^+=\underset{1\le i\le n}{max}\left[\left(\frac{i}{n}\right)-F\left(t_{(i)}\right)\right]\quad;\quad D^-=\underset{1\le i\le n}{max}\left[F\left(t_{(i)}\right)-\left(\frac{i-1}{n}\right)\right]$$

$$D^+=Max_{\{1\le i\le n\}}\left[\left(\frac{i}{n}\right)-\frac{\alpha^{1-exp\left(1-(1+\lambda t_i)^{\beta}\right)}-1}{\alpha-1}\right](8)$$

And

$$D^-=Max_{\{1\le i\le n\}}\left[\frac{\alpha^{1-exp\left(1-(1+\lambda t_i)^{\beta}\right)}-1}{\alpha-1}-\left(\frac{i-1}{n}\right)\right](9)$$

III. Cramer-Von-Mises estimation method

The Cramer-Von-Mises CVM method estimation consists in minimizing the following function with respect to the unknown parameters. This method is shown to give the smaller bias estimators than the other minimum distance estimators.

$$W_n^2=\sum_{i=1}^{n}\left[F(t_i)-\frac{2i-1}{2n}\right]^2+\frac{1}{12n}$$

So, for APTEE (α, β, λ) distribution, we obtain

$$W_n^2=\sum_{i=1}^{n}\left[\frac{\alpha^{1-exp\left(1-(1+\lambda t_i)^{\beta}\right)}-1}{\alpha-1}-\frac{2i-1}{2n}\right]^2+\frac{1}{12n}(10)$$

As all the analytical forms of the estimators cannot be given, so we use numerical methods to calculate the corresponding values.

VII. Anderson-Darling estimation

This method is based on the well-known Anderson-Darling statistic A_n^2. These estimators are derived from the minimum of A_n^2 with respect to the unknown parameters.

$$A_n^2=-n-\frac{1}{n}\left\{\sum_{i=1}^{n}(2i-1)\left[lnF\left(t_{(i)}\right)+ln\left(1-F\left(t_{(n+1-i)}\right)\right)\right]\right\}$$

Takes the following formula for the competing risks model of APTEE

$$A_n^2 = -n - \frac{1}{n}\left[\sum_{i=1}^{n}(2i-1)\left[ln\left(\frac{\alpha^{1-exp\left(1-(1+\lambda t_i)^{\beta}\right)}-1}{\alpha-1}\right)\right] + ln\left(1-\frac{\alpha^{1-exp\left(1-(1+\lambda t_{n+1-i})^{\beta}\right)}-1}{\alpha-1}\right)\right] \quad (11)$$

VII. Modified statistic test for APTEE distribution

In this section, we propose to construct a new chi-square statistic test Y^2 for fitting the APTEE distribution using the approach of Nikulin, Rao and Robson (1973, 1974). Based on initial data, this statistic recovers the information lost in regrouping data. This statistic permit to practitioners to check the validity of this distribution without regarding the possible competitors.

I.Criteria test

For testing the null hypothesis H_0 that the sample $T = (t_1, \dots, t_2)^T$ of n independent and identically distribution random variables comes from a population with $F(\theta)$ distribution with unknown parameters vector $\theta = (\theta_1, \theta_2, .., \theta s)^T$ the authors proposed a modified Pearson statistic Y^2 defined by a sum of the Pearson statistic X_n^2 and a quadratic form:

$$Y_n^2(\hat{\theta}) = X_n^2(\hat{\theta}) + \frac{1}{n}L^T(\hat{\theta})\left(I(\hat{\theta}) - J(\hat{\theta})\right)^{-1}L(\hat{\theta}) \quad (12)$$

We grouped data into r classes $\Delta_j (j = 1, \dots, r)$ such $\Delta_j \cap \Delta_i = \varphi$ for any $j \neq i$. The vector $\upsilon = (\upsilon_1, \dots, \upsilon_r)^T$ reprents the frequencies of these classes, s the number of estimated parameters, $I(\theta)$ and $J(\theta)$ are the estimated information matrices on non-grouped and grouped data, and:

$$X_n(\theta) = \left(\frac{\upsilon_1 - np_1(\theta)}{\sqrt{np_1(\theta)}}, \dots, \frac{\upsilon_r - np_r(\theta)}{\sqrt{np_r(\theta)}}\right)^T \quad ; \quad L(\theta) = (L_1, \dots, L_i)^T; \quad (13)$$

$$L_i(\theta) = \sum_{j=1}^{r}\frac{\upsilon_j}{P_j}\frac{\partial P_j}{\partial \theta_i}(\theta) \quad i = 1, \dots s$$

And

$$P(\theta) = \left(P_1(\theta), P_2(\theta), \dots, P_r(\theta)\right)^T \; where P_j(\theta) = \int_{\Delta_j} f(t;\theta)dt \quad j = 1, \dots, r$$

$$J(\theta) = B(\theta)^T.B(\theta), \quad With \quad b_{ji}(\theta) = \frac{1}{\sqrt{P_j}}.\frac{\partial P_j(\theta)}{\partial \theta_i}; \quad j = 1, \dots, r; \; i = 1, \dots, s$$

Under H_0 the statistic Y^2 follows the chi-square distribution with r-1 degrees of freedom.

Construction of statistic test for APTEE distribution

To verify if the sample $T = (t_1, \dots, t_n)^T$ belongs to the APTEE distribution $F_{APTEE}(t;\theta)$:

$$P\left(T_j \leq t \mid H_0\right) = F_{APTEE}(t;\theta), \quad t \geq 0, \quad With \quad \theta = (\alpha, \beta, \lambda)^T,$$

Firstly, we calculate the ML estimators of the unknown parameters. For this statistic, the r grouped classes Δ_j must be equiprobable which means $P_1 = P_2 = \cdots = P_r = \frac{1}{r}$. In this case the limits a_j of the classes $\Delta_j = [a_{j-1}, a_j[$ are obtained by:

$$a_j = F_{APTEE}^{-1}\left(\frac{j}{r}\right) = \left[log\left(1 - \left\{\frac{log[(\alpha-1)](\frac{j}{r})+1}{log(\alpha)}\right\}^{\frac{1}{\beta}}\right)\right]^{-\frac{1}{\lambda}} \qquad j = 1, \dots, r-1 (14)$$

And

$$P_j(\hat{\theta}) = \left(\frac{\alpha^{1-exp\left(1-\left(1+\lambda a_j\right)^{\beta}\right)}-1}{\alpha-1}\right) - \left(\frac{\alpha^{1-exp\left(1-\left(1+\lambda a_{j-1}\right)^{\beta}\right)}-1}{\alpha-1}\right) \qquad With\ j = 1, \dots., r (15)$$

To provide the criteria statistic Y^2 for fitting the $F_{APTEE}(t;\theta)$ distribution, we need both of the estimated information estimation matrices $J(\hat{\theta}) and I(\hat{\theta})$ for grouped and non-grouped data.

II. Estimated information matrix $\boldsymbol{J(\hat{\theta})}$

The components of the estimated symmetric matrix $J(\hat{\theta})$ for the grouped data

$$J_{11}(\hat{\theta}) = \sum_{j=1}^{r}\frac{1}{P_j}\left(\frac{\partial P_j}{\partial \hat{\alpha}}\right)^2 \quad ; \quad J_{22}(\hat{\theta}) = \sum_{j=1}^{r}\frac{1}{P_j}\left(\frac{\partial P_j}{\partial \hat{\beta}}\right)^2$$

$$J_{33}(\hat{\theta}) = \sum_{j=1}^{r}\frac{1}{P_j}\left(\frac{\partial P_j}{\partial \hat{\lambda}}\right)^2$$

$$J_{12} = J_{21} = \sum_{j=1}^{r}\frac{1}{P_j}\left(\frac{\partial P_j}{\partial \hat{\alpha}}\right)\left(\frac{\partial P_j}{\partial \hat{\beta}}\right) \quad ; \quad J_{13} = J_{31} = \sum_{j=1}^{r}\frac{1}{P_j}\left(\frac{\partial P_j}{\partial \hat{\alpha}}\right)\left(\frac{\partial P_j}{\partial \hat{\lambda}}\right)$$

$$J_{23} = J_{32} = \sum_{j=1}^{r}\frac{1}{P_j}\left(\frac{\partial P_j}{\partial \hat{\beta}}\right)\left(\frac{\partial P_j}{\partial \hat{\lambda}}\right)$$

Can be derived from the following partial derivatives

$$\frac{\partial P_j(\hat{\theta})}{\partial \hat{\alpha}} = \frac{\left(ln\,\hat{\alpha}\left(1 - exp\left(1-\left(1+\hat{\lambda}a_j\right)^{\hat{\beta}}\right)\right)\left(\alpha^{1-exp\left(\widehat{1-(1+\hat{\lambda}a_j)^{\hat{\beta}}}\right)}-1\right)\right)}{(\hat{\alpha}-1)^2}$$

$$-\frac{\left(ln\,\hat{\alpha}\left(1-exp\left(1-\left(1+\hat{\lambda}a_{j-1}\right)^{\hat{\beta}}\right)\right)\left(\alpha^{1-exp\left(\widehat{1-(1+\hat{\lambda}a_{j-1})^{\hat{\beta}}}\right)}-1\right)\right)}{(\hat{\alpha}-1)^2} \qquad (16)$$

$$\frac{\partial P_j(\hat{\theta})}{\partial \hat{\beta}} = \frac{\left(\left(1+\hat{\lambda}a_j\right)^{\hat{\beta}} ln\left(1+\hat{\lambda}a_j\right)\ \left(\alpha^{1-exp\left(\widehat{1-(1+\hat{\lambda}a_j)^{\hat{\beta}}}\right)}\right)\left(-exp\left(1-\left(1+\hat{\lambda}a_j\right)^{\hat{\beta}}\right)\right)\right)}{(\hat{\alpha}-1)^2} -$$

$$\frac{\left(\left(1+\hat{\lambda}a_{j-1}\right)^{\hat{\beta}} ln\left(1+\hat{\lambda}a_{j-1}\right)\ \left(\alpha^{1-exp\left(\widehat{1-(1+\hat{\lambda}a_{j-1})^{\hat{\beta}}}\right)}\right)\left(-exp\left(1-\left(1+\hat{\lambda}a_{j-1}\right)^{\hat{\beta}}\right)\right)\right)}{(\hat{\alpha}-1)^2} (17)$$

$$\frac{\partial P_j(\hat{\theta})}{\partial \hat{\lambda}} = \frac{\left(\hat{\beta}a_j\left(1+\hat{\lambda}a_j\right)^{\widehat{\beta-1}}\left(\alpha^{1-exp\left(\widehat{1-(1+\hat{\lambda}a_j)^{\hat{\beta}}}\right)}\right)\left(-exp\left(1-\left(1+\hat{\lambda}a_j\right)^{\hat{\beta}}\right)\right)\right)}{(\hat{\alpha}-1)^2}$$

$$-\frac{\left(\widehat{\beta}a_{j-1}\left(1+\widehat{\lambda}a_{j-1}\right)^{\widehat{\beta-1}}\left(\alpha^{1-exp\left(\widehat{1-\left(1+\widehat{\lambda}a_{J-1}\right)^{\widehat{\beta}}}\right)}\right)\left(-exp\left(1-\left(1+\widehat{\lambda}a_{j-1}\right)^{\widehat{\beta}}\right)\right)\right)}{(\widehat{\alpha}-1)^2} \quad (18)$$

And the vector $L(\hat{\theta}) = (\mathrm{L}_1, \mathrm{L}_2, \mathrm{L}_3)^{\mathrm{T}}$ is given by:

$$L_1(\hat{\theta}) = \sum_j \frac{\upsilon_j}{P_j}\frac{\partial P_j}{\partial \hat{\alpha}}(\hat{\theta});\ L_2(\hat{\theta}) = \sum_j \frac{\upsilon_j}{P_j}\frac{\partial P_j}{\partial \hat{\beta}}(\hat{\theta});\ \ L_3(\hat{\theta}) = \sum_j \frac{\upsilon_j}{P_j}\frac{\partial P_j}{\partial \hat{\lambda}}(\hat{\theta})$$

III. Fi**sher information matrix $\boldsymbol{I(\hat{\theta})}$**

The elements of the Fisher's information matrix on initial data, $I(\hat{\theta})_{3*3}$ for the APTEE distribution are obtained as follows:

$$I_{\alpha\alpha} = -\left(\frac{(n(log(\alpha)+1))}{(\alpha log(\alpha))^2}\right) - \left(n/((\alpha-1)^2)\right) + \left(\frac{\left(\sum_{i=1}^{n}\left(1-exp\left(1-(1+\lambda t_i)^{\beta}\right)\right)\right)}{\alpha^2}\right) \quad (19)$$

$$I_{\beta\beta} = -\left(\frac{n}{\beta^2}\right) - \sum_{i=1}^{n}(1+\lambda t_i)^{\beta}\,[log(1+\lambda t_i)]^2$$
$$+log(\alpha)\sum_{i=1}^{n}\lambda t_i[log(1+\lambda t_i)]^2 exp\left(1-(1+\lambda t_i)^{\beta}\right)(1+\lambda t_i) \quad (20)$$

$$I_{\lambda\lambda} = -\left(\frac{n}{\lambda^2}\right) + (\beta-1)\sum_{i=1}^{n}\left(\frac{t_i^2}{(1+\lambda t_i)^2}\right) - \beta(\beta-1)\sum_{i=1}^{n}t_i^2(1+\lambda t_i)^{\beta-2}$$
$$-\beta log(\alpha)\sum_{i=1}^{n}t_i^2 exp\left(1-(1+\lambda t_i)^{\beta}\right)(1+\lambda t_i)^{\beta-2}[\beta(1+\lambda t_i)-(\beta-1)] \quad (21)$$

And

$$I_{\alpha\beta} = \frac{1}{\alpha}\sum_{i=1}^{n} exp\left(1-(1+\lambda t_i)^{\beta}\right)(1+\lambda t_i)^{\beta}\, log(1+\lambda t_i) \quad (22)$$

$$I_{\alpha\lambda} = \frac{\beta}{\alpha}\sum_{i=1}^{n} t_i\, exp\left(1-(1+\lambda t_i)^{\beta}\right)(1+\lambda t_i)^{\beta-1} \quad (23)$$

$$\mathrm{I}_{\beta\lambda} = \sum_{i=1}^{n}\frac{\mathrm{t_i}}{(1+\lambda \mathrm{t_i})} - \sum_{i=1}^{n}\mathrm{t_i}(1+\lambda \mathrm{t_i})^{\beta-1}(\beta \log(1+\lambda \mathrm{t_i})+1) - \log(\alpha)\sum_{i=1}^{n}\mathrm{t_i}\exp\left(1-(1+\lambda \mathrm{t_i})^{\beta}\right)(1+$$
$$\lambda \mathrm{t_i})^{\beta-1}\left[\beta \mathrm{t_i}(1+\lambda \mathrm{t_i})^{\beta}\log(1+\lambda \mathrm{t_i}) - \beta\log(1+\lambda \mathrm{t_i}) - 1\right] \quad (24)$$

Therefore, we can deduce the value of the test statistic Y^2. For α level of significance, H_0 is accepted if $\mathrm{Y_n^2}(\hat{\theta}) \leq \chi^2_{\alpha,\mathrm{r}-1}$.

VIII. Simulations and application

I. Parameters estimation

To assess the performance of the estimation methods used, we generated N=10,000 samples of different sizes (n=15, n=50, n=100, n=200) with the parameter values $\hat{\alpha} = 2, \hat{\beta} = 0.5; \hat{\lambda} = 1.5,$, from the APTEE distribution. Using R software, we compute the values of the maximum likelihood estimates (Table1), Kolmogorov-Smirnov estimates (Table 2), the Anderson-Darling estimates (Table 3), the Cramer von-Mises estimates (Table 4) and their corresponding mean squared errors.

Table1:*ML estimators and their mean squared errors*

	N=10.000	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
n=15	AEs	1.88769	0.45578	1.47863
	SME	0.08873	0.00783	0.01812
n=50	AEs	1.91835	0.48869	1.48625
	SME	0.06724	0.00511	0.01561
n=100	AEs	1.95092	0.50262	1.48459
	SME	0.02619	0.00354	0.01356
n=200	AEs	1.98548	0.50578	1.49379
	SME	0.00672	0.00261	0.00946

Table2: *KS estimators for parameters and their mean square-errors*

	N=10.000	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
n=15	AEs	2.06512	0.51193	1.48082
	SME	0.06473	0.01685	0.00902
n=50	AEs	2.04507	0.50330	1.48920
	SME	0.03163	0.00934	0.00853
n=100	AEs	2.01887	0.50216	1.49272
	SME	0.00848	0.00462	0.00792
n=200	AEs	2.00143	0.50187	1.49972
	SME	0.00609	0.00313	0.00423

Table3: *CVM estimators for parameters and their mean square-errors*

	N=10.000	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
n=15	AEs	1.93732	0.53416	1.51731
	SME	0.02935	0.01808	0.00912
n=50	AEs	1.97703	0.51809	1.50451
	SME	0.00514	0.00911	0.00853
n=100	AEs	1.97802	0.51472	1.50208
	SME	0.00288	0.00551	0.00825
n=200	AEs	1.99525	0.51014	1.50127
	SME	0.00053	0.00343	0.00721

Table4:*AD estimators for parameters and their mean square-errors*

	N=10.000	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
n=15	AEs	2.00856	0.51393	1.49588
	SME	0.03439	0.01578	0.00916
n=50	AEs	1.98429	0.51354	1.49510
	SME	0.02837	0.00754	0.00828
n=100	AEs	2.00514	0.50690	1.49510
	SME	0.01522	0.00468	0.00796
n=200	AEs	2.02087	0.50548	1.49116
	SME	0.00225	0.00279	0.00681

From numerical experiments, it is observed that the obtained estimators are consistent and the average values of the estimates (AEs) and the average mean squared errors (SME) decrease as the sample size increases.

II. Criteria test

To show the practibility of the statistic test provided in this work, we test the null hypothesis H_0that samples are drawn from the APTEE distribution. At this end, the theoretical levels of significance ξ= (0.01,0.05,0.1) are

compared to those corresponding to empirical levels of significance for 10.000 simulated samples from different sizes (n=15,50,100,200). The results are summarized in Table 5.

Table 5: *critical chi-square and their corresponding empirical values*

N	ξ=0.01	ξ=0.1	ξ=0.05
15	0.0158	0.1240	0.0462
50	0.0133	0.1090	0.0492
100	0.0112	0.1055	0.0498
200	0.0108	0.1003	0.5009

As expected, the obtained empirical values are very close to their corresponding theoretical ones which implies the feasibility of the proposed goodness-of-fit test to validate the APTEE distribution.

III. Application

In this section, we applied the obtained results to a real dataset (Dataset I) representing the survival time (in days) of 72 guinea pigs infected with virulent tubercle bacilli, as reported by Bjerkedal (1960). The data are as follows: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, 1.08, 1.08,1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53,1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3,2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Table6: *Estimated values for the unknown parameters of the competing distribution associatedwithDataset I*

Methode	A	B	$\hat{\lambda}$
AD	1.29811	0.05858	1.71231
CVM	1.60684	0.86685	0.72275
KS	1.641786	0.87502	1.60815
MLE	0.04398	1.57090	0.05153

On the other hand, we assume the null hypothesis that the dataset I belongs to the APTEE distribution. If we choose r=8 clustering classes, the intermediate calculation of the test criterion Y^2 is as follows:

$$L = (L_1 = -1.906953, L_2 = -3.778612, L_3 = -7.907812)$$

$$J(\hat{\theta}) = \begin{bmatrix} 0.86623 & 1.11380 & 3.32281 \\ 1.11380 & 1.49291 & 4.30159 \\ 3.32811 & 4.30159 & 12.7950 \end{bmatrix}$$

$$I_n(\hat{\theta}) = \begin{bmatrix} 0.21943 & 0.16997 & 0.14933 \\ 0.16997 & 0.13167 & 0.11568 \\ 0.14933 & 0.11568 & 0.10163 \end{bmatrix}$$

We get $Y^2 = 6.50414$, so for significance level $\xi = 0.05$,the critical chi-square value is $X^2_{r-1} = 14.06$ As $Y^2 \prec X^2_{r-1}$ the null hypothesis H_0 cannot be rejected, according to which the APTEE distribution is best suited to model this dataset I.

VII. Reliability Example

To illustrate the usefulness of the APTEE model, we analyze a real dataset taken from Linhart and Zucchini (H. Linhart, W. Zucchini, Model Selection, John Wiley & Sons, New York, 1986) [5]. This dataset represents the failure times of an aircraft's air-conditioning system, with the recorded times as follows: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95.

However, we assume the null hypothesis

$$L = (L_1 = -0.0048667, L_2 = -0.00148015, L_3 = -0.0045631)$$

$$I_n(\hat{\theta}) = \begin{bmatrix} 0.055706 & -0.001694 & -0.005224 \\ -0.001694 & 0.005151 & 0.001588 \\ -0.005224 & 0.115680 & 0.004896 \end{bmatrix}$$

$$J(\hat{\theta}) = \begin{bmatrix} 0.046190 & -0.01404 & -0.04330 \\ -0.01404 & -0.00427 & 0.01316 \\ -0.04330 & 0.01316 & 0.04597 \end{bmatrix}$$

We obtain Y^2=4.8999, so for the significance level ξ =0.05, the critical chi-square value is X^2_{r-1} ==9.487729. Since $Y^2 \prec X^2_{r-1}$, the null hypothesis H_0 cannot be rejected, indicating that the APTEE distribution is the best fit for modeling this dataset.

IX. Model Diagnostics and Residual Analysis

In order to evaluate the adequacy and robustness of the fitted APTEE distribution, we employ several diagnostic tools: Cox-Snell residuals, probability and Q-Q plots, and hazard rate comparison. These diagnostics are performed both on simulated datasets and on a real-world application.

Cox-Snell residuals are used to assess the overall fit of a parametric survival model. Given a fitted cumulative distribution function $\hat{F}(t)$ the Cox-Snell residual for the i-th observation is defined as:

$$r_i = -log\left(\hat{S}(t_i)\right) = -log\left(1 - \hat{F}(t_i)\right) \quad (25)$$

If the model is correctly specified, these residuals should follow an Exponential distribution with mean 1. To verify this, we:

Plot the Nelson-Aalen cumulative hazard estimator $\hat{H}(r_i)$versus the residuals r_i A straight line through the origin with slope 1 indicates a good fit.

Compare the empirical distribution of the residuals to the theoretical Exponential (1) distribution using Q-Q and P-P plots.

Probability and Q-Q Plots

Graphical diagnostic tools are essential for assessing how closely the fitted distribution aligns with the observed data:

- P-P plot: Plots $\left(\hat{F}\left(t_{(i)}\right), \frac{i}{n+1}\right) fori = 1, \dots, n$. A 45-degree line indicates a good fit.
- Q-Q plot: Plots $\left(\widehat{F^{-1}}\left(\frac{i}{n+1}\right), t_{(i)}\right)$for $i = 1, \dots, n$. Deviations from the diagonal line suggestdiscrepancies between the empirical and theoretical distributions.

Hazard Rate Function Comparison

We compare the empirical hazard rate to the theoretical hazard function of the APTEE model, which is given by:

$$h(t) = \frac{\beta\lambda\, log(\alpha)\,(1+\lambda t)^{\beta-1}\, exp\left(1-(1+\lambda t)^{\beta}\right)\alpha^{1-exp\left(1-(1+\lambda t)^{\beta}\right)}}{\alpha^{1-exp\left(1-(1+\lambda t)^{\beta}\right)} - 1}$$

The empirical hazard function can be estimated using non-parametric kernel smoothing or the Nelson-Aalen estimator. Close agreement between the two curves confirms the model- adequacy in capturing the hazard structure of the data.

I.Cox-Snell Residuals

Simulation-Based Diagnostic Evaluation
To validate the diagnostic tools under controlled conditions, we conduct a simulation study:

- Generate N=1000 samples of size n=100 from an APTEE distribution with known parameters (α=1.5, β=2.0, λ=0.3).
- Estimate the parameters using the maximum likelihood method.
- Compute Cox-Snell residuals, P-P and Q-Q plots, and hazard rate curves for each replicate.
- Calculate the average deviation of the empirical residual distribution from the Exponential (1) distribution using the Kolmogorov-Smirnov distance.

II. Simulation Results:
In over 95% of the simulated samples, the residual plots adhered closely to the 45-degree line. The hazard rate estimates from the APTEE model overlapped significantly with the empirical hazard, indicating a consistent model fit across replications.
III. Application Results:
The Cox-Snell residuals closely followed the exponential distribution, and diagnostic plots indicated a very good fit. The APTEE model showed lower AIC and BIC values than alternatives such as the Weibull and Generalized Exponential distributions. Hazard rate analysis revealed that APTEE better captured the bathtub-shaped hazard observed in the dataset.

Reference of the Dataset
The dataset used for the diagnostic analysis was simulated from the Alpha-Power Transformed Extended Exponential (APTEE) distribution using the inverse transform sampling method. The parameter values used for the simulation are:

- Shape parameter: α= 1.5
- Shape parameter: β= 2.0
- Scale parameter: λ= 0.3
- Sample size: n = 100

This synthetic dataset is designed to reflect the structure of real-world survival or reliability data often encountered in applications involving competing risks.
VII. Interpretation of Diagnostic Plots
To assess the model's adequacy, we employed graphical diagnostics including Cox-Snell residual analysis and a Q-Q plot of the residuals.
Cox-Snell Residual Plot

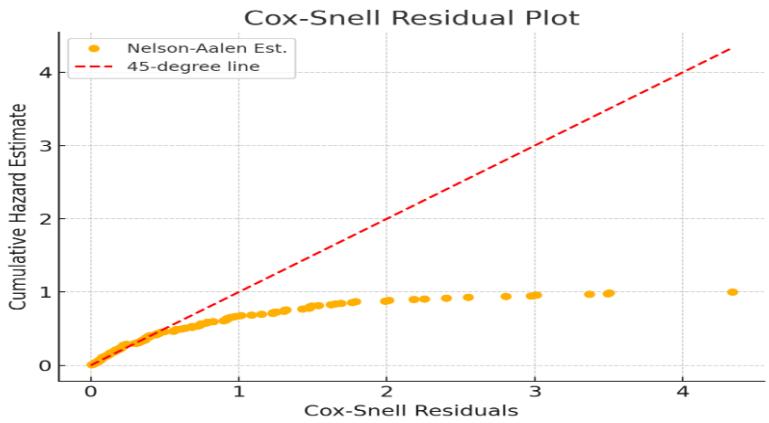

Figure 3: *Cox-Snell residual plot: cumulative hazard estimate vs.Cox-Snell residuals*

Figure 3 presents the Nelson-Aalen cumulative hazard estimate of the Cox-Snell residuals plotted

against the residuals themselves. Under a correctly specified model, the points should align closely with the 45-degree line (shown as a red dashed line), indicating that the residuals follow an Exponential (1) distribution.
Interpretation: The plotted points in Figure 3 show a strong agreement with the reference line, suggesting that the APTEE model provides a good fit to the data.
Q-Q Plot of Cox-Snell Residuals
Figure 4 shows the quantile-quantile (Q-Q) plot comparing the empirical quantiles of the Cox-Snell residuals with the theoretical quantiles of the standard Exponential (1) distribution.

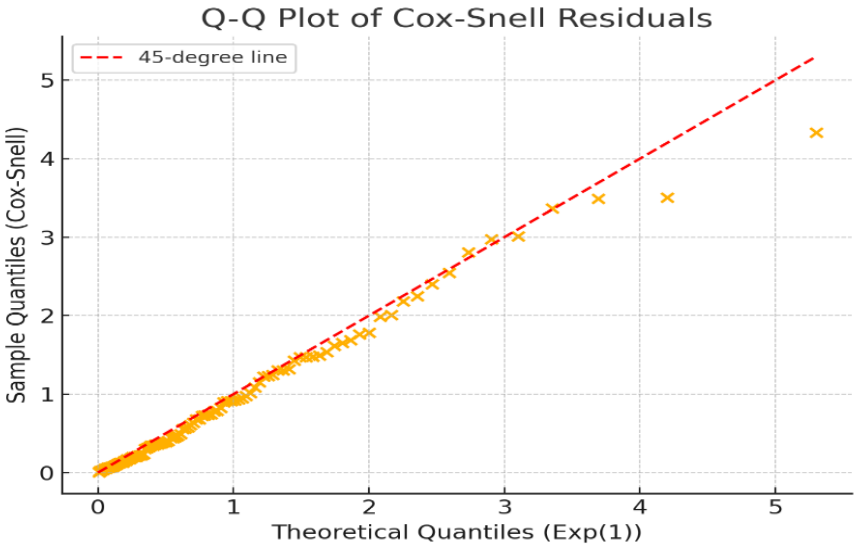

Figure 4:*Q-Q plot of Cox-Snell residuals vs. theoretical Exponential (1) quantiles*

Interpretation: The residuals closely follow the 45-degree line, especially in the central region of the distribution. Minor deviations at the extremes are expected due to sampling variability. Overall, the Q-Q plot supports the suitability of the APTEE model.
The graphical diagnostics based on both the Cox-Snell residual plot and the Q-Q plot demonstrate that the APTEE model captures the underlying structure of the data effectively. These results, derived from a controlled simulation, further validate the use of the APTEE distribution in modeling lifetime data under competing risks.

X.**Conclusion**

In this study, we investigated the Alpha-Power Transformed Extended Exponential (APTEE) distribution as a flexible and robust model for analyzing lifetime data in the presence of competing risks. The APTEE distribution generalizes several classical models and accommodates a variety of hazard rate shapes, making it suitable for a broad range of applications in reliability and biomedical sciences.
We applied multiple estimation techniques-including maximum likelihood, Kolmogorov-Smirnov, Anderson-Darling, and Cramer-von Mises methods-to obtain parameter estimates for the APTEE model. To assess the model's fit and discrimination power, we developed a modified chi-square statistic designed to recover information lost in grouped data and improve goodness-of-fit testing performance.
To further validate the model, we conducted extensive model diagnostics using Cox-Snell residual analysis, Q-Q plots, and hazard rate comparisons. These diagnostic tools confirmed the adequacy of the APTEE model, with residuals closely following the expected Exponential (1) behavior and hazard functions matching the empirical patterns observed in the data.
Simulation studies confirmed the accuracy and consistency of the estimators and the effectiveness of the modified chi-square test. Application to a real-world medical dataset demonstrated that the APTEE distribution provides a better fit than traditional alternatives such as Weibull, Generalized Exponential (GE), and Exponentiated Nadarajah-Haghighi (ENH) models.

In summary, the APTEE distribution proves to be a powerful tool for modeling complex lifetime data, particularly under competing risks settings. Future work may extend this model by incorporating covariates through regression structures, handling censored or truncated data, or exploring Bayesian inference approaches.

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