

Fractional Order Three-Dimensional Generalized Boundary Value Problem for Rectangular Plate Moving with Heat Source

Changdev B. Kothule¹, Tarachand L. Holambe², Satish G. Khavale^{1*}, Bhausaheb R. Sontakke³

^{1,3} Department of Mathematics, Pratishtan Mahavidyalaya Paithan, Chhatrapati Sambhajnagar (M.S.) India

² Department of Mathematics, Kai. Shankarrao Gutte ACS College, Dharmapuri, Beed, (M.S.) India

¹ Department of Mathematics, Statistics & Computer Science, Chetna's Hazarimal Somani College of Commerce & Economics, Smt. Kusumtai Chaudhari College of Arts, Bandra(E), Mumbai, (M.S.) India

kothulepatil1995@gmail.com, tarachandholambe@gmail.com, khavalesatish8@gmail.com, satish.khavale@chetanacollege.in

Article History:

Received: 19-09-2024

Revised: 24-10-2024

Accepted: 11-11-2025

Abstract: This paper investigates a three-dimensional generalized boundary value problem (BVP) for a rectangular plate subjected to a moving heat source within the framework of fractional-order thermoelasticity. The study employs a combination of Laplace and double Fourier transform techniques to obtain analytical solutions in the transform domain. These are subsequently inverted using numerical methods, including Fourier expansion and Riemann-sum approximation, to derive physical field quantities such as temperature increment, stress, strain, and displacement. The model accounts for the memory and non-local effects characteristic of fractional calculus, providing a more accurate description of thermal and mechanical responses. Numerical results are presented for various fractional orders and heat source velocities, demonstrating the significant influence of these parameters on the physical behaviour of the plate. The findings have direct implications for materials and processes involving transient thermal loading, such as in aerospace, electronics cooling, and advanced manufacturing systems.

Introduction: In recent years, fractional-order thermoelasticity has emerged as a powerful framework for modelling materials exhibiting memory and non-local effects, where traditional integer-order theories fail to capture the true dynamic response. The present study extends the generalized thermoelastic theory to a three-dimensional rectangular plate subjected to a moving heat source, using fractional calculus. Earlier works by Ezzat, Youssef, and Gaikwad established the groundwork for fractional thermoelastic behaviour in simpler geometries. This research aims to advance that understanding by solving a fractional-order boundary value problem (BVP) that accounts for transient thermal and mechanical interactions in a continuously moving thermal environment. The results provide valuable insights for engineering systems involving rapid thermal loading, such as in aerospace components, electronic cooling, and high-precision manufacturing.

Objectives:

- To formulate and analyse a three-dimensional generalized fractional-order boundary value problem for a rectangular plate under a moving heat source.
- To apply Laplace and double Fourier transform techniques to derive analytical expressions for temperature, stress, strain, and displacement fields.
- To evaluate the influence of fractional order (α), heat source velocity (v), and time (t) on thermoelastic field quantities.
- To validate the significance of fractional-order parameters in accurately describing memory-dependent and non-local thermoelastic effects.

Methods: The model is developed under the assumptions of homogeneous and isotropic material behaviour using the generalized thermoelasticity theory with one relaxation time. The governing equations of motion, heat conduction, and constitutive relations are

expressed in fractional differential form. The analytical solution is obtained through the Laplace and double Fourier transforms for dimensional reduction and solution in the transform domain. Also the numerical inversion of the transforms using Fourier expansion and Riemann-sum approximation techniques to obtain results in the physical domain. Copper is used as the reference material, and its standard thermophysical constants are employed for numerical simulation. Field variables such as temperature increment, stress, strain, and displacement are evaluated for varying heat source speeds and fractional orders.

Results: The numerical analysis demonstrates that the fractional-order parameter, heat source velocity, and time significantly influence the thermoelastic behaviour of the rectangular plate. The temperature increment is found to depend strongly on the speed of the moving heat source. As the heat source velocity increases, the temperature initially rises rapidly and then decreases once the source moves beyond a particular region, indicating the transient nature of the thermal field. For lower fractional orders, the temperature distribution exhibits sharper gradients and slower diffusion, which highlights the nonlocal and memory-dependent characteristics of fractional thermoelasticity. The stress distribution shows a dual-phase response: it decreases initially with an increase in heat source velocity due to reduced thermal gradients, but later rises because of rapid cooling and thermal mismatch effects. Similarly, the displacement component along the x-axis decreases with increasing source velocity, showing that higher velocities allow less time for thermal expansion, resulting in smaller mechanical deformation. Spatially, the effects are more prominent near the centre ($y = z = 0$) due to direct exposure to the heat source, while off-centre regions ($y = z = 0.5$) experience attenuated responses. These results collectively confirm that fractional-order modelling provides deeper insight into the coupled thermal and mechanical behaviour of materials under moving heat loads.

Conclusions: The present investigation establishes that the fractional-order generalized thermoelastic model effectively captures the nonlocal and memory-dependent behavior of materials subjected to a moving heat source. The study concludes that the fractional-order parameter (α), heat source velocity, and time play vital roles in determining temperature, displacement, and stress distributions in the rectangular plate. As the fractional order increases toward unity, the system behavior gradually approaches that predicted by classical thermoelastic theory, while smaller values of α produce stronger thermal gradients and higher stress magnitudes due to enhanced memory effects. The heat source velocity influences both the magnitude and distribution of thermal and mechanical responses—higher speeds result in lower displacements and reduced peak temperatures. These findings demonstrate that the fractional-order approach provides a more accurate and generalized framework than traditional models, particularly for materials and processes that involve transient heat transfer and time-dependent deformation. The proposed model can therefore be effectively applied to engineering systems in aerospace, electronics cooling, and precision manufacturing, where rapid and localized thermal loading plays a significant role in material performance.

Keywords: Fractional-order, Thermoelasticity, Boundary Value Problem, Rectangular Plate, Moving Heat Source, Integral Transforms.

Nomenclature

λ, μ	Lame's parameters
ρ	Density
C_E	Specific heat of the material with constant strain
t	Time
T	Absolute temperature
T_0	Reference temperature
θ	$= (T - T_0)$ Temperature increment $ T - T_0 / T_0 \ll 1$.
ωT	Linear thermal expansion coefficient
γ	$= \omega T (3\lambda + 2\mu)$
σ_{ij}	Stress tensor
e_{ij}	Strain tensor
u_i	Displacement components
K	Thermal conductivity
τ_0	Relaxation time
c_0	$= \sqrt{\frac{\lambda + 2\mu}{\rho}}$
η	$= \frac{\rho C_E}{K}$
ε	$= \frac{\gamma^2 T_0}{\rho C_E (\lambda + 2\mu)}$
β	$= \left(\frac{\lambda + 2\mu}{\mu}\right)^{1/2}$

1. Introduction

De Chant [1] introduced a numerical approach to model impulsive displacement in quasi-linear viscoelastic materials, a topic with broad implications in dynamic systems where viscoelastic effects are significant. Duan et al. [3] expanded on the fundamental principles of fluid-structure interaction under dynamic loading by examining the nonlinear stability of rarefaction waves in compressible fluids. These approaches are essential for understanding how complex material responses influence the design and performance of structures subjected to thermal and mechanical loads.

Ezzat and Youssef focused on generalized thermoelastic models, extending classical theories to include more complex material behaviours. Youssef [4] presented a two-temperature generalized thermoelastic medium subjected to a moving heat source, emphasizing the roles of heat conduction and mechanical stress in the material. Ezzat and Youssef's work [5] on three-dimensional thermal shock problems in generalized thermoelastic half-spaces provided deeper insights into the impact of rapid thermal loading on material responses. These studies are critical for applications in heat treatment processes, aerospace engineering, and industries involving rapid temperature variations.

Gaikwad and Ghadle [28, 29] investigated the thermoelastic deformation of thin circular disks and plates with internal heat generation and non-uniform heat supply. Their studies are particularly relevant for structural health monitoring and the design of thermally stressed components in engineering applications. The practical applications of these theoretical studies are vast. For instance, Parnell et al.

[8] addressed transient thermal mixed boundary value problems in half-spaces, applying advanced computational methods to solve these systems. Numerical solutions are essential for real-world problems where exact analytical solutions are unavailable. Tahouneh and Naei [9] examined the effect of multi-directional nanocomposite materials on the vibrational response of thick shell panels, highlighting the interaction between thermal and mechanical loading in structures with complex material properties. The work by Parnell et al. [8] on transient thermal problems in half-spaces and Youssef and Al-Lehaibi [12] on thermoelastic half-spaces subjected to moving heat sources are directly applicable to scenarios in aerospace engineering, electronics cooling, and geophysics.

Gaikwad et.al. [13-26] focused on time-fractional heat conduction and thermoelastic stress analysis in various geometries, such as thin hollow circular disks and rectangular plates. Their studies demonstrate the application of fractional calculus to address complex boundary conditions and material responses. Fractional-order models provide a more accurate description of memory effects and non-local behaviour, which are not captured by traditional integer-order models. Khavale and Gaikwad [14, 18] explored transient thermoelastic problems, using fractional derivatives to analyse temperature distributions and thermal deflections in thin circular disks. These models are particularly useful in cases where materials exhibit anomalous diffusion or time-dependent behaviour.

Youssef and Al-Lehaibi [12] studied generalized thermoelastic diffusion, where thermal and mass diffusion processes are coupled, applying this model to a thermoelastic half-space subjected to thermal pulses. This work is important for analysing materials undergoing both heat transfer and mass transport, such as semiconductors and porous materials. Similarly, Khavale and Gaikwad [21, 24] investigated magneto-thermoviscoelasticity, considering the interaction between thermal, mechanical, and magnetic fields in materials exhibiting fractional-order behaviour. These studies are vital for understanding the behaviour of materials in applications like electromagnetic heating and advanced materials science. The use of fractional-order models, as demonstrated in the studies by Khavale and Gaikwad [14, 25], can enhance predictions of material performance in systems where heat conduction does not follow classical diffusion laws.

2. The Basic Equations

The system a homogeneous and isotropic Boundary Value Problem on the generalized thermoelasticity with one relaxation time and without any external body forces in undefined coordinates $\{i, j, k = 1, 2, 3\}$ takes the (Ezzat, Youssef [5]; Youssef, Al-Lehaibi [12]):

The equations of motion are

$$\mu u_{i,jj} + (\mu + \lambda)u_{i,jj} - (3\lambda - 2\mu)\alpha\theta_i = \rho\ddot{u}_i \quad (1)$$

The heat equation is,

$$K\theta_{ii} = \left(\frac{\partial^\alpha}{\partial t^\alpha} + \tau_0 \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} \right) [\rho C_e \theta + \gamma T_0 e] - \left(1 + \tau_0 \frac{\partial^\alpha}{\partial t^\alpha} \right) Q \quad (2)$$

The constitutive relations are

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{kk} \delta_{ij} - \gamma \theta \delta_{ij} \quad (3)$$

The displacement relation with the strain takes the form

$$e_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}). \quad (4)$$

3. Problem Formulation

Assume an isotropic, homogeneous boundary value problem in three dimensions occupies the space $\psi = \{x, y, z : 0 \leq x < \infty, -\infty < y < \infty, -\infty < z < \infty\}$ where the body is quiescent initially and is loaded thermally by a moving heat source with constant speed v , which starts from the bounding plane of the surface $x = 0$ and moves along the x-axis when these surfaces are traction-free as in Fig. 1

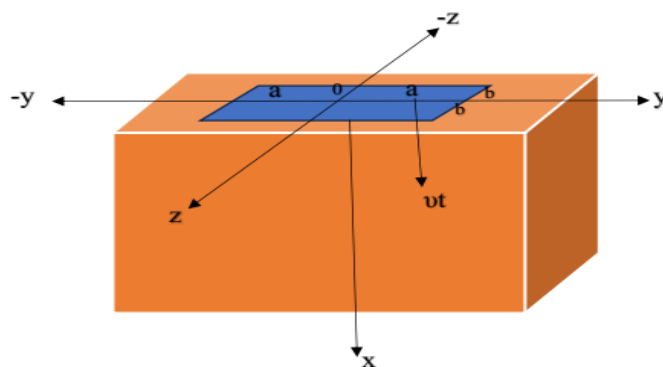


Fig.1. Graphical representation of problem

The equations of motion are

$$(\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + (\lambda + \mu) \left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z} \right) - \gamma \frac{\partial \theta}{\partial x} = \rho \ddot{u} \quad (5)$$

$$(\lambda + 2\mu) \frac{\partial^2 v}{\partial y^2} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} \right) + (\lambda + \mu) \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 w}{\partial z \partial y} \right) - \gamma \frac{\partial \theta}{\partial y} = \rho \ddot{v} \quad (6)$$

$$(\lambda + 2\mu) \frac{\partial^2 w}{\partial z^2} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + (\lambda + \mu) \left(\frac{\partial^2 u}{\partial z \partial x} + \frac{\partial^2 v}{\partial z \partial y} \right) - \gamma \frac{\partial \theta}{\partial z} = \rho \ddot{w} \quad (7)$$

The heat equation is

$$K \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \right) = \rho C_E \left(\frac{\partial^\alpha \theta}{\partial t^\alpha} + \tau_0 \frac{\partial^{\alpha+1} \theta}{\partial t^{\alpha+1}} \right) \theta + \gamma T_0 \left(\frac{\partial^\alpha}{\partial t^\alpha} + \tau_0 \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} \right) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) - \left(1 + \tau_0 \frac{\partial^\alpha}{\partial t^\alpha} \right) Q \quad (8)$$

The stress-strain relations as:

$$\sigma_{xx} = 2\mu e_{xx} + \lambda e - \gamma \theta, \quad (9)$$

$$\sigma_{yy} = 2\mu e_{yy} + \lambda e - \gamma \theta, \quad (10)$$

$$\sigma_{zz} = 2\mu e_{zz} + \lambda e - \gamma\theta, \quad (11)$$

$$\sigma_{xy} = 2\mu e_{xy} \quad (12)$$

$$\sigma_{xz} = 2\mu e_{xz} \quad (13)$$

$$\sigma_{yz} = 2\mu e_{yz} \quad (14)$$

The strain components are:

$$\sigma_{xx} = \frac{\partial u}{\partial x}, \quad \sigma_{yy} = \frac{\partial v}{\partial y}, \quad \sigma_{zz} = \frac{\partial w}{\partial z} \quad (15)$$

$$\sigma_{xy} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad \sigma_{xz} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \quad \sigma_{yz} = \frac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \quad (16)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = e. \quad (17)$$

Equations (5)-(7) can be rewritten by using Eq. (17) as:

$$\mu \nabla^2 \frac{\partial u}{\partial x} + (\lambda + \mu) \frac{\partial^2 e}{\partial x^2} - \gamma \frac{\partial^2 \theta}{\partial x^2} = \rho \frac{\partial \ddot{u}}{\partial x} \quad (18)$$

$$\mu \nabla^2 \frac{\partial v}{\partial y} + (\lambda + \mu) \frac{\partial^2 e}{\partial y^2} - \gamma \frac{\partial^2 \theta}{\partial y^2} = \rho \frac{\partial \ddot{v}}{\partial y} \quad (19)$$

$$\mu \nabla^2 \frac{\partial w}{\partial z} + (\lambda + \mu) \frac{\partial^2 e}{\partial z^2} - \gamma \frac{\partial^2 \theta}{\partial z^2} = \rho \frac{\partial \ddot{w}}{\partial z} \quad (20)$$

where,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The heat conduction equation takes the form

$$K \nabla^2 \theta = \rho C_E \left(\frac{\partial^\alpha}{\partial t^\alpha} + \tau_0 \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} \right) \theta + \gamma T_0 \left(\frac{\partial^\alpha}{\partial t^\alpha} + \tau_0 \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} \right) e - \left(1 + \tau_0 \frac{\partial^\alpha}{\partial t^\alpha} \right) Q \quad (21)$$

Now, the following dimensionless variables will be applied (Ezzat, Youssef [5-6-10]):

$$(x^*, y^*, z^*) = c\eta(x, y, z), \quad (t^*, \tau_0^*) = c^2\eta(t, \tau_0), \quad \theta^* = \frac{\gamma\theta}{(\gamma + 2\mu)}$$

$$Q^* = \frac{Q}{Kc^2\eta^2T_0}, \quad \sigma_{ij}^* = \frac{\sigma_{ij}}{\gamma + 2\mu}, \quad c = \sqrt{\frac{\gamma + 2\mu}{\rho}}, \quad \eta = \frac{\rho C_E}{K}.$$

Thus, we obtain

$$\beta \nabla^2 \frac{\partial u}{\partial x} + (1 - \beta) \frac{\partial^2 e}{\partial x^2} - \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \ddot{u}}{\partial x} \quad (22)$$

$$\beta \nabla^2 \frac{\partial v}{\partial y} + (1 - \beta) \frac{\partial^2 e}{\partial y^2} - \frac{\partial^2 \theta}{\partial y^2} = \frac{\partial \ddot{v}}{\partial y} \quad (23)$$

$$\beta \nabla^2 \frac{\partial w}{\partial z} + (1 - \beta) \frac{\partial^2 e}{\partial z^2} - \frac{\partial^2 \theta}{\partial z^2} = \frac{\partial \ddot{w}}{\partial z} \quad (24)$$

$$\nabla^2 \theta = \left(\frac{\partial^\alpha}{\partial t^\alpha} + \tau_0 \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} \right) \theta + \varepsilon_1 \left(\frac{\partial^\alpha}{\partial t^\alpha} + \tau_0 \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} \right) e - \varepsilon_2 \left(1 + \tau_0 \frac{\partial^\alpha}{\partial t^\alpha} \right) Q \quad (25)$$

$$\sigma_{xx} = 2\beta e_{xx} + (1 - 2\beta)e - \theta, \quad (26)$$

$$\sigma_{yy} = 2\beta e_{yy} + (1 - 2\beta)e - \theta, \quad (27)$$

$$\sigma_{zz} = 2\beta e_{zz} + (1 - 2\beta)e - \theta, \quad (28)$$

$$\sigma_{xy} = 2\beta e_{xy} \quad (29)$$

$$\sigma_{xz} = 2\beta e_{xz} \quad (30)$$

$$\sigma_{yz} = 2\beta e_{yz} \quad (31)$$

where,

$$\beta = \frac{\mu}{\lambda + 2\mu}, \quad \varepsilon_1 = \frac{\gamma^2 T_0}{\rho C_E (\lambda + 2\mu)}, \quad \varepsilon_2 = \frac{\gamma T_0}{(\lambda + 2\mu)}.$$

By taking the sum of the eq. (22)-(24) and substituting eq. (17), we get

$$\nabla^2 e - \nabla^2 \theta = \ddot{e} \quad (32)$$

Assume that the function σ is the invariant stress, as follows:

$$\sigma = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}. \quad (33)$$

By using eq. (26)-(28), we obtain

$$\sigma = \omega e - \theta \quad (34)$$

where, $\omega = \frac{3-4\beta}{3}$.

4. Solution of The Problem

Using Laplace transform, which is defined for any function $G(t)$ as follows:

$$\bar{G}(s) = \int_0^\infty G(t) e^{-st} dt. \quad (35)$$

Applying the transform to (35), we obtain:

$$\beta \nabla^2 \frac{\partial \bar{u}}{\partial x} + (1 - \beta) \frac{\partial^2 \bar{e}}{\partial x^2} - \frac{\partial^2 \bar{\theta}}{\partial x^2} = s^2 \frac{\partial \bar{u}}{\partial x} \quad (36)$$

$$\beta \nabla^2 \frac{\partial \bar{v}}{\partial y} + (1 - \beta) \frac{\partial^2 \bar{e}}{\partial y^2} - \frac{\partial^2 \bar{\theta}}{\partial y^2} = s^2 \frac{\partial \bar{v}}{\partial y} \quad (37)$$

$$\beta \nabla^2 \frac{\partial \bar{w}}{\partial z} + (1 - \beta) \frac{\partial^2 \bar{e}}{\partial z^2} - \frac{\partial^2 \bar{\theta}}{\partial z^2} = s^2 \frac{\partial \bar{w}}{\partial z} \quad (38)$$

$$\nabla^2 \bar{e} - \nabla^2 \bar{\theta} = s^2 \bar{e} \quad (39)$$

$$\nabla^2 \bar{\theta} = (s^\alpha + \tau_0 s^{\alpha+1}) \bar{\theta} + \varepsilon_1 (s^\alpha + \tau_0 s^{\alpha+1}) \bar{e} - \varepsilon_2 (1 + \tau_0 s^\alpha) \bar{Q} \quad (40)$$

$$\bar{\sigma}_{xx} = 2\beta \bar{e}_{xx} + (1 - 2\beta) \bar{e} - \bar{\theta}, \quad (41)$$

$$\bar{\sigma}_{yy} = 2\beta \bar{e}_{yy} + (1 - 2\beta) \bar{e} - \bar{\theta}, \quad (42)$$

$$\bar{\sigma}_{zz} = 2\beta \bar{e}_{zz} + (1 - 2\beta) \bar{e} - \bar{\theta}, \quad (43)$$

$$\bar{\sigma}_{xy} = 2\beta \bar{e}_{xy} \quad (44)$$

$$\bar{\sigma}_{xz} = 2\beta \bar{e}_{xz} \quad (45)$$

and

$$\bar{\sigma}_{yz} = 2\beta \bar{e}_{yz} \quad (46)$$

$$\bar{\sigma} = \omega \bar{\sigma} - \bar{\theta} \quad (47)$$

Using the Laplace transform, we applied the following initial conditions:

$$u(r, t)|_{t=0} = v(r, t)|_{t=0} = w(r, t)|_{t=0} = \theta(r, t)|_{t=0} = 0$$

$$\left. \frac{\partial u(r, t)}{\partial t} \right|_{t=0} = \left. \frac{\partial v(r, t)}{\partial t} \right|_{t=0} = \left. \frac{\partial w(r, t)}{\partial t} \right|_{t=0} = \left. \frac{\partial \theta(r, t)}{\partial t} \right|_{t=0} = 0 \quad r = x, y, z. \quad (48)$$

We assumed the medium is subjected to a rectangular heat source moving with constant speed and constant strength, releasing its energy continuously on a band of constant dimensions $2a \times 2b$ centered on the y-axis and z-axis, respectively, while moving with constant speed v along the x-axis and being zero elsewhere as in fig.1. Thus, the rectangular moving heat source is assumed to be of the following dimensionless form [4]:

$$Q = Q_0 \delta(x - vt) H(a - |y|) H(b - |z|), \quad (49)$$

Where a and b are constants, Q_0 is the heat source strength (constant), δ is the Dirac delta function, and $H(t)$ is the Heaviside function.

Applying the Laplace transform, we get

$$Q = \frac{Q_0}{v} H(a - |y|) H(b - |z|) e^{-\left(\frac{s}{v}\right)x}. \quad (50)$$

Eliminating \bar{e} in eq. (39), (40) and (47), we get

$$\nabla^2 \bar{\sigma} = \alpha_1 \bar{\theta} + \alpha_2 \bar{\sigma} - \gamma_1 H(a - |y|) H(b - |z|) e^{-\left(\frac{s}{v}\right)x}. \quad (51)$$

and

$$\nabla^2 \bar{\theta} = \alpha_3 \bar{\theta} + \alpha_4 \bar{\sigma} - \gamma_2 H(a - |y|) H(b - |z|) e^{-\left(\frac{s}{v}\right)x}. \quad (52)$$

where,

$$\alpha_1 = \frac{s^2 \omega - (s^\alpha + \tau_0 s^{\alpha+1})(1 - \omega)(\omega + \varepsilon_1)}{\omega}, \quad \alpha_2 = \frac{s^2 \omega - \varepsilon_1(1 - \omega)}{\omega},$$

$$\alpha_3 = \frac{(s^\alpha + \tau_0 s^{\alpha+1})(\alpha + \varepsilon_1)}{\omega}, \quad \alpha_4 = \frac{\varepsilon_1(s^\alpha + \tau_0 s^{\alpha+1})}{\omega},$$

$$\gamma_1 = \varepsilon_2(1 + \tau_0 s^\alpha)(\omega - 1) \frac{Q_0}{v}, \quad \gamma_2 = \varepsilon_2(1 + \tau_0 s^\alpha) \frac{Q_0}{v}.$$

The double Fourier transform for any function $f(x, y, z)$ is defined as follows:

$$F[\bar{f}(x, y, z, s)] = \tilde{\tilde{f}}(x, p, q, s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{f}(x, y, z, s) e^{-i(py+qz)} dy dz, \quad (53)$$

Where the inverse of the double Fourier transform takes the form

$$F^{-1}[\tilde{\tilde{f}}(x, p, q, s)] = \bar{f}(x, y, z, s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\tilde{f}}(x, p, q, s) e^{-i(py+qz)} dp dq, \quad (54)$$

Thus, we have

$$F[\nabla^2 \bar{f}(x, y, z, s)] = \left(\frac{\partial^2}{\partial x^2} - q^2 - p^2 \right) \tilde{\tilde{f}}(x, p, q, s) \quad (55)$$

Applying the transforms to (53) and (55), we get the following ordinary differential equations:

$$\frac{\partial^2 \tilde{\tilde{\sigma}}}{\partial x^2} = \alpha_1 \tilde{\tilde{\theta}} + \beta_1 \tilde{\tilde{\sigma}} - \gamma_3 e^{-\left(\frac{s}{v}\right)x} \quad (56)$$

And

$$\frac{\partial^2 \tilde{\tilde{\theta}}}{\partial x^2} = \beta_2 \tilde{\tilde{\theta}} + \alpha_4 \tilde{\tilde{\sigma}} - \gamma_4 e^{-\left(\frac{s}{v}\right)x} \quad (57)$$

where

$$\beta_1 = p^2 + q^2 + \alpha_2, \quad \beta_2 = p^2 + q^2 + \alpha_3$$

$$\gamma_3 = \frac{2}{\pi} \frac{\gamma_1 \sin(qa) \sin(pa)}{qp} \quad \text{and} \quad \gamma_4 = \frac{2}{\pi} \frac{\gamma_2 \sin(qb) \sin(pb)}{qp}$$

Eliminating $\tilde{\tilde{\sigma}}$ from eq. (56) and (57), we get

$$\left[\frac{\partial^4}{\partial x^4} - (\beta_1 + \beta_2) \frac{\partial^2}{\partial x^2} + (\beta_1\beta_2 - \alpha_1\alpha_4) \right] \tilde{\theta} = -\beta_5 e^{-\left(\frac{s}{v}\right)x} \quad (58)$$

Similarly, eliminating $\tilde{\theta}$ from (56) and (57) we obtain

$$\left[\frac{\partial^4}{\partial x^4} - (\beta_1 + \beta_2) \frac{\partial^2}{\partial x^2} + (\beta_1\beta_2 - \alpha_1\alpha_4) \right] \tilde{\sigma} = -\beta_6 e^{-\left(\frac{s}{v}\right)x} \quad (59)$$

where

$$\beta_5 = \gamma_4 \left(\frac{s^2}{v^2} - \beta_1 \right) + \alpha_4 \gamma_3, \quad \beta_6 = \gamma_3 \left(\frac{s^2}{v^2} - \beta_2 \right) + \gamma_4 \alpha_1$$

The solution of eq. (58) takes the form

$$\tilde{\theta} = A_1 e^{-k_1 x} + A_2 e^{-k_2 x} - A_3 e^{-\left(\frac{s}{v}\right)x} \quad (60)$$

where $A_3 = \frac{\beta_5}{\left(\frac{s^2}{v^2} - k_1^2\right)\left(\frac{s^2}{v^2} - k_2^2\right)}$.

The solution of eq. (59) takes the form

$$\tilde{\sigma} = B_1 e^{-k_1 x} + B_2 e^{-k_2 x} - B_3 e^{-\left(\frac{s}{v}\right)x} \quad (61)$$

Where $B_3 = \frac{\beta_6}{\left(\frac{s^2}{v^2} - k_1^2\right)\left(\frac{s^2}{v^2} - k_2^2\right)}$

With A_1, A_2, B_1 and B_2 being some parameters, and $\pm k_1^2$ and $\pm k_2^2$ the roots of the characteristic equation:

$$k^4 - Lk^2 + M = 0 \quad (62)$$

Where $L = \beta_1 + \beta_2$ and $M = \beta_1\beta_2 - \alpha_1\alpha_4$, which satisfy the relations:

$$k_1^2 + k_2^2 = \beta_1 + \beta_2, \quad k_1^2 k_2^2 = \beta_1\beta_2 - \alpha_1\alpha_4. \quad (63)$$

From eq. (60), (61) and (56), we get

$$B_1 = \frac{(k_1^2 - \alpha_1)}{\beta_1} A_1 \quad B_2 = \frac{(k_2^2 - \alpha_1)}{\beta_1} A_2 \quad (64)$$

Hence, we have

$$\tilde{\sigma} = \frac{(k_1^2 - \alpha_1)A_1}{\beta_1} e^{-k_1 x} + \frac{(k_2^2 - \alpha_1)A_2}{\beta_1} e^{-k_2 x} + B_3 e^{-\left(\frac{s}{v}\right)x}. \quad (65)$$

To finalize the solution, we have to obtain the parameters A_1, A_2 by applying certain boundary conditions. We consider that the bounding plane of the surface is traction-free and it has no external thermal loading which gives, by using all the above transformations, the following conditions:

$$\tilde{\sigma}(0, q, p, s) = \tilde{\theta}(0, y, z, t) = 0 \quad (66)$$

Substituting (66) into eq. (60) and (61), we get the following system:

$$A_1 + A_2 = A_3 \quad (67)$$

$$(k_1^2 - \alpha_1)A_1 + (k_2^2 - \alpha_1)A_2 = \beta_1 B_3 \quad (68)$$

The solution of the above system gives

$$A_1 = \frac{A_3(k_2^2 - \alpha_1) - \beta_1 B_3}{k_2^2 - k_1^2}, \quad A_2 = -\frac{A_3(k_1^2 - \alpha_1) - \beta_1 B_3}{k_2^2 - k_1^2} \quad (69)$$

Hence, we have the solutions in Fourier and Laplace transform domain as follows:

$$\tilde{\theta}(x, p, q, s) = \frac{A_3(k_2^2 - \alpha_1) - \beta_1 B_3}{(k_2^2 - k_1^2)} e^{-k_1 x} - \frac{A_3(k_1^2 - \alpha_1) - \beta_1 B_3}{(k_2^2 - k_1^2)} e^{-k_2 x} - A_3 e^{-\left(\frac{s}{v}\right)x} \quad (70)$$

$$\begin{aligned} \tilde{\sigma}(x, p, q, s) &= \frac{(k_1^2 - \alpha_1)[A_3(k_2^2 - \alpha_1) - \beta_1 B_3]}{\beta_1(k_2^2 - k_1^2)} e^{-k_1 x} - \frac{(k_2^2 - \alpha_1)[A_3(k_1^2 - \alpha_1) - \beta_1 B_3]}{\beta_1(k_2^2 - k_1^2)} e^{-k_2 x} \\ &\quad - B_3 e^{-\left(\frac{s}{v}\right)x} \end{aligned} \quad (71)$$

And

$$\tilde{e}(x, p, q, s) = \frac{1}{\alpha} \left[\tilde{\sigma}(x, p, q, s) + \tilde{\theta}(x, p, q, s) \right] \quad (72)$$

5. Inversion of Fourier and Laplace Transforms

To get the final solution in its original variables, we should calculate the inverse of the double Fourier and Laplace transforms in eq. (70)-(72). These expressions may be formally written as functions of x , and all the parameters of the Fourier and Laplace transforms, namely p , q and s in the form $\tilde{f}(x, q, p, s)$ (Ezzat, Youssef [5, 6, 10]).

In the beginning we invert the double Fourier transforms using the formula in (54) which gives the expression $\tilde{f}(x, y, z, s)$ in the Laplace transform domain as follows (Ezzat, Youssef [5]):

$$\begin{aligned} \tilde{f}(x, y, z, s) &= F^{-1}[\tilde{f}(x, p, q, s)] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(x, q, p, s) e^{i(py+qz)} dp dq \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\cos(py + qz)\tilde{f}_e + i\sin(py + qz)\tilde{f}_o] dp dq \end{aligned} \quad (73)$$

Where \tilde{f}_o and \tilde{f}_e denote the odd and even parts of the function $\tilde{f}(x, q, p, s)$, respectively. To invert the Laplace, transform, the technique of the Riemann-sum approximation will be applied as (Tzou [31]):

$$g(t) = \frac{e^{kt}}{t} \left[\frac{1}{2} \bar{g}(k) + Re \sum_{n=1}^N (-1)^n \bar{g} \left(K + \frac{in\pi}{t} \right) \right] \quad (74)$$

Here Re means the real part and $i = \sqrt{-1}$. Almost all the computational experiments have shown that the value of k comes from the relation $Kt \approx 4.7$, which gives faster convergence [31].

6. Numerical Result and Discussion

Copper plate was taken for the numerical calculations and the constants of the material have been taken as follows [7]:

Physical constant		Value
Thermal conductivity (K)		386 N/K sec
Linear thermal expansion coefficient (ωT)		$1.78 \times 10^{-5} K^{-1}$
Specific heat of the material with constant strain (C_E)		$383.1 m^2/K$
Reference temperature (T_0)		293 K
Lame's parameters (μ)		$3.86 \times 10^{10} N/m^2$
Lame's parameters (λ)		$7.79 \times 10^{10} N/m^2$
Density (ρ)		$8954 kg/m^2$
Relaxation time (τ_0)		$0.33 \times 10^{-15} sec$
$\tau_0 = 0.002$	$\beta = 0.25$	$\alpha = 0.67$
$\varepsilon_1 = 0.0168$	$\varepsilon_2 = 0.010444$	$\eta = 8886.73 m/sec^2$

The study investigates the distribution of temperature increment, stress, strain and displacement (u_x) components over a wide range of distance x (from 0 to 1) for varying heat source speeds v (values 0.1, 0.3, 0.5, 0.7). specifically, the analysis forces on two distinct cases for the positions y and z where $y = z = 0.0$ and $y = z = 0.5$.

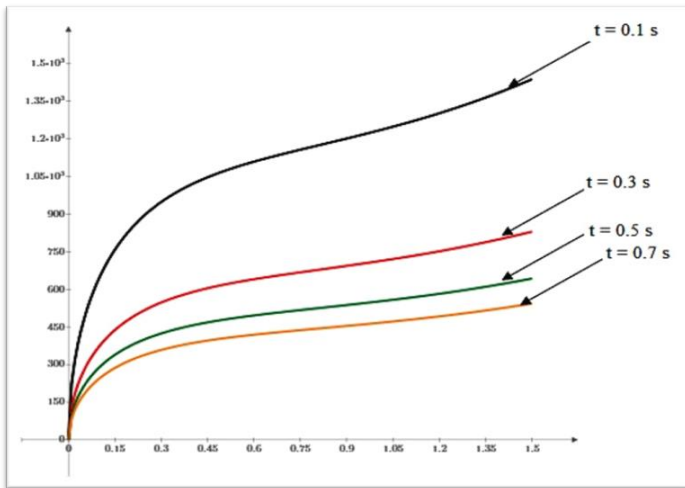
Temperature Increment: -The temperature increment shows a significant dependence on the heat speed v . As the speed increases, the temperature increment values are initially large but decrease after reaching a critical position $x = v t$ where t is time. For position $x < v t$, the heat source is still influencing the region, leading to rapid increases in temperature increment. For position $x \geq v t$ the heat source is no longer present which leads to rapid decrease in temperature increment.

Stress Distribution: -Thermal stress initially decreases with increasing heat source speed, due to a lower thermal gradient. After the peak stress point, further increases in speed lead to higher stress, possibly because of rapid cooling and thermal mismatch. This behaviour illustrates a dual-phase response: lower stress due to milder gradients early on, but higher stress post-heating due to quicker relaxation/cooling.

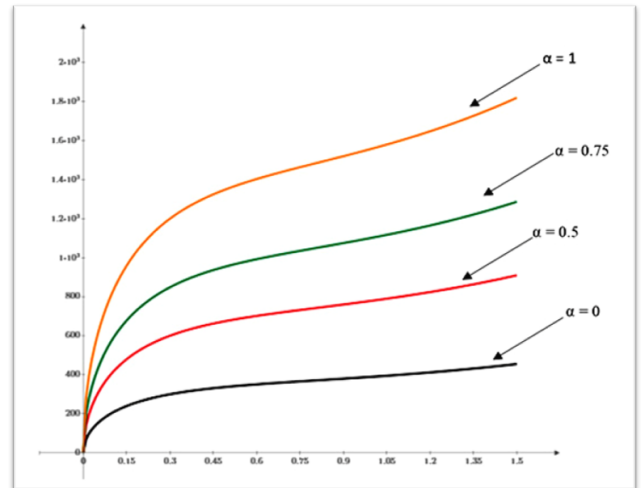
Displacement (u_x): - Displacement decreases with increasing speed of the heat source. Higher speeds give the material less time to undergo thermal expansion, resulting in smaller net displacement. This

shows that rapid heating minimizes mechanical deformation, which could be important in precision thermal processing applications.

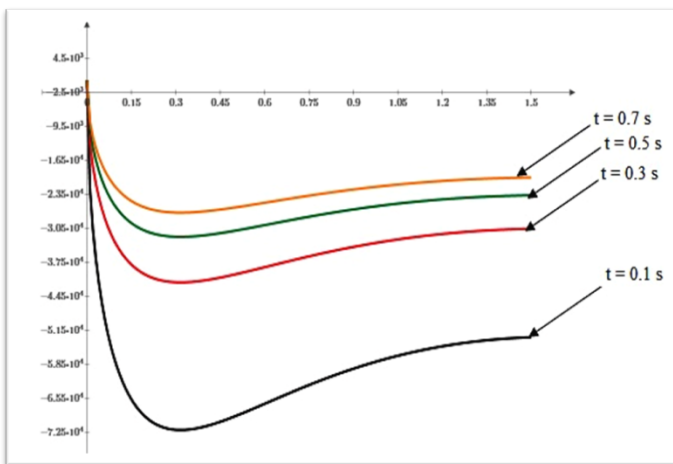
Influence of Position ($y = z = 0$ vs $y = z = 0.5$): - At the center ($y = z = 0$), effects are more pronounced in all aspect's temperature, stress, and displacement due to direct exposure to the heat source. At off-centre positions ($y = z = 0.5$), responses are attenuated, confirming the localized nature of thermal effects in the analysed geometry.



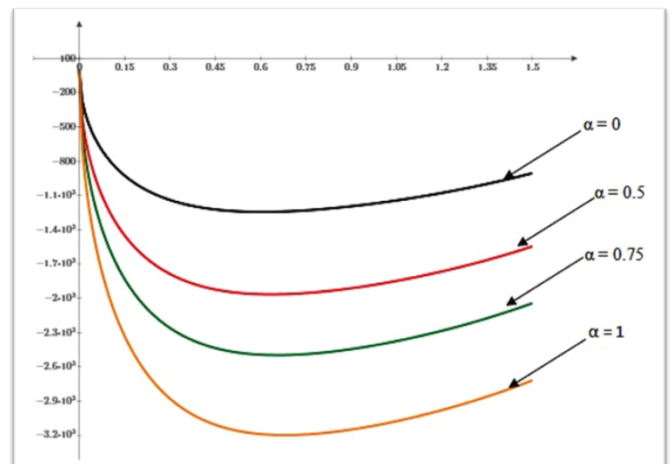
Temperature for different values of time



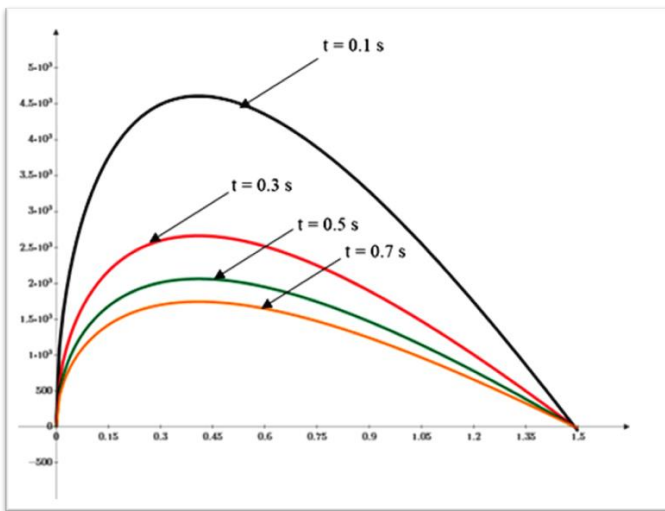
Temp for different values of alpha



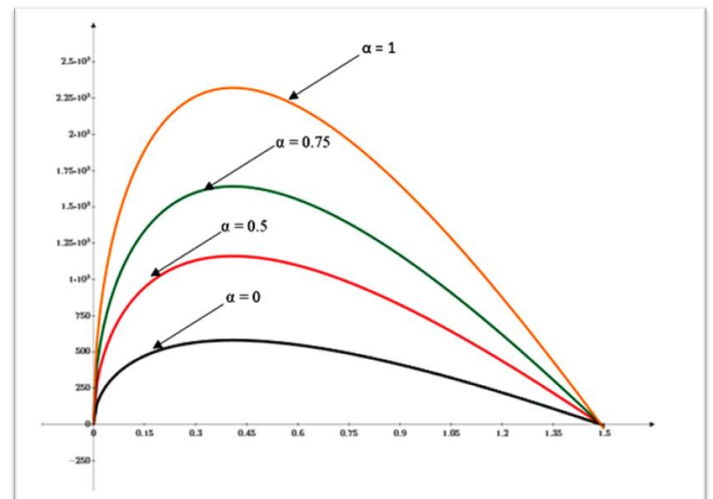
Displacement for Different values of time



Displacement of different values of alpha



Stresses for different values of time



Stresses for different values of alpha

7. Concluding Remarks

Effect of Time (t):

1. Temperature (θ):

- Temperature rises sharply at early times due to active heating by the moving source.
- Over time, the temperature profile shifts along the x-axis in the direction of heat source motion, indicating the transient nature of thermal propagation.
- After the source passes, cooling is evident as temperature falls significantly.

2. Displacement (u_x):

- Displacement increases initially with time due to thermal expansion.
- Later, it stabilizes or decreases as the material begins to cool.
- Peak displacement shifts with time, indicating delayed mechanical response to thermal input.

3. Stress (σ):

- Stress shows an initial increase, followed by a decrease over time, aligned with temperature and strain variations.
- Peak stress shifts downstream over time, mimicking the thermal gradient progression.
- Temporal evolution reflects thermoelastic coupling and relaxation behaviour.

Effect of Fractional Order Parameter (α):

1. Temperature (θ):

- As α increases (approaching classical behaviour), the temperature distribution becomes smoother and more diffused.
- Lower α values (more fractional behaviour) produce sharper temperature gradients and delayed diffusion, highlighting memory effects.

2. Displacement (u_x):

- Displacement decreases with increasing α .

- Fractional models (lower α) capture stronger nonlocal and history-dependent mechanical responses, leading to higher displacements.

3. Stress (σ):

- Stress magnitude reduces with increasing α .
- This suggests that fractional models (lower α) result in stronger stress responses due to enhanced memory and time-delayed effects.

3. General Observations:

- Heat source speed, time, and fractional order α all significantly influence the physical behaviour.
- Fractional order models provide more realistic and flexible descriptions of thermoelastic responses, especially in materials exhibiting non-Fourier heat conduction or memory effects.
- These insights are critical for advanced materials and structures used in aerospace, electronics, and thermal management systems.

Acknowledgment

The authors are grateful thanks to Chhatrapati Shahu Maharaj Research, Training and Human Development Institute (**SARTHI**) for awarding the Chhatrapati Shahu Maharaj National Research Fellowship - 2022 (**CSMNRF - 2022**).

References

- [1] **De Chant, L.:** Impulsive displacement of a quasi-linear viscoelastic material through accurate numerical inversion of the Laplace transform. *Comput. Math Appl.* 43(8-9), pp. 1161-1170, (2002).
- [2] **Musll, R.:** Equations in stresses for two-dimensional dynamic problems of thermoelasticity in spherical coordinates. *Mater. Sci.* 39(1), pp. 48-53, (2003).
- [3] **Duan, R., Liu, H., Zhao H.:** Nonlinear stability of rarefaction waves for the compressible Navier-Stokes equations with large initial perturbation. *Trans. Am. Math. Soc.* 361(1), pp. 453-493, (2009).
- [4] **Youssef, H.:** A two-temperature generalized thermoelastic medium subjected to a moving heat source and ramp-type heating. *A state-space approach. J. Mech Mater. Struct.* 4(9) pp. 1637-1649, (2010).
- [5] **Ezzat, M.A., Youssef, H.M.:** Three-dimensional thermal shock problem of generalized thermoelastic half-space. *Appl. Math. Model* 34(11), pp. 3608-3622, (2010).
- [6] **Ezzat M. A., Youssef, H. A.:** Two-temperature theory in three-dimensional problem for thermoelastic half space subjected to ramp type heating. *Mech. Mat Struct.* 21(4), pp. 293-304, (2014).
- [7] **Abbas, I. A., Youssef, H. M.:** Two-dimensional fractional order generalized thermoelastic porous material. *Lat Am. J. Solids struct.* 12(7), pp. 14-31, (2015).
- [8] **Parnell, W. J., Nguyen, V. H., Assier, R. Naili, S., Abrahams, I. D.:** Transient thermal mixed boundary value problems in the half-space. *SIAM J. Appl. Math.* 76(3), pp. 845-866, (2016).
- [9] **Tahounch, V., Naei, M. H.:** The effect of multi-directional nanocomposite materials on the vibrational response of thick shell panels with finite length and rested on two-parameter elastic foundations. *Int. J. Adv. Struct. Eng.* 8(1), pp. 11-28, (2016).
- [10] **Ezzat M. A., Youssef, H. A.:** Three-dimensional thermos-viscoelastic material. *Mech. Adv. Mat. Struct.* 23(1), pp. 108-116, (2016).
- [11] **Marin, M.:** An approach of a heat flux dependent theory for micropolar porous media. *Meccanica* 51(5), pp. 1127-1133, (2016).

- [12] **Youssef, H. M., Al-Lehaibi, E. A.:** Three-dimensional generalized thermoelastic diffusion and application for a thermoelastic half-space subjected to rectangular thermal pulse. *J. Therm. Stresses* 41(8), pp. 1008-1021, (2018).
- [13] **Gaikwad K. R. & S. G. Khavale.:** Time fractional heat conduction problem in a thin hollow circular disk and its thermal deflection, *Easy chair preprint No. 1672*, pp.1-11, (2019).
- [14] **Khavale S. G. & K. R. Gaikwad.:** Generalized theory of magneto-thermoviscoelastic spherical cavity problem under fractional order derivative: state space, approach, *Advances in Mathematics: Scientific Journal*, Vol. 9 (11), pp. 9769 -9780, (2020).
- [15] **Hamdy M. Youssef & Eman A.N. Al-Lehaibi.:** The boundary value problem of a three-dimensional generalized thermoelastic half-space subjected to moving rectangular heat source, *Boundary Value Problems*, pp. 1-15, (2019).
- [16] **Gaikwad K. R. & S. G. Khavale.:** Time fractional 2d thermoelastic problem of thin hollow circular disk associated thermal stresses, *Bulletin of Marathwada mathematical Society*, Vol. 21(1&2), June & Dec, pp. 37-47, (2020).
- [17] **Gaikwad K. R., Y. U. Naner, S. G. Khavale.:** Time fractional thermoelastic stress analysis of a thin rectangular plate, *NOVYI MIR Research Journal*, Vol. 6(1), pp. 42-56, (2021).
- [18] **Khavale S. G. & K. R. Gaikwad.:** Fractional order thermoelastic problem of thin hollow circular disk and its thermal stresses under axi-symmetric heat supply, *Design Engineering (Toronto)*, Vol. 2021(9), pp. 13851-13862, (2021).
- [19] **Khavale S. G. & K. R. Gaikwad.:** 2D problem for a sphere in the fractional order theory thermoelasticity to axisymmetric temperature distribution, *Advances in Mathematics: Scientific Journal*, Vol. 11(1), pp. 1–15, (2022).
- [20] **Gaikwad K. R. & S. G. Khavale.:** Fractional order transient thermoelastic stress analysis of a thin circular sector disk, *International Journal of Thermodynamics*, Vol. 25(1), pp. 1–8, (2022).
- [21] **Khavale S. G. & K. R. Gaikwad.:** Two-dimensional generalized magneto-thermoviscoelasticity problem for a spherical cavity with one relaxation time using fractional derivative, *International Journal of Thermodynamics*, Vol. 25(2), pp. 89–97, (2022).
- [22] **Khavale S. G. & K. R. Gaikwad.:** Analysis of non-integer order thermoelastic temperature distribution and thermal deflection of thin hollow circular disk under the axisymmetric heat supply, *J. Korean Soc. Ind. Appl. Math.*, Vol. 26(1), pp. 67–75, (2022).
- [23] **Gaikwad K. R., Y. U. Naner, S. G. Khavale.:** Transient thermoelastic bending analysis of a rectangular plate with a simply supported edge under heat source: green's function approach, *Int. J. Nonlinear Anal. Appl.*, Vol. 14(1), pp. 805–818, (2023).
- [24] **Khavale S. G. & K. R. Gaikwad.:** Fractional thermoelasticity: A Review, *Easy chair preprint No. 9531*, pp. 1 – 9, (2023).
- [25] **Khavale S. G. & K. R. Gaikwad.:** Fractional ordered thermoelastic stress analysis of a thin circular plate under axi-symmetric heat supply, *International Journal of Nonlinear Analysis and Applications*, Vol. 14(4), pp. 207–219, (2023).
- [26] **C. B. Kothule, T. L. Holambe, Khavale S. G., B. R Sontakke.:** Analysis of fractional order heat conduction problems in noncylindrical domains with different boundary conditions, *Econophysics, Sociophysics & Other Multidisciplinary Sciences Journal (ESMSJ)*, Vol. 12(2), pp. 23-26, (2023).
- [27] **N. B. Jadhav, K. R. Gaikwad & S. G. Khavale,** Fractional order thermoelastic problem for a thin circular plate with uniform internal heat generation, *JP Journal of Heat and Mass Transfer*, Vol. 35, pp. 107-124, (2023).

- [28] **Kishor R. Gaikwad, Kirtiwant P. Ghadle.:** Nonhomogeneous heat conduction problem and its thermal deflection due to internal heat generation in a thin hollow circular disk, *Journal of Thermal Stresses* 35(6), pp. 485-498, (2012).
- [29] **K. R. Gaikwad.:** Analysis of thermoelastic deformation of a thin hollow circular disk due to partially distributed heat supply. *Journal of Thermal Stresses* 36(2), pp.207-224, (2013).
- [30] **K. R. Gaikwad, Y. U. Naner.:** Analysis of transient thermoelastic temperature distribution of a thin circular plate and its thermal deflection under uniform heat generation. *Journal of Thermal Stresses* 44(1), pp. 75-85, (2020).
- [31] **Tzou, D.,** Macro to micro heat transfer. *Taylor & Francis, Washington* (1996).