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Classical and Bayesian Estimation for a New Long Tailed Distribution: Modi-Lomax Distribution

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Abstract:
This paper presents the new Modi-Lomax distribution (MOLD), a three-parameter distribution obtained by using the Modi family generator to include one extra shape parameter in the standard two-parameter Lomax distribution. We establish extensive statistical properties such as survival function, hazard rate function, moment generating function, quantile function with quartile representations, mean residual lifetime, order statistics, and stress-strength reliability. For parameter estimation, we use maximum likelihood estimation and Bayesian methods with Gamma priors under squared error loss functions, comparing their performances based on bias and mean square error criteria. The practical usefulness of MOLD is shown through the analysis of one real data set of bladder cancer patients and a comparative analysis with a number of alternative distributions. Results validate the superior flexibility and goodness-of-fit of the suggested distribution under different test criteria, emphasizing its flexibility to model data with diverse failure rate shapes and aging behaviours. This paper makes MOLD a great contribution to lifetime distributions with prominent practice in reliability engineering and survival analysis.

Keywords: Lomax Distribution, Modi Family, Maximum Likelihood Estimation, Bayesian Estimation, Stress-Strength Model

1. Introduction

Probability distributions serve as fundamental tools in statistical modelling across various scientific fields. With the growing complexity of observed phenomena, statistical research has increasingly focused on developing more adaptable distributions through mathematical transformations, weighting schemes, and mixture techniques. In recent years, the generalization of probability models has become particularly prominent, with various approaches available, such as Alpha Power Transformation (APT), exponentiation, mixture and weighted techniques, power transformations, among others.

The Lomax distribution, or Lomax-Pareto Type II distribution, has found its place as a useful statistical model in reliability studies and as well as distribution theory. The distribution gained popularity because it is easily flexible in representing and modelling heavy-tailed behaviours in various fields of mathematics [(Harris, 1968); (Bryson, 1974)].

Initial theoretical contributions by (Balkema & De Haan, 1974) laid the foundation of the Lomax distribution's application in reliability modelling and life testing. (Arnold, 1983) further developed a detailed framework that linked the Lomax distribution to the rest of the Pareto family. (Johnson et al., 1994) continued from such foundations, providing elaborate details about the properties and features of the distribution that cemented its place in statistical literature.

The statistical properties of Lomax distribution have been well explored. (Ahsanullah, 1991) made noteworthy contributions by studying record values that came from Lomax-distributed variables. (Balakrishnan & Ahsanullah, 1994) derived significant recurrence relations among moments of record values from the Lomax distribution. (Atkinson & Harrison, 1978) illustrated the Lomax distribution's suitability in modelling income and wealth data. (Tadikamalla, 1980) formulated significant relationships among the Lomax and Burr family of distributions. The failure rate characteristics of Lomax model were also comprehensively studied by (Chahkandi et al., 2009), who identified that it comes under the class of decreasing failure rate distributions.

The literature has seen significant development in extensions of the standard Lomax distribution. (Ashour & Eltehiwy, 2013) presented Transmuted Exponential Lomax (TEL) model. (Rady et al., 2016) used the power transformation on Lomax distribution (PoL). (Shabbir et al., 2018) contributed to Lomax family by creating Rayleigh Lomax distribution (RaL). (Joshi & Kumar, 2021) came up with the Poisson Inverted Lomax (PoIL) distribution. Another four-parameter distribution is Exponentiated odd Lomax, developed by (Dhungana & Kumar, 2022). There is another four-parameter distribution competitive to our study is Power XLindley (PXLin) by (Elgarhy et al., 2025) Some of the recent contributions to the Lomax family is Log-Lomax (LoL) by (Ishaq et al., 2025).

(Modi et al., 2020) introduced the Modi family of distributions, demonstrating its adaptability for modelling diverse phenomena in engineering, economics, and finance. Subsequent studies have expanded this family - (Kumawat et al., 2024) developed the Modi-Weibull distribution, examining its statistical properties, while (Ndayisaba et al., 2023) proposed the Modi exponentiated exponential distribution, analysing its mathematical features and practical applications. In a recent study, (Kumar et al., 2025) proposed a new Modi-Rayleigh distribution, demonstrating its superior performance relative to existing distributions generated through the identical Modi transformation approach. These methods enable statisticians to refine existing distributions to better represent complex data characteristics, such as heavier tails, greater skewness, or unique hazard rate patterns, which conventional distributions may fail to capture adequately.

This manuscript introduces the Modi-Lomax distribution (MOLD) using the Modi-family generator method. Our primary motivation stems from adding one new shape parameters to an existing long-tailed distribution, enhancing its ability to fit failure time data containing extremely small values. We compare our proposed model with well-established lifetime distributions to demonstrate its flexibility in fitting real-world datasets. The article employs both classical and Bayesian estimation procedures to estimate the parameters of our proposed model. For classical estimation, we utilize the maximum likelihood method, while our Bayesian approach incorporates gamma priors and square error loss function (SELF).

The manuscript is organized as follows: Section 2 introduces the Modi-Lomax (MOLD) distribution. Section 3 presents reliability analysis of MOLD. Section 4 covers the statistical properties, moment generating function, quantile function and order statistics, respectively. Section 6 covered the detailed parameter estimation using both maximum likelihood and Bayesian approaches with gamma priors and SELF. Section 7 presents simulation studies, Section 8 demonstrates applications to real-world datasets with comparative analysis, and Section 9 concludes with a summary of findings. In section 10, we discussed the future scope of the article.

2. Modi-Lomax Distribution

The Modi family generator (Modi et al., 2020) is widely employed across various fields due to its capacity to introduce two additional shape parameters into existing lifetime models, thereby enhancing their flexibility for modelling real-world datasets. If f(t) and F(t) represent the probability density function (pdf) and cumulative distribution function (cdf), respectively, of an existing lifetime distribution, then the pdf g(.) and cdf G(.) of the new Modi family distribution are given by:

$$g(x) = \frac{\alpha^{\beta}(1+\alpha^{\beta})f(x)}{\left(\alpha^{\beta}+F(x)\right)^2};\ x > 0, \alpha > 0, \beta > 0 \qquad (1)$$

$$G(x) = \frac{(1+\alpha^{\beta})F(x)}{\alpha^{\beta}+F(x)};\ x > 0, \alpha > 0, \beta > 0 \qquad (2)$$

In the Modi family generator, there are two shape parameters viz. α and β. We combine these two shape parameters and introduce single shape parameter $\xi = \alpha^{\beta}$. The motivation behind this substitution is that, we want to check the joint effect of these two-shape parameter after the fusion with Lomax distribution.

If t follows the Lomax distribution, so the pdf and cdf of t are as follows:

$$f(t, \lambda, \theta) = \frac{\lambda}{\theta}\left(1 + \frac{t}{\theta}\right)^{-(\lambda+1)};\ t > 0, \lambda > 0, \theta > 0 \qquad (3)$$

$$F(t, \lambda, \theta) = 1 - \left(1 + \frac{t}{\theta}\right)^{-\lambda};\ t > 0, \lambda > 0, \theta > 0 \qquad (4)$$

The MOLD is created from the Lomax distribution by putting Eq. (3) and Eq. (4), respectively, into Eq. (1) and Eq. (2). As a result, the MOLD's pdf is ascertained as

$$g(t) = \frac{\xi(1+\xi)\lambda\theta^{-\lambda}(\theta+t)^{-(\lambda+1)}}{\left[(1+\xi)\theta^{-\lambda}-(\theta+t)^{-\lambda}\right]^2};t > 0, \xi > 0, \lambda > 0, \theta > 0 \qquad (5)$$

The corresponding cdf of MOLD is given as

$$G(t) = \frac{(1+\xi)\left[\theta^{-\lambda}-(\theta+t)^{-\lambda}\right]}{\left[(1+\xi)\theta^{-\lambda}-(\theta+t)^{-\lambda}\right]};t > 0, \xi > 0, \lambda > 0, \theta > 0 \qquad (6)$$

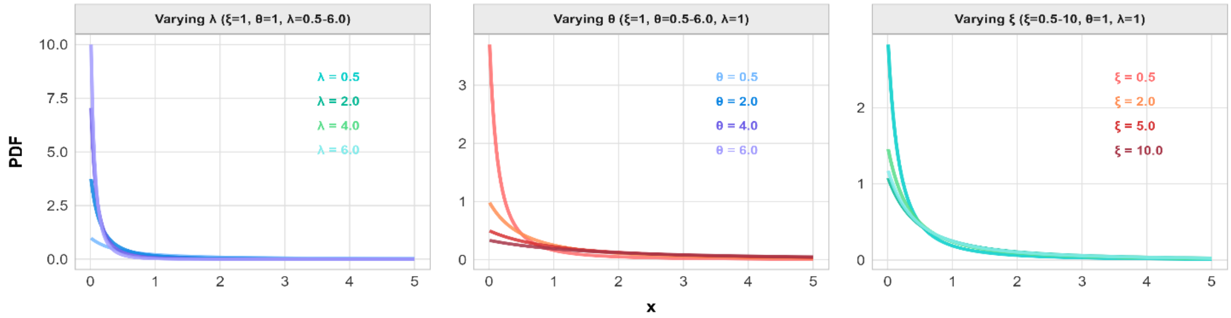

Figure 1. *PDF plots of MOLD for different parametric values of ξ, λ and θ.*

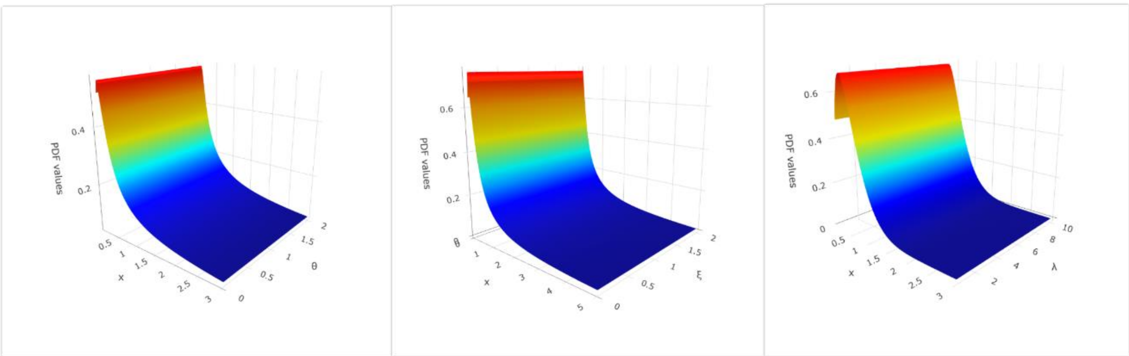

Figure 2. *3D PDF plots of MOLD for varying ξ, λ and θ values*.

3. Reliability Characteristics of MOLD

This section presents the derivation of essential reliability characteristics for the MOLD distribution, specifically addressing survival function, hazard rate, reverse hazard rate, cumulative failure rate, and the Mills ratio.

3.1. Survival Function

The survival function S(t) is one of important reliability measure in the reliability realm. It is defined as

$$S(t) = 1 - G(t)$$

where G(t) is the cumulative density function.

From Eq. (6), we get the survival function for MOLD as following

$$S(t) = \frac{\xi(\theta+t)^{-\lambda}}{\theta^{-\lambda}(1+\xi)-(\theta+t)^{-\lambda}} \tag{7}$$

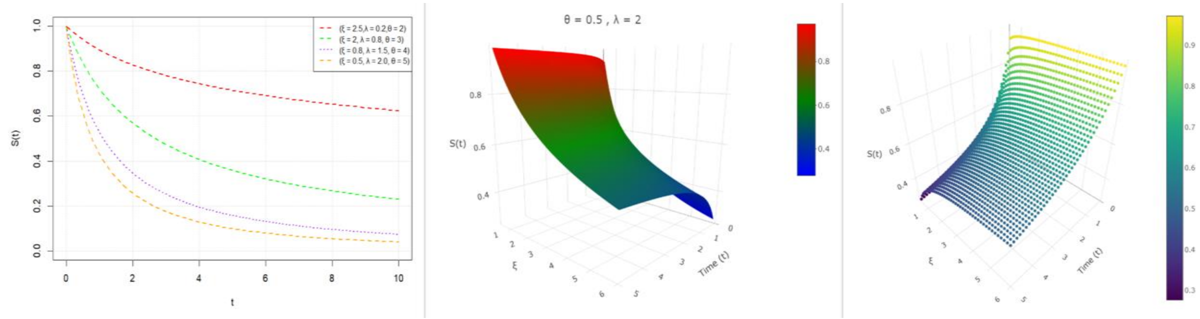

Figure 3. *Survival function plots for MOLD distribution*.

3.2. Hazard Rate and Inverse Hazard Rate Function

The hazard rate function $\mathrm{H(t)} = \frac{\mathrm{g(t)}}{\mathrm{S(t)}}$ and the reversed hazard rate function $\mathrm{H'(t)} = \frac{\mathrm{g(t)}}{\mathrm{G(t)}}$ are important quantities characterizing life time phenomena. For the MOLD distribution, they are given as

$$H(t) = \frac{\lambda(1+\xi)}{(\theta+t)(1+\xi)-\theta^{\lambda}(\theta+t)^{1-\lambda}} \qquad (8)$$

and

$$H'(t) = \frac{\xi\lambda\theta^{-\lambda}(\theta+t)^{-(\lambda+1)}}{[(1+\xi)\theta^{-\lambda}-(\theta+t)^{-\lambda}][\theta^{-\lambda}-(\theta+t)^{-\lambda}]} \qquad (9)$$

3.3. Cumulative Hazard Rate

The cumulative hazard rate, measures the total risk that an event will occur by some specified time. The function is actually the integration of the hazard rate over time intervals, which illustrates how probability of the event occurrence increases with time. It is defined as

$$\Delta(t) = -\log\big(S(t)\big)$$

Hence for the MOLD, it is given as

$$\Delta(t) = -\log\left[\frac{\xi(\theta+t)^{-\lambda}}{\theta^{-\lambda}(1+\xi)-(\theta+t)^{-\lambda}}\right]$$

$$\Delta(t) = -\log(\xi) + \lambda\log(\theta+t) + \log\left[\theta^{-\lambda}(1+\xi)-(\theta+t)^{-\lambda}\right] \qquad (10)$$

3.4. Mills Ratio

The Mills ratio (M.R) for the MOLD is obtained as

$$\mathrm{M.R} = \frac{G(t)}{S(t)}$$

$$\mathrm{M.R} = (1+\xi)\left[\left(1+\frac{t}{\theta}\right)^{\lambda} - 1\right] \qquad (11)$$

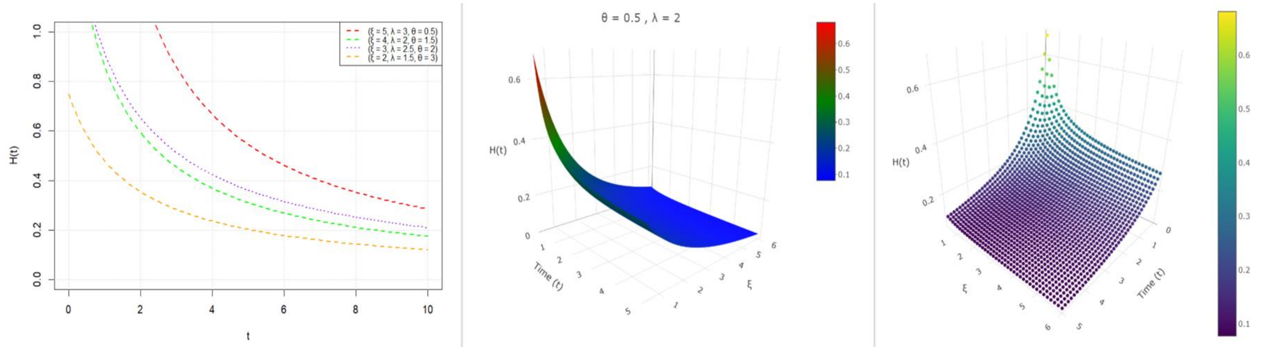

Figure 4. *Hazard function plots for MOLD distribution*

3.5. Cumulative Hazard Rate

The mean residual lifetime (MRL) represents the expected remaining lifespan of a subject given that it has survived to a particular time point. This measure provides valuable insight into the conditional life expectancy and is particularly useful for understanding survival patterns in different phases of the study period.

$$\text{MRL} = \mathrm{E}[(x-t)|x>t]$$

$$= \frac{\int_t^{\infty}(x-t)f(x)dx}{S(t)}$$

For the MOLD, MRL is derived as

$$\text{MRL} = \frac{(\theta+t)}{\lambda-1}\ ;\lambda>1$$

This shows that the MRL for the MOLD increases linearly with t, with the rate determined by the parameter λ. This is a characteristic of distributions with the lack of memory property similar to the exponential distribution, but with a different structure.

4. Statistical Properties of MOLD

4.1. Moment Generating Function

Theorem 1. *If X is the MOLD variable having pdf given in Eq. (5), then its moment generating function $M_x(s)$ is given by*

$$M_x(s) = \xi(1+\xi)\lambda\theta^{-\lambda}e^{-s\theta}\int_{\theta}^{\infty}\frac{e^{su}u^{-(\lambda+1)}}{[(1+\xi)\theta^{-\lambda}-u^{-\lambda}]^2}du \qquad (12)$$

Proof. The MGF of MOLD is obtained from

$$M_x(s) = E[e^{sx}] = \int_0^{\infty} e^{st}\,g(t)dt$$

If we define L[g(t)](s)as the Laplace transformation of g(t), then

$$M_x(s) = L[g(t)](-s)$$

where $L[g(t)](s) = \int_0^\infty e^{-st}\, g(t)dt$

Hence, we get

$$M_x(s) = \xi(1+\xi)\lambda\theta^{-\lambda}e^{-s\theta}\int_\theta^\infty \frac{e^{su}u^{-(\lambda+1)}}{[(1+\xi)\theta^{-\lambda}-u^{-\lambda}]^2}du$$

Theorem 2. The rth moment about origin of MOLD is defined as

$$\mu_r' = E(t^r) = \frac{\theta^{r+\lambda}}{\lambda}\sum_{k=0}^{r}\binom{r}{k}(-1)^k B\left(\frac{1-(r-k)}{\lambda}+1,1\right)(1+\xi)^{1-(r-k)/\lambda-2}$$

where, B(x,y) is the beta function.

Proof. The rth moment about origin for the MOLD is obtained from the following expression using Eq.(5)

$$\mu_r' = E(t^r) = \int_0^\infty t^r g(t)dt$$

Let $u = \theta + t$

$$E(t^r) = \int_\theta^\infty \frac{(u-\theta)^r\xi\lambda\theta^{-\lambda}(u)^{-(\lambda+1)}(1+\xi)}{[(1+\xi)\theta^{-\lambda}-u^{-\lambda}]^2}du$$

Now we use second substitution $v = \left(\frac{\theta}{u}\right)^\lambda$

$$E(t^r) = \frac{\theta^{r+\lambda}}{\lambda}\int_0^1 \frac{\left(v^{-1/\lambda}-1\right)^r v^{1/\lambda}}{(1+\xi-v)^2}dv$$

Now expanding $\left(v^{-1/\lambda}-1\right)^r = \sum_{k=0}^{r}\binom{r}{k}\left(v^{-1/\lambda}\right)^{r-k}(-1)^k$ using binomial expression, we get

$$E(t^r) = \frac{\theta^{r+\lambda}}{\lambda}\sum_{k=0}^{r}\binom{r}{k}(-1)^k B\left(\frac{1-(r-k)}{\lambda}+1,1\right)(1+\xi)^{1-(r-k)/\lambda-2}$$

4.2 Quantile Function

Let $t \sim \text{MOLD}(\xi,\lambda,\theta)$ and $\text{G}(\text{t}) = \text{p}$ be the cdf of t, so the quantile function is defined as

$$Q(p) = G^{-1}(p), \quad \text{for } 0 < p < 1$$

Hence the quantile function of MOLD is obtained as follows

$$Q(p) = \theta\left[\left(\frac{1+\xi-p}{(1+\xi)(1-p)}\right)^{1/\lambda}-1\right], \quad 0 < p < (1+\xi) \tag{13}$$

The first, second and third quartile can be obtained from Eq. (13) by putting $\text{p} = 1/4, 1/2$ and $3/4$, respectively.

The first quartile

$$Q_1 = \theta\left[\left(\frac{3+4\xi}{3(1+\xi)}\right)^{1/\lambda} - 1\right] \tag{14}$$

The second quartile

$$Q_2 = \theta\left[\left(\frac{1+2\xi}{(1+\xi)}\right)^{1/\lambda} - 1\right] \tag{15}$$

The third quartile

$$Q_3 = \theta\left[\left(\frac{1+4\xi}{(1+\xi)}\right)^{1/\lambda} - 1\right] \tag{16}$$

4.3 Order Statistics

Consider a finite random sample consisting of observations $t_1, t_2, \ldots, t_n$ drawn from the MOLD with pdf g(t) and cdf G(t). When these sample values are rearranged in non-decreasing sequence, the resulting ordered observations $t_1, t_2, \ldots, t_n$ are called the order statistics of the sample, where:

$$t_{(1)} \le t_{(2)} \le \cdots \le t_{(n)}$$

The order statistic $t_{(r)}$ corresponds to the r^{th} smallest observation among the sorted sample values. Specifically, $t_{(1)}$ is the minimum value, $t_{(2)}$ is the second minimum, and this pattern continues until $t_{(n)}$ which represents the maximum value in the sample. For any r where $1 \le r \le n$ the pdf of the r^{th} order statistic is expressed as:

$$q_{(r)}(t) = \frac{n!}{(r-1)!(n-r)!} g(t)[G(t)]^{r-1}[1 - G(t)]^{n-r} \tag{17}$$

Putting the expression of pdf and cdf of MOLD from Eq. (5) and Eq. (6) respectively in Eq.(17),

$$q_{(r)}(t) = m\frac{\lambda\xi^{(1+n-r)}(1+\xi)^r(\theta+t)^{\lambda(r+n-1)-1}\theta^{-\lambda r}\left[1-\left(1+\frac{t}{\theta}\right)^{-\lambda}\right]^{r-1}}{\left[(1+\xi)\theta^{-\lambda}-(\theta+t)^{-\lambda}\right]^{n+1}} \tag{18}$$

where, $m = \frac{n!}{(r-1)!(n-r)!}$

The pdf for first order statistics of MOLD, when the value of r is 1, and given by

$$q_{(1)}(t) = \frac{n\lambda\xi^n(1+\xi)(\theta+t)^{-(\lambda n+1)}\theta^{-\lambda}}{\left[(1+\xi)\theta^{-\lambda}-(\theta+t)^{-\lambda}\right]^{n+1}} \tag{19}$$

The pdf of n^{th} order statistics of the MOLD when the value of r is n, and is given by

$$q_{(n)}(t) = \frac{n\lambda\xi(1+\xi)^n(\theta+t)^{-(\lambda+1)}\theta^{-\lambda n}\left[1-\left(1+\frac{t}{\theta}\right)^{-\lambda}\right]^{n-1}}{\left[(1+\xi)\theta^{-\lambda}-(\theta+t)^{-\lambda}\right]^{n+1}} \tag{20}$$

4.4 Stress-Strength Reliability

The reliability model based on stress-strength analysis is expressed as $R = P(X > Y)$, where X denotes the random strength capacity of a system or its component, while Y represents the random stress applied to the system or component at any given moment. In this study, we compute the stress-strength reliability by assuming that the stress Y follows an exponential distribution and the strength X follows a MOLD distribution. The rationale for modeling Y with an exponential distribution is that in many practical engineering scenarios, stress tends to grow exponentially over time.

Let $X \sim MOLD(\xi_1, \lambda, \theta)$ and $Y \sim MOLD(\xi_2, \lambda, \theta)$ with the pdf and cdf function given in Eq. (5) and (6) respectively.

Hence

$$R = P(X > Y)$$

$$R = \int_0^\infty P\,(X > Y) f(x) dx$$

$$R = \int_0^\infty \frac{(1+\xi_2)\left[\theta^{-\lambda} - (\theta + x)^{-\lambda}\right]}{\left[(1+\xi_2)\theta^{-\lambda} - (\theta + x)^{-\lambda}\right]} \cdot \frac{\xi_1(1+\xi_1)\lambda\theta^{-\lambda}(\theta + x)^{-(\lambda+1)}}{\left[(1+\xi_1)\theta^{-\lambda} - (\theta + x)^{-\lambda}\right]^2} dx$$

Putting two substitutions in sequence, $u = (\theta + x)^{-\lambda}$ and $v = u/\theta^{-\lambda}$, we get

$$R = \xi_1(1+\xi_1)(1+\xi_2)\int_0^1 \frac{(1-v)}{[(1+\xi_1) - v]^2[(1+\xi_2) - v]} dv$$

After using By-Part method from calculus, we get

$$R = \frac{\xi_1(1+\xi_2)}{(\xi_1 - \xi_2)} + \frac{\xi_1\xi_2(1+\xi_1)(1+\xi_2))}{(\xi_1 - \xi_2)^2} log\left(\left[\frac{(1+\xi_1)\xi_2}{(1+\xi_2)\xi_1}\right]\right) \tag{21}$$

6. Parameter Estimation

6.1 Classical estimation procedure

The method of maximum likelihood estimation (MLE) is a key classical statistical methodology for the inference of parameters of probability distributions. The method operates by identifying the set of parameters which maximizes the value of the likelihood function, which is defined as the probability of the observed data under a specified model. For a series of i.i.d. observations $t_1, t_2, \dots, t_n$ constituting a sample of size n from the MOLD (ξ, λ, θ) distribution, we can write the likelihood function as:

$$L\,(t, \xi, \lambda, \theta) = \xi^n \lambda^n \theta^{-n\lambda}(1+\xi)^n \frac{\prod_{i=1}^n (\theta + t_i)^{-(\lambda+1)}}{\prod_{i=1}^n \left[(1+\xi)\theta^{-\lambda} - (\theta + t_i)^{-\lambda}\right]^2} \tag{22}$$

Using the Eq. (22) the log-likelihood function is given as

$$\log(L) = n\log(\xi) + n\log(\lambda) - (\lambda + 1)\sum_{i=1}^n \log(\theta + t_i) - n\lambda\log(\theta)$$

$$+n\, log(1+\xi) - 2\sum_{i=1}^n log\left((1+\xi)\theta^{-\lambda} - (\theta + t_i)^{-\lambda}\right) \tag{23}$$

By partially differentiating log likelihood in Eq. (23) with respect to each parameter, we get the following closed form equations for the MLE for the parameters ξ, λ and θ, respectively as

$$\frac{\partial\, log(L)}{\partial \xi} = \frac{n}{\xi} + \frac{n}{(1+\xi)} - 2\sum_{i=1}^{n} \frac{\theta^{-\lambda}}{(1+\xi)\theta^{-\lambda} - (\theta+t_i)^{-\lambda}} \qquad (24)$$

$$\frac{\partial\, log(L)}{\partial \lambda} = \frac{n}{\lambda} - n\, log(\theta) - \sum_{i=1}^{n} log(\theta + t_i) + 2\sum_{i=1}^{n} \frac{(1+\xi)\theta^{-\lambda} log(\theta) - (\theta+t_i)^{-\lambda} log(\theta+t_i)}{(1+\xi)\theta^{-\lambda} - (\theta+t_i)^{-\lambda}} \qquad (25)$$

$$\frac{\partial\, log(L)}{\partial \theta} = \frac{-n\lambda}{\theta} - \frac{(\lambda+1)}{\theta}\sum_{i=1}^{n} \frac{1}{1+t_i/\theta} + 2\sum_{i=1}^{n} \frac{\lambda(1+\xi)\theta^{-(\lambda+1)} - \lambda(\theta+t_i)^{-(\lambda+1)}}{(1+\xi)\theta^{-\lambda} - (\theta+t_i)^{-\lambda}} \qquad (26)$$

The above non-linear equations cannot be solved analytically so, the MLE of the parameters ξ, λ and θ can be obtained as the numerical solution of the Eq. (24), (25) and (26), respectively.

6.2 Bayesian estimation

In this section we used Bayesian parameter estimation methods for the Modi-Lomax distribution's parameters. Bayesian inference demands appropriate prior distribution selection to incorporate available information or parameter assumptions. Here, we employ gamma prior distributions for estimating parameters, which are suitable due to positive parameter space restrictions. The parameters are assumed to be independent and have gamma prior distributions such that $\xi \sim \mathrm{Gamma}(a_1, b_1)$, $\lambda \sim \mathrm{Gamma}(a_2, b_2)$ and $\theta \sim \mathrm{Gamma}(a_3, b_3)$.

Hence the joint prior density of ξ, λ and θ is given by

$$\psi(\xi, \lambda, \theta) \propto \xi^{a_1-1} e^{-b_1\xi} \lambda^{a_2-1} e^{-b_2\lambda} \theta^{a_3-1} e^{-b_3\theta} \qquad (27)$$

Where a_1, b_1, a_2, b_2, a_3 and b_3 are the hyper parameters. The hyper-parameters in the prior distributions are assumed to be known.

The choice of loss function is pivotal in Bayesian decision theory, as it quantifies the penalty for estimation errors. Here, we adopt the squared error loss function (SELF), defined as:

$$L_s(\alpha, \widehat{\alpha}) = (\alpha, \widehat{\alpha})^2$$

where α is the true parameter and $\widehat{\alpha}$ is its estimator. The SELF is symmetric and penalizes over and under estimation equally, yielding the posterior mean as the optimal estimator. This approach balances computational tractability with theoretical rigor, making it widely used in reliability and survival analysis.

By employing the likelihood function in Eq. (22) with the joint prior distribution in Eq. (28), the joint posterior distribution of ξ, λ and θ can be expressed in the following form:

$$\pi(\xi, \lambda, \theta) \propto \xi^{a_1+n-1} \lambda^{a_2+n-1} \theta^{a_3-n\lambda-1} e^{-b_1\xi - b_2\lambda - b_3\theta} (1+\xi)^n$$
$$\frac{\prod_{i=1}^{n} (\theta+t_i)^{-(\lambda+1)}}{\prod_{i=1}^{n} \left[(1+\xi)\theta^{-\lambda} - (\theta+t_i)^{-\lambda}\right]^2} \qquad (28)$$

The Bayes estimators under the SELF loss function is the posterior mean of the posterior distribution, are given as,

$$\hat{\xi} = \frac{1}{\delta}\int_{\xi}\int_{\lambda}\int_{\theta} \xi^{a_1+n}\,\lambda^{a_2+n-1}\theta^{a_3-n\lambda-1}e^{-b_1\xi-b_2\lambda-b_3\theta}(1+\xi)^n$$

$$\frac{\prod_{i=1}^{n}(\theta+t_i)^{-(\lambda+1)}}{\prod_{i=1}^{n}\left[(1+\xi)\theta^{-\lambda}-(\theta+t_i)^{-\lambda}\right]^2}\,d\xi\,d\lambda\,d \tag{29}$$

$$\hat{\lambda} = \frac{1}{\delta}\int_{\xi}\int_{\lambda}\int_{\theta} \xi^{a_1+n-1}\,\lambda^{a_2+n}\theta^{a_3-n\lambda-1}e^{-b_1\xi-b_2\lambda-b_3\theta}(1+\xi)^n$$

$$\frac{\prod_{i=1}^{n}(\theta+t_i)^{-(\lambda+1)}}{\prod_{i=1}^{n}\left[(1+\xi)\theta^{-\lambda}-(\theta+t_i)^{-\lambda}\right]^2}\,d\xi\,d\lambda\,d \tag{30}$$

and

$$\hat{\theta} = \frac{1}{\delta}\int_{\xi}\int_{\lambda}\int_{\theta} \xi^{a_1+n-1}\,\lambda^{a_2+n-1}\theta^{a_3-n\lambda}e^{-b_1\xi-b_2\lambda-b_3\theta}(1+\xi)^n$$

$$\frac{\prod_{i=1}^{n}(\theta+t_i)^{-(\lambda+1)}}{\prod_{i=1}^{n}\left[(1+\xi)\theta^{-\lambda}-(\theta+t_i)^{-\lambda}\right]^2}\,d\xi\,d\lambda\,d \tag{31}$$

respectively, where

$$\delta = \int_{\xi}\int_{\lambda}\int_{\theta} \xi^{a_1+n-1}\,\lambda^{a_2+n-1}\theta^{a_3-n\lambda-1}e^{-b_1\xi-b_2\lambda-b_3\theta}(1+\xi)^n$$

$$\frac{\prod_{i=1}^{n}(\theta+t_i)^{-(\lambda+1)}}{\prod_{i=1}^{n}\left[(1+\xi)\theta^{-\lambda}-(\theta+t_i)^{-\lambda}\right]^2}\,d\xi\,d\lambda\,d$$

The mathematical intricacy of posterior expectations from mathematically formulated expressions inherently renders it difficult to obtain closed-form analytical solutions. To avoid this computational hurdle, researchers can utilize a number of Bayesian approximation techniques, including Lindley's approximation, Tierney-Kadane's method, or Markov Chain Monte Carlo (MCMC) techniques. In our study, we use MCMC sampling to approximate the posterior expectations, as described in the next section.

6.3 Markov Chain Monte Carlo (MCMC) method

We must first determine the complete conditional distributions of the ξ, λ, and θ in order to apply the MCMC approach, as indicated below.

$$\pi^*(\xi|\lambda,\theta,data) \propto \xi^{a_1+n-1}e^{-b_1\xi}(1+\xi)^n\frac{1}{\prod_{i=1}^{n}\left[(1+\xi)\theta^{-\lambda}-(\theta+t_i)^{-\lambda}\right]^2} \tag{32}$$

$$\pi^*(\lambda|\xi,\theta,data) \propto \lambda^{a_2+n-1}\theta^{-n\lambda}e^{-b_2\lambda}\frac{\prod_{i=1}^{n}(\theta+t_i)^{-(\lambda+1)}}{\prod_{i=1}^{n}\left[(1+\xi)\theta^{-\lambda}-(\theta+t_i)^{-\lambda}\right]^2} \tag{33}$$

and

$$\pi^*(\theta|\xi,\lambda,data) \propto \theta^{a_3-n\lambda-1}e^{-b_3\theta}\frac{\prod_{i=1}^{n}(\theta+t_i)^{-(\lambda+1)}}{\prod_{i=1}^{n}\left[(1+\xi)\theta^{-\lambda}-(\theta+t_i)^{-\lambda}\right]^2} \tag{34}$$

The conditional posterior distributions in Eq. (32), (33) and (34) of the parameters under the Bayesian framework do not follow any standard or closed-form distribution. This non-standard nature arises due to the complexity of our likelihood function combined with the Gamma prior distributions, which prevents the use of direct sampling methods. As a result, conventional analytical approaches become impractical. To address this, the Metropolis-Hastings (MH) algorithm is employed as a flexible and efficient Markov Chain Monte Carlo (MCMC) technique to generate samples from the intractable conditional distributions and obtain the Bayesian estimates of the parameters. MH algorithm is carried out in following steps:

1. Set the starting value $\xi^{(0)} = \hat{\xi}, \lambda^{(0)} = \hat{\lambda}$ and $\theta^{(0)} = \hat{\theta}$ through MLE.
2. Set $i = 1$.
3. Create xi*, λ^* and θ^* from $N\left(\hat{\xi}, \widehat{\sigma^2}_{\xi}\right), N\left(\hat{\lambda}, \widehat{\sigma^2}_{\lambda}\right)$ and $N\left(\hat{\theta}, \widehat{\sigma^2}_{\theta}\right)$, respectively.
4. Find A_ξ ,A_λ and A_θ as

$$A_\xi = min\left\{1, \frac{\pi^*\left(\xi^*\middle|\lambda^{(i-1)}, \theta^{(i-1)}, data\right)}{\pi^*\left(\xi^{(i-1)}\middle|\lambda^{(i-1)}, \theta^{(i-1)}, data\right)}\right\},$$

$$A_\lambda = min\left\{1, \frac{\pi^*\left(\lambda^*\middle|\xi^{(i-1)}, \theta^{(i-1)}, data\right)}{\pi^*\left(\lambda^{(i-1)}\middle|\xi^{(i-1)}, \theta^{(i-1)}, data\right)}\right\}$$

and

$$A_\theta = min\left\{1, \frac{\pi^*\left(\theta^*\middle|\xi^{(i-1)}, \lambda^{(i-1)}, data\right)}{\pi^*\left(\theta^{(i-1)}\middle|\xi^{(i-1)}, \lambda^{(i-1)}, data\right)}\right\}$$

5. Utilizing the uniform U (0,1) distribution, generate samples u_1, u_2 and u_3 respectively for ξ, λ and θ.
6. If u_1, u_2 and u_3 are less than A_ξ, A_λ and A_θ, respectively than set $\xi^{(i)} = \xi^*, \lambda^{(i)} = \lambda^*$ and $\theta^{(i)} = \theta^*$, respectively. Otherwise, set $\xi^{(i)} = \xi^{i-1}, \lambda^{(i)} = \lambda^{i-1}$ and $\theta^{(i)} = \theta^{i-1}$, respectively.
7. Set $i = i + 1$
8. Redo steps 3 - 7, L times to get $\xi^{(i)}, \lambda^{(i)}$ and $\theta^{(i)}$for $i = 1,2,---, L$.
9. After a suitable burn-in phase, say D, obtain the Bayes estimate of the parameter ξ, λ and θ as an example, as

$$\tilde{\xi} = \frac{1}{L-D}\sum_{i=D+1}^{L}\xi^{(i)}, \tilde{\lambda} = \frac{1}{L-D}\sum_{i=D+1}^{L}\lambda^{(i)}, \tilde{\theta} = \frac{1}{L-D}\sum_{i=D+1}^{L}\theta^{(i)}.$$

7. Simulation Study

In this section, to determine the effectiveness of MOLD estimators, we conduct a Monte Carlo simulation. Using specific selected variable values and the MOLD distribution, a number of random numbers are generated. The following procedures to investigate the new model are now presented:

[Step:1] Fix the values of MOLD (ξ, λ, θ) as

Set-1 MOLD (9, 5, 0.25) and

Set-2 MOLD (0.02, 0.05, 10).

[Step:2] Fix the sample sizes n as

n = 10, 30, 50, 200.

[Step:3] Fix the number of replications as 1000.

[Step:4] Generate the random samples using Quantile function given in Eq. (13).

$$x_i = \theta\left[\left(\frac{1+\xi-u_i}{(1+\xi)(1-u_i)}\right)^{1/\lambda} - 1\right], \text{where} \quad u_i \sim U(0,1), i = 1,2,\dots,n.$$

[Step:5] The MLE of each parameter are obtained using the optim function in R software using the log-likelihood function from Eq. (23).

[Step:6] Bayes estimates of the parameters are obtained based on two informative prior sets as

Prior-1 (P-1) $(a_1, a_2, a_4) = (18,10,0.5)$ and $b_i = 2$ for i = 1, 2, 3.

Prior-2 (P-2) $(a_1, a_2, a_4) = (0.08,0.20,40)$ and $b_i = 4$ for i = 1, 2, 3.

[Step:7] The performance of the point estimates is assessed using two criteria: mean squared error (MSE) and absolute bias (AB).

$$MSE = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}-\theta)^2$$

$$AB = |(\hat{\theta}-\theta)|$$

where, θ= Parameter true value, $\hat{\theta}$ = Estimated value of the parameter.

Table 1: *Mean Estimate, Absolute Bias (AB) and Mean Squared Error (MSE) of the parameters of MOLD under MLE and Bayesian Method (BM) for Case-1*

n	Properties	MLE			BM		
		ξ = 9	λ = 5	θ = 0.25	ξ = 9	λ = 5	θ = 0.25
10	Mean Estimate	8.35711	7.12690	0.26704	8.96335	4.97687	0.25111
	AB	0.64289	2.12690	0.01704	0.03665	0.02313	0.00111
	MSE	10.91290	16.11558	0.01407	0.34820	0.50040	0.00636
30	Mean Estimate	8.33814	7.05030	0.25669	9.08599	4.98748	0.25240
	AB	0.66187	2.05030	0.00669	0.08599	0.01252	0.00240

	MSE	9.48808	12.27414	0.00397	0.55615	0.50785	0.00424
50	Mean Estimate	8.20359	6.85089	0.25723	9.04125	5.11707	0.25892
	AB	0.79641	1.85089	0.00723	0.04125	0.11707	0.00892
	MSE	9.52439	9.91493	0.00253	0.42477	0.64209	0.00343
200	Mean Estimate	8.81283	6.18065	0.25527	9.02399	5.00547	0.25215
	AB	0.18717	1.18065	0.00527	0.02399	0.00547	0.00215
	MSE	3.23149	3.64873	0.00051	0.74400	0.69949	0.00272

Table 2: *Mean Estimate, Absolute Bias (AB) and Mean Squared Error (MSE) of the parameters of MOLD under MLE and Bayesian method (BM) for Case-2*

n	Properties	MLE			BM		
		$\xi = 0.02$	$\lambda = 0.05$	$\theta = 10$	$\xi = 0.02$	$\lambda = 0.05$	$\theta = 10$
10	Mean Estimate	0.09588	0.02575	9.99064	0.05352	0.09236	10.04951
	AB	0.07588	0.02425	0.00936	0.03352	0.04236	0.04951
	MSE	0.01284	0.00188	0.08164	0.00241	0.00580	2.46946
30	Mean Estimate	0.06400	0.03510	9.99969	0.05335	0.10310	9.92382
	AB	0.04400	0.01490	0.00031	0.03335	0.05310	0.07618
	MSE	0.00439	0.00141	0.00000	0.00344	0.01054	2.12657
50	Mean Estimate	0.05590	0.03890	9.99959	0.05349	0.09945	9.82858
	AB	0.03590	0.01110	0.00041	0.03349	0.04945	0.17142
	MSE	0.00321	0.00128	0.00001	0.00476	0.01133	1.42323
200	Mean Estimate	0.02828	0.05120	9.97097	0.02491	0.05684	9.84556
	AB	0.00828	0.00120	0.02904	0.00491	0.00684	0.15444
	MSE	0.00063	0.00071	0.10122	0.00038	0.00140	0.43880

Tables (1) and (2) present the parameter estimates for the MOLD distribution obtained through maximum likelihood estimation and Bayesian methodology, respectively. A comparative analysis of the results indicates that the Bayesian approach yields superior parameter estimates relative to the maximum likelihood method, consistent with findings reported in the existing literature.

8. Application

To demonstrate the practical applicability of the proposed MOLD distribution, we analyse a medical dataset comprising remission times (measured in months) for 137 bladder cancer patients. This dataset Table (3), originally reported by (Lee & Wang, 2003), provides survival times that exhibit the characteristics commonly observed in oncological studies. The observed remission times range from 0.08 to 79.05 months, with the complete dataset presented as follows:

Table 3: *Remission time (in month) of 137 cancer patients*

4.5	32.15	3.88	13.8	19.13	4.87	3.02	5.85	14.24	5.71
19.36	7.09	7.87	7.59	20.28	5.32	5.49	3.02	46.12	4.33
2.02	4.51	5.17	2.83	9.22	1.05	0.2	8.37	3.82	9.47
36.66	14.77	26.31	79.05	10.06	8.53	4.65	2.02	4.98	11.98
2.62	4.26	5.06	1.76	0.9	11.25	16.62	4.4	21.73	10.34
12.07	34.26	0.87	10.66	6.97	2.07	0.51	12.03	0.08	17.12

3.36	2.64	1.4	12.63	43.01	14.76	2.75	7.66	0.81	1.19
7.32	4.18	3.36	8.66	1.26	13.29	1.46	14.83	6.76	23.63
24.8	5.62	8.6	3.25	10.86	18.1	7.62	7.63	17.14	25.74
3.52	2.87	15.96	17.36	9.74	3.31	7.28	1.35	0.4	2.26
4.33	9.02	5.41	2.69	22.69	6.94	2.54	11.79	2.46	7.26
2.69	5.34	3.48	4.7	8.26	6.93	4.23	3.7	0.5	10.75
6.54	3.64	5.32	13.11	8.65	3.57	5.09	7.39	5.41	11.64
2.09	2.23	6.25	7.93	4.34	25.82	12.02			

We applied MLE and Bayesian method to fit the MOLD parameters and conducted a comparative study against seven competing distributions from the Lomax family. The comparative models included Log-Lomax (LoL) by (Ishaq et al., 2025), Power Lomax (PoL) by (Rady et al., 2016), Rayleigh Lomax (RaL) by (Shabbir et al., 2018), Power X-Lindey (PXLin) by (Elgarhy et al., 2025), Poisson Inverted Lomax (PoIL) by (Joshi & Kumar, 2021), Exponentiated odd Lomax exponential distribution (EOLE) by (Dhungana & Kumar, 2022) and Transmuted Exponential Lomax (TEL) by (Ashour & Eltehiwy, 2013). Distribution comparison is performed using a comprehensive set of information criteria, specifically AIC, BIC, CAIC, and HQIC, representing the Akaike, Bayesian, Consistent Akaike, and Hannan-Quinn information criteria, respectively. We also used the Kolmogorov-Smirnov (K-S) test to better check how well each distribution worked. We picked the best distribution by looking for the one with the highest log-likelihood value, the best p-value from the K-S test, and the lowest AIC, BIC, and CAIC scores.

Table 4: *Goodness of fit tests for cancer patient dataset*

Distributions	-2logL	AIC	BIC	HQIC	CAIC	K-S	p-value
MOLDx	885.4448	891.4448	900.2047	895.0046	891.6253	0.17	0.000726
RaL	885.8124	891.8124	900.5723	895.3722	891.9929	0.1961	5.33e-05
LoL	888.3446	894.3446	903.1045	897.9044	894.5251	0.2103	1.09e-05
PXLin	912.2588	918.2588	927.0187	921.8186	918.4393	0.2052	1.94e-05
PoIL	989.5734	995.5734	1004.333	999.1332	995.7539	0.3391	4.18e-14
EOLE	997.8764	997.5734	997.5734	1009.253	1002.32	0.1955	5.64e-05
TEL	1017.408	1025.408	1037.087	1030.154	1025.711	0.3384	4.72e-14
PoL	1111.972	1117.972	1126.732	1121.531	1118.152	0.2587	2.17e-08

Table (4) presents a comparative analysis of the MOLD against seven alternative distribution functions using goodness-of-fit measures. The results demonstrate that MOLD exhibits superior performance, achieving the minimum values for AIC, BIC, HQIC, and CAIC relative to the seven competing distributions. Additionally, we computed the Kolmogorov Smirnov test statistic alongside its associated p-value. Optimal model fit is characterized by minimal K-S values and maximal p-values. The MOLD yielded a K-S statistic of 0.17 with a corresponding p-value of 0.000726, representing the most favourable performance among all competing models, thereby establishing MOLD as the most appropriate distribution for practical applications, particularly in modelling cancer patient remission times.

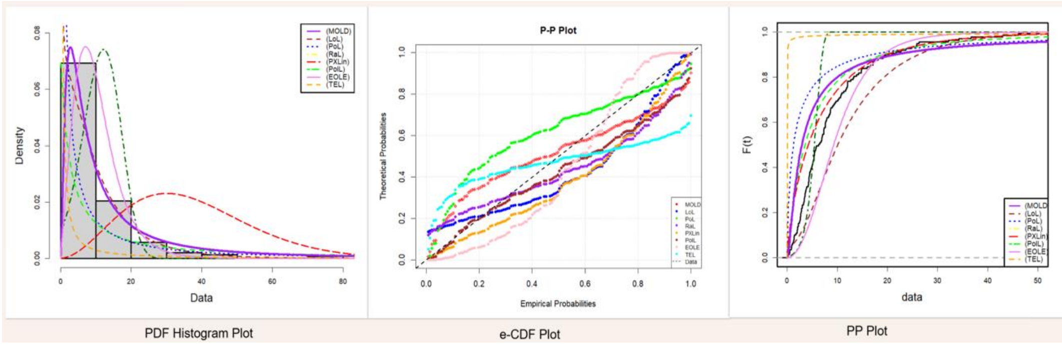

Figure 5. *Plots for the Cancer patient's dataset*

Figure (5) displays three visual diagnostic tools—probability density function histogram, empirical cumulative distribution function (ECDF), and probability-probability plot utilized to assess model adequacy across various distributions for the given dataset. Throughout all three evaluation methods, MOLD exhibits optimal fitting characteristics, as evidenced by its curves closely following the observed data patterns.

The newly developed MOLD distribution achieved superior performance when compared to existing competitive models, showing better goodness-of-fit measures and diminished information criteria scores. These empirical outcomes emphasize its capability in modelling non-symmetric and sophisticated data configurations. The findings of this investigation present the MOLD as a significant analytical resource for reliability engineering, survival research, and risk management applications.

9. Conclusion

This research introduces the MOLD, a new three-parameter extension of the standard Lomax distribution, created to overcome challenges in studying complex survival and reliability data. The distribution features heavy tails and a decreasing hazard rate, making it suitable for long-term survival analysis. By combining the flexible properties of the Modi family with the basic structure of the Lomax distribution, the MOLD shows improved adaptability through its hazard function behaviour, quantile features, and tail properties.

We developed a complete mathematical framework for the distribution, including moments, survival functions, and entropy measures, which confirms its theoretical strength. For parameter estimation, we applied both classical maximum likelihood estimation (MLE) and Bayesian methods using gamma priors. Monte Carlo simulations validated that the MLE estimators are consistent and asymptotically efficient. Our simulation studies examined estimator performance using mean squared error, showing that estimation accuracy improves as sample sizes increase.

To test practical usefulness, we applied the MOLD to a bladder cancer patient dataset and compared it with existing distributions using standard measures like AIC, BIC, CAIC, log-likelihood, and the Kolmogorov-Smirnov test. The MOLD consistently performed better than competing models, showing superior fit quality and lower information criteria values. These results demonstrate its effectiveness in modelling skewed and complex data patterns.

The study establishes the MOLD as a useful tool for reliability analysis, survival studies, and risk assessment, particularly when dealing with heavy-tailed data that exhibits decreasing failure rates over time.

10. Future Scope

Future work could develop the MOLD for multiple variables and use Bayesian approaches for complicated data analysis. Researchers could also improve how it handles incomplete data and combine it with artificial intelligence for better predictions. Testing the MOLD against newer statistical models and adapting it for specific fields like weather studies or smart device networks would show how widely it can be used and help advance survival analysis and risk assessment.

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