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A Study about Isolate Domination & Isolate Inclusive Set in Graphs

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Abstract:

This paper investigates the concept of isolate domination in graphs, focusing on how structural changes affect the isolate domination number. The impact of removing both isolated and non-isolated vertices is examined, and necessary and sufficient conditions are established for when such removals lead to an increase or decrease in the isolate domination number. It is shown that if a graph Gcontains an isolated vertex v such that γ_0 (G-v)>γ_0 (G), then vmust be the only isolated vertex in G. The study further explores the effect of edge removal, identifying conditions under which the isolate domination number increases. Additionally, the isolate inclusive set number is analyzed, with criteria provided for its increase following edge deletion. Special attention is given to graphs where the isolate domination number equals one or two, offering characterizations that enhance understanding of their structural properties. These results contribute to the broader theory of domination in graphs by clarifying how isolate domination responds to local modifications, and they provide useful tools for analyzing graph resilience and optimization in network structures.

Keywords: Isolate dominating set, Minimal isolate dominating set, Minimum isolate dominating set, Isolate domination number, Isolate inclusive set, 1-maximal isolate inclusive set, Private neighbourhood.

1. Introduction:

The concept of isolate domination in graphs has attracted considerable interest due to its relevance in structural graph theory and its applications in network analysis. An *isolate dominating set* is a subset of vertices such that every isolated vertex in the graph is either in the set or adjacent to a vertex in the set. Building on this foundation, the notion of an *isolate inclusive set* was introduced to further refine the understanding of domination in graphs. It was established that every 1-maximal isolate inclusive set qualifies as an isolate dominating set, thereby linking these two concepts in a meaningful way.

This paper extends the study of isolate domination by examining how various graph operations influence the isolate domination number and the isolate inclusive set number. Specifically, the effect of vertex removal is analyzed in detail. The removal of an isolated vertex is considered first, and necessary and sufficient conditions are derived under which the isolate domination number either increases or decreases. The analysis is then extended to the removal of non-isolated vertices, with similar conditions established.

Further, the paper investigates the behavior of isolate inclusive sets under vertex removal, offering insights into how such operations affect the isolate inclusive set number. The study also explores the impact of edge removal on both the isolate domination number and the isolate inclusive set number. Conditions are provided to determine when these parameters increase following the deletion of an edge.

Finally, graphs with isolate domination numbers equal to one or two are characterized, contributing to a deeper understanding of their structural properties and the role of isolate domination in graph classification.

2. Preliminaries and Notations

Let Gbe a graph. The vertex set of $G$is denoted by $V(G)$, and the edge set is denoted by $E(G)$. For an edge $e \in E(G)$, the graph $G - e$represents the subgraph obtained by removing the edge $e$from G. For a vertex $v \in V(G)$, the graph $G - v$denotes the subgraph induced by all vertices in $V(G) \setminus \{v\}$. If $x \in V(G)$, then $d(x)$denotes the degree of the vertex $x$in G, which is the number of edges incident to x. Throughout this paper, only simple, undirected graphs with finite vertex sets are considered.

3. Definitions and Examples

Definition 3.1: (Isolate Inclusive set)
Let Gbe a graph and Sa nonempty subset of $V(G)$. The set $S$is called an *isolate inclusive set* if the subgraph induced by S, denoted by $\langle S \rangle$, contains at least one isolated vertex. An isolate inclusive set of maximum cardinality is referred to as a *maximum isolate inclusive set*, and its cardinality is denoted by $\beta_{\text{is}}(G)$.

Definition 3.2: (1-maximal Isolate inclusive set)
Let G be a graph and S be a Isolate inclusive set of G then S is said to be a 1-maximal Isolate inclusive set if $S \cup \{v\}$ is not an Isolate inclusive set, for every $v \in V(G) - S$.

Definition 3.3: (Minimal set)
A maximal Isolate inclusive set with minimum cardinality is called a minimal set and its cardinality is denoted as $m_{is}(G)$ and it's called the minimal number of the graph G.

Let G be a graph & $v \in V(G)$ such that $d(v) = \nabla(G)$.
Now $V(G) - N(v)$ is an Isolate inclusive set of G but it did not be a 1-maximal Isolate inclusive set of G. This can be observed in following example.

Example 1: Consider the path graph P_5 with 5 vertices $\{1, 2, 3, 4, 5\}$

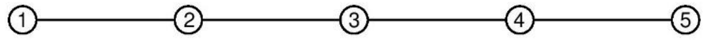

Figure 1. Path Graph

Consider the vertex 3 .
$d(3) = 2 = \nabla(G)$.

$N(3) = \{2, 4\}$ & $V(G) - N(3) = \{1, 3, 5\}$
This set is an Isolate inclusive set but it is not 1-maximal.

Definition 3.4: (Isolate Dominating Set)
Let G be a graph and $S \subset V(G)$then S is said to be an isolate dominating set if
1. S is a dominating set and
2. $< S >$ contains an isolated vertex.

An isolate dominating set with minimum cardinality is called a minimum isolate dominating set.

The cardinality of a minimum isolate dominating set is called the isolate domination number of the graph G and it is denoted as $\gamma_0(G)$.

Obviously for any graph G, $\gamma(G) \leq \gamma_0(G)$ where $\gamma(G)$ denotes the domination number of the graph G.

Remark: Note that every 1-maximal Isolate inclusive set is an isolate dominating set, but converse is not true.

We introduce the following symbols:

$$V_0^+ = \{x \in V(G) \ni \gamma_0(G - x) > \gamma_0(G)\}$$

$$V_0^- = \{x \in V(G) \ni \gamma_0(G - x) < \gamma_0(G)\}$$

$$V_0^0 = \{x \in V(G) \ni \gamma_0(G - x) = \gamma_0(G)\}$$

4. Main Result

Proposition 4.1: Let G be a graph & S be a 1-maximal Isolate inclusive set of G.

(1) For each isolated vertex v of S, $N(v) = V(G) - S$.
(2) If u & v are isolates of S then $d(u) = d(v)$.

Proof:
(1) Let v be an isolated vertex of Sthen $N(v) \subset V(G) - S$.
Let $x \in V(G) - S$.Since S is 1-maximal, x is adjacent to every isolated vertex of S and therefore x is adjacent to v which implies that $x \in N(v)$.
Thus, $(v) = V(G) - S$.

(2) Let u & v be to isolate sets of Sthen $d(u) = |N(u)| = |V(G) - S| = |N(v)| = d(v)$.
Thus, $d(u) = d(v)$. ▌

Proposition 4.2: Let G be a graph and $v \in V(G)$ then $\beta_{is}(G - v) \leq \beta_{is}(G)$.

Proof: Let M be a maximum Isolate inclusive set of $G - v$.
By the above proposition 4.2, M is also an Isolate inclusive set of G .
Therefore, $\beta_{is}(G - v) \leq \beta_{is}(G)$ ▌
Example 2: Consider the path graph P_5 with vertices $\{1, 2, 3, 4, 5\}$

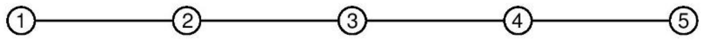

Figure 2. Path Graph

Here, $\beta_{is}(G) = 4$.

Now consider the subgraph $G - 3$.

Figure 3. Path Graph

Here, $\beta_{is}(G - 3) = 3$.
Therefore, in this example $\beta_{is}(G - 3) < \beta_{is}(G)$.

Theorem 4.3: Let G be a graph and $v \in V(G)$ then $\beta_{is}(G - v) = \beta_{is}(G)$ if and only if there is a maximum Isolate inclusive set $M(G)$ such that $v \notin M$.

Proof: First suppose that $\beta_{is}(G - v) = \beta_{is}(G)$ that M be a maximum Isolate inclusive set of $G - v$.
Obviously, M is an Isolate inclusive set of G .
Since $\beta_{is}(G - v) = \beta_{is}(G)$, M must be a maximum Isolate inclusive set of G .
Note that $v \notin M$.

Conversely, suppose that M is a maximum Isolate inclusive set of G such that $v \notin M$.
Now M is a subset of $G - v$ & it is also an Isolate inclusive set of $G - v$.
Therefore, $\beta_{is}(G - v) \geq |M| = \beta_{is}(G)$.
It is also true that $\beta_{is}(G - v) \leq \beta_{is}(G)$.
Therefore, $\beta_{is}(G - v) = \beta_{is}(G)$ ▌

Now we consider the effect of removing a vertex from a graph on the isolate domination number.

Example 3: Consider the cycle graph C_7 with 7 vertices $\{1, 2, 3, 4, 5, 6, 7\}$

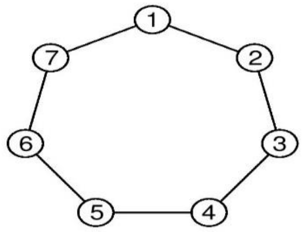

Figure 4. Cycle Graph

In this graph the set $\{1, 2, 5\}$ is a minimum isolate dominating set and therefore the isolate domination number is 3.

Now consider the graph $G - 7$ which is the path graph with 6 vertices $\{1, 2, 3, 4, 5, 6\}$ the isolate domination number is 2.

Thus, the isolate domination number decreases in this graph.

Example 4: Consider the path graph P_5 with 5 vertices $\{1, 2, 3, 4, 5\}$

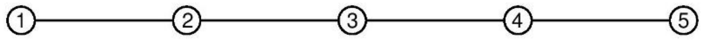

Figure 5. Path Graph

The isolated domination number of this graph is 2. If we remove any vertex from the graph the isolate domination number of the resulting graph remains unchanged.

Theorem 4.4: Let G be a graph and v be an isolated vertex in G then $\gamma_0(G) < \gamma_0(G - v)$if and only if for any minimum isolate dominating set S. The following two conditions are satisfied.

(1) $v \in S$
(2) v is the only isolate in the $< S >$.

Proof: Suppose $\gamma_0(G - v) > \gamma_0(G)$.
If $v \notin S$ then v is not adjacent to any vertex of S which implies that S is not a dominating set. Which is a contradiction.
Thus, $v \in S$.
Hence condition (1) is satisfied.
Suppose u is anther vertex in S. Which is an isolate in the $< S >$.
Let $S_1 = S - \{v\}$.
Note that $u \in S_1$ and u is an isolate in $< S_1 >$.
Thus, S_1 is an isolate dominating set in $-v$.
Therefore, $\gamma_0(G - v) \leq |S_1| < |S| = \gamma_0(G)$.
This is a contradiction and therefore (2) holds.

Conversely, Suppose (1) and (2) hold.
Let T be a set of vertices of $G - v$ such that $|T| < \gamma_0(G)$ and $< T >$ contains an isolate.
If T is an isolate dominating set in $G - v$ then $T_1 = T \cup \{v\}$ is an isolate dominating set of G.
Note that T is a minimum isolate dominating set in $G - v$ then T_1 is also a minimum isolate dominating set in G. Then T_1 is a minimum isolate dominating set of G containing atleast two distinct isolate in G .
This contradicts condition (2).
Therefore, any set T with $|T| < \gamma_0(G)$ cannot be a minimum isolate dominating set of $G - v$.

Suppose T is a set of vertices of $G - v$ such that $|T| = \gamma_0(G)$ and suppose T is a minimum isolate dominating set of $G - v$.

Now $T_1 = T \cup \{v\}$ is an isolate dominating set of G with $|T_1| = \gamma_0(G) + 1$. Also T is a proper subset of T_1 and therefore T_1 is not a minimal isolate dominating set of G.
Hence there is a vertex u in T_1 such that $T_1 - u$ is an isolate dominating set of G.
Also $|T_1 - u| = |T| = \gamma_0(G)$.
Further note that u cannot be an isolate in T_1 because otherwise $T_1 - u$ would not be a dominating set.
Thus, $T_1 - u$ is a minimum isolate dominating set of G containing at least two isolate vertices.
Which is a contradiction.

Thus, there is no set of T of vertices of $G - v$ such that $|T| = \gamma_0(G)$ and T is an isolate dominating set of $G - v$.

Therefore, any isolate dominating set of $G - v$ must have cardinality $> \gamma_0(G)$.
Therefore, $\gamma_0(G - v) > \gamma_0(G)$ ▌

Theorem 4.5: Let G be a graph and v be an isolated vertex in G then $\gamma_0(G - v) < \gamma_0(G)$ if and only if there is a minimum isolate dominating set S which contains v and it also contains some other isolate.

Proof: Suppose $\gamma_0(G - v) < \gamma_0(G)$.
Let S_1 be a minimum isolate dominating set of $G - v$ then S_1 contain an isolate.
Let $S_1 = S \cup \{v\}$.
Then S is a minimum isolate dominating set of Gand $v \in S$.
Further, S contains two isolates one of them is v.

Conversely, suppose that condition is satisfied.
Let $S_1 = S - \{v\}$ then by assumption S_1 contains an isolate. It is also isolate dominating set of $G - v$.

Therefore, $\gamma_0(G-v) \leq |S_1| < |S| = \gamma_0(G)$ ■

Theorem 4.6: Let G be a graph and v be a non isolated vertex in G then $\gamma_0(G-v) > \gamma_0(G)$ if and only if the following two conditions are satisfied.

(1) $v \in S$, for every minimum isolate dominating set S of G .
(2) There is no subset S of $G-v$ such that $|S| \leq \gamma_0(G)$, S is a subset of $V(G)-N[v]$ and S is an isolate dominating set of $G-v$.

Proof: suppose $\gamma_0(G-v) > \gamma_0(G)$.

(1) Suppose there is a minimum isolate dominating set S of G such that $v \notin S$ then S is an isolate dominating set of $G-v$.
Therefore, $\gamma_0(G-v) \leq |S| = \gamma_0(G)$.
That is $\gamma_0(G-v) \leq \gamma_0(G)$
Which is a contradiction.
Therefore, $v \in S$, for every minimum isolate dominating set S of G .

(2) Suppose there is a subset S of $G-v$ such that $|S| \leq \gamma_0(G)$, S is a subset of $V(G)-N[v]$ and S is an isolate dominating set of $G-v$.
Then $\gamma_0(G-v) \leq |S| = \gamma_0(G)$ and therefore $\gamma_0(G-v) \leq \gamma_0(G)$.
Which is a contradiction.
Therefore, condition (2) is also satisfied.

Conversely, suppose condition (1) and (2) are satisfied.
First suppose that $\gamma_0(G-v) = \gamma_0(G)$.
Let S be a minimum isolate dominating set of $G-v$ then $|S| = \gamma_0(G-v) = \gamma_0(G)$.

Case (1): Suppose v is adjacent to some vertex of S then S is a minimum isolate dominating set of G not containing v.
Which contradiction is condition (1).

Case (2): Suppose v is not adjacent any vertex of S, then $N[v] \cap S = \emptyset$, which is equivalent to the fact that S is subset of $V(G)-N[v]$ also $|S| \leq \gamma_0(G)$ and S is an isolate dominating set of $G-v$.
This is again contradiction condition (2).

From case (1) and case (2) it follows that $\gamma_0(G-v) = \gamma_0(G)$ is not possible.
Suppose $\gamma_0(G-v) < \gamma_0(G)$.
Let S be a minimum isolate dominating set of $G-v$ that S cannot be isolate dominating set of G because $|S| < \gamma_0(G)$ this means that v is not adjacent to any other vertex of S then S is subset of $V(G)-N[v]$, $|S| \leq \gamma_0(G)$ and S is an isolate dominating set of $G-v$.
This is again contradiction condition (2).
Therefore, $\gamma_0(G-v) < \gamma_0(G)$ is also not possible.

Hence $\gamma_0(G-v) > \gamma_0(G)$ ▌

Proposition 4.7: Let G be a graph and v be a non isolated vertex of G if $\gamma_0(G-v) < \gamma_0(G)$ then $\gamma_0(G-v) = \gamma_0(G) - 1$.

Proof: Let S_1 be a minimum isolate dominating set of $G-v$ then S_1cannot be an isolate dominating set of G because $|S_1| < \gamma_0(G)$.

Let $S = S_1 \cup \{v\}$ then S is an isolate dominating set of G .
Since $\gamma_0(G-v) < \gamma_0(G)$, S must be a minimum isolate dominating set of G .
Thus, $\gamma_0(G) = |S| = |S_1| + 1 = \gamma_0(G-v) + 1$ ▌

Now we state and prove a necessary and sufficient condition under which the isolate domination number decreases when a non-isolated vertex is removed from the graph.

Theorem 4.8: Let G be a graph and v be a non isolated vertex in G then $\gamma_0(G-v) < \gamma_0(G)$ if and only if there is a minimum isolate dominating set S containing v and some other isolate such that $P_n[v, S] = \{v\}$.

Proof: Suppose $\gamma_0(G-v) < \gamma_0(G)$.
Let S_1 be a minimum isolate dominating set of $G-v$. Then S_1 cannot be an isolate dominating set of G.
It follows that v cannot be adjacent to any vertex of S_1 .
Let $S = S_1 \cup \{v\}$ then S is aminimum isolate dominating set of Gand $v \in S$.
Since v is not adjacent to any other vertex of S, $v \in P_n[v, S]$.
Suppose $x \neq v$ & $x \in P_n[v, S]$ then $x \notin S_1$ .Since x is a vertex of $G-v$, x is adjacent to some vertex y of S_1 .
Thus, x is adjacent to v also.
Thus, x is adjacent to two distinct vertices of G.
Which contradict the fact that $x \in P_n[v, S]$.
Therefore, $P_n[v, S] = \{v\}$.
Obviously, S contains atleast two isolates and one of them is v .

Conversely, suppose that there is a minimum isolate dominating set S of G such that $v \in S$ and $P_n[v, S] = \{v\}$ and S contains atleast two isolates.
Let $S_1 = S - \{v\}$ then S_1 is an isolate dominating set of $G-v$.
Therefore, $\gamma_0(G-v) \leq |S_1| < |S| = \gamma_0(G)$.

Thus, $\gamma_0(G-v) < \gamma_0(G)$ ▌

Theorem 4.9: Let G be a graph and v be a non isolated vertex in G then $\gamma_0(G-v) > \gamma_0(G)$ and let S be a minimum isolate dominating set of G which contains an isolate different from v then $v \in S$ and $P_n[v, S]$ contains two non-adjacent vertices .

Proof: Since $\gamma_0(G-v) > \gamma_0(G)$, $v \in S$
Since S is a minimal isolate dominating set $P_n[v, S] \neq \emptyset$.

If $P_n[v, S] = \{x\}$, then $\gamma_0(G-v) < \gamma_0(G)$
Therefore, there is a vertex $x \neq v$ & $x \in P_n[v, S]$.

Suppose $P_n[v, S] = \{x\}$ then $x \notin S$.
Let $S_1 = S - \{v\} \cup \{x\}$ then S_1 is a minimum isolate dominating set of G not containing v .
This is a contradiction.

Suppose $P_n[v, S] = \{v, y\}$for some vertex $y \neq v$.
Let $S_1 = S - \{v\} \cup \{y\}$ then S_1 is a minimum isolate dominating set of G not containing v .
Which is again a contradiction.

Therefore, $P_n[v, S]$ contains atleast two distinct vertices different from v .

Suppose any two vertices in the $P_n[v, S]$ which are different from v are adjacent. Then let y_1, y_2 be two distinct vertices in the $P_n[v, S]$ such that $y_1 \neq v$, $y_2 \neq v$.

Now y_1 & y_2 are adjacent.
Let $S_1 = S - \{v\} \cup \{y_1\}$ then S_1 is a minimum isolate dominating set of G not containing v .
Which is again a contradiction.

Therefore, there must be exist two distinct vertices in $P_n[v, S]$ which are non-adjacent.
Thus, the theorem is proved ▌

Proposition 4.10: Let G be a graph and e be an edge of G then $\beta_{is}(G-e) \geq \beta_{is}(G)$.

Proof: Let S be a maximum Isolate inclusive sets of G. Then S is also Isolate inclusive set of $G - e$.

Therefore, $\beta_{is}(G-e) \geq |S| = \beta_{is}(G)$.

Thus, $\beta_{is}(G-e) \geq \beta_{is}(G)$ ▌

Example 5: Consider the path graph P_5 with 5 vertices $\{1, 2, 3, 4, 5\}$

Figure 6. Path Graph

Here, $\beta_{is}(G) = 4$.
Now consider the subgraph $G - e$ where $= \{45\}$.

Figure 7. Path Graph

Here, $\beta_{is}(G - e) = 4$.
Therefore, for this graph $\beta_{is}(G - e) = \beta_{is}(G)$.

Example 6: Consider the cycle graph C_5 with 5 vertices $\{1 , 2 , 3 , 4 , 5\}$

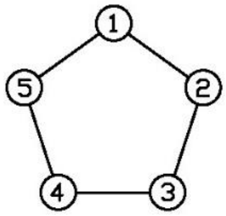

Figure 8. Cycle Graph

Here, $\beta_{is}(G) = 3$.
Now consider the subgraph $G - e$ where $= \{15\}$.

Figure 9. Path Graph

Here $\beta_{is}(G - e) = 4$.
Therefore, for this graph $\beta_{is}(G - e) > \beta_{is}(G)$.

Now we state and prove a necessary and sufficient condition under which Isolate inclusive set number of a graph increases when an edge is removed from the graph.

Theorem 4.11: Let G be a graph and $e = \{uv\}$ be an edge of G then $\beta_{is}(G - e) > \beta_{is}(G)$ if and only if there is a subset S of $V(G)$ such that S has no isolated vertices, $|S| > \beta_{is}(G)$, $u, v \in S$ and atleast one of u & v is a pendent vertex in the $< S >$.

Proof: First suppose that $\beta_{is}(G - e) > \beta_{is}(G)$.
Let S be any maximum Isolate inclusive set of $G - e$.

Claim: $u \in S$ & $v \in S$

Now consider the set S in the graph G . Since $|S| > \beta_{is}(G)$ is cannot be Isolate inclusive set of G but S is an Isolate inclusive set in $G - e$.

Therefore, u or v must be an isolated vertex in S when S is regard as a vertex set of $G - e$.
Therefore, u or v must be a pendent vertex in the $< S >$ when S is regard as a set of vertices of G .
Since S is not Isolate inclusive set of G. The $< S >$ does not have any isolated vertices.

Conversely, suppose condition is satisfied.
Let S be a set of vertices of G such that $|S| > \beta_{is}(G), < S >$ has no isolated vertex and $u, v \in S$ and atleast one of u & v is a pendent vertex in the $< S >$.
Suppose u is a pendent vertex in the $< S >$. Now consider S is in the graph $G - e$. Then obviously S is an Isolate inclusive set in $G - e$.
Then $\beta_{is}(G - e) \geq |S| > \beta_{is}(G)$
Thus, $\beta_{is}(G - e) > \beta_{is}(G)$ ▌

Theorem 4.12: Let G be a regular graph and e be any edge of G then $\beta_{is}(G - e) > \beta_{is}(G)$.

Proof: Suppose G is a k_regular graph, $k \geq 1$

Let $e = \{uv\}$ is any edge of G.
Now $(v) = k = \delta(G)$.
Therefore, by the above remark, $\beta_{is}(G - e) > \beta_{is}(G)$ ▌

Now we consider the operation of removing an edge of a graph on the isolate domination number of a graph.

Remark: Let G be a graph and $e = \{uv\}$ be any edge of G then any of the following three possibilities exists

(i) $\gamma_0(G - e) = \gamma_0(G)$
(ii) $\gamma_0(G - e) < \gamma_0(G)$
(iii) $\gamma_0(G - e) > \gamma_0(G)$

Example 7: Let G be a graph with 4 vertices {1 , 2 , 3 , 4}.

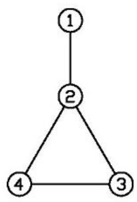

Figure 10. Graph G

Here, $\gamma_0(G) = 1$.

Now we state and prove a necessary and sufficient condition under which the isolate domination number of a graph increases when an edge is removing from the graph.

Theorem 4.13: Let G be a graph with $\gamma_0(G) \geq 2$ and $e = \{uv\}$ be an edge of G then the following statement are equivalent.

(1) $\gamma_0(G - e) > \gamma_0(G)$
(2) There is a minimum isolate dominating set S of $G - e \ni u, v \in S, v$ is an isolate in $S, P_{extn}[v,\ S]$ is empty and as has an isolate different from v.
(3) For every minimum isolate dominating set T of $G, u \in T, v \notin T$ & $v \in P_{extn}[u, T]$.

Proof: (1)$\Rightarrow$ (3)
Let T be any minimum isolate dominating set of G.
If $u, v \in T$ or $u, v \notin T$ then obviously T is an isolate dominating set in $G - e$.
Therefore, $\gamma_0(G - e) \leq |T| = \gamma_0(G)$.
Which is a contradiction.
Therefore, $u \in T$ & $v \notin T$ or $v \in T$ & $u \notin T$.

We may assume that $u \in T$ & $v \notin T$.
Now $|T| = \gamma_0(G) < \gamma_0(G - e)$.
Therefore, T cannot be an isolate dominating set in $-e$.
Now e has isolated vertices then regarded as a set of vertices of $G - e$.
Therefore, T cannot be dominating set of $G - e$.
Therefore, v is not adjacent to any vertex of T in $G - v$ but v is adjacent to some vertex of T in G.
$\therefore v \in P_n[u, T]$.
Thus, (1)$\Rightarrow$ (3) is proved.

(3)$\Rightarrow$ (2)
Let T be any minimum isolate dominating set of G then $u \in T,\ v \notin T$ & $v \in P_n[u, T]$ in G.
Obviously T cannot be an isolate dominating set in $G - e$.

Let $S = T \cup \{v\}$.
Then $v \in S$ & $u \in S$ & v is an isolate in S.
Obviously, S is an isolate dominating set of $G - e$.

If there is no vertex w outside of S which is adjacent to v then $P_{extn}[v, S] = \emptyset$.
Suppose there is a vertex $w \ni$ w not in G , w is adjacent to v in $G - e$.
Now $w \notin T$ & T is a dominating set of G.
Therefore, w is adjacent to some vertex z of G .
Thus, w is adjacent to two distinct vertices of S in $G - e$.
Thus, $P_{extn}[v, S] = \emptyset$.
Note that an isolate of T in G is also an isolate of S in $G - e$ and it is different from v.

Thus, (3)$\Rightarrow$ (2) is proved.

(2)$\Rightarrow$ (1)
Let S be minimum isolate dominating set of $G - e$ such that $u \in S$ & $v \in S$ & v is an isolate of S, $P_{extn}[v, S] = \emptyset$ and suppose S has an isolate different from v.
Let $T = S - \{v\}$.
Let z be any vertex of G which is not in $G - e$.
If $z = v$ then z is adjacent to u in G.
If $z \neq v$ then $\notin S$.
Suppose z is adjacent to v in $G - e$ then z must be adjacent to some other vertex v' of S because $z \notin P_{extn}[v, S]$.
Then $v' \in T$ & z is adjacent to v' in G.
Therefore, T is a dominating set in G.

Note that T has an isolate because S has an isolate different from v.
Thus, T is an isolate dominating set of G.
Therefore, $\gamma_0(G) \leq |T| < |S| = \gamma_0(G - e)$.
Thus, $\gamma_0(G - e) > \gamma_0(G)$ ▌

Theorem 4.14: Let G be a graph with $\gamma_0(G) = 1$. Let $e = \{uv\}$ be any edge of G. Then $\gamma_0(G - e) > \gamma_0(G)$ if and only if $\{z\}$ is an isolate dominating set of G, then $z \in \{u, v\}$.

Proof: Suppose condition is satisfied.
If $\gamma_0(G - e) = \gamma_0(G)$ then $\gamma_0(G - e) = 1$.
Suppose $\{z\}$ is a minimum isolate dominating set of $G - e$.
Then $z \neq u$ because v is not adjacent to u in $G - e$.

Similarly, if $z \neq v$.
This contradicts condition C.
Therefore, $\gamma_0(G - e) > \gamma_0(G)$ ($\gamma_0(G - e) < \gamma_0(G)$ is not possible because $\gamma_0(G) = 1$).

Conversely, suppose $\gamma_0(G - e) > \gamma_0(G)$.
Let z be a vertex of G $\ni z \notin \{u, v\}$.
If $\{z\}$ is a minimum isolate dominating set of G then $\{z\}$ is also an isolate dominating set of $G - e$. $\Rightarrow \gamma_0(G - e) = \gamma_0(G)$.
Which is a contradiction.
Therefore, $z \in \{u, v\}$ ▌

Theorem 4.15: Let G be a graph with $\gamma_0(G) = 2$ and $e = uv$ be an edge of G. Then $\gamma_0(G - e) > \gamma_0(G)$ if and only if for every minimum dominating set S of G. The following conditions holds:

(1) If S is an independent dominating set then $u \in S$ & $v \notin S$ & $v \in P_{extn}[u, S]$ or $v \in S$ & $u \notin S$ & $u \in P_{extn}[v, S]$.

(2) If S is a total dominating set of G then $u \notin S$ or $v \notin S$.

Proof: Suppose $\gamma_0(G - e) > \gamma_0(G)$.
Suppose S is a minimum dominating set.

(1) Suppose S is an independent set.
If $u \in S$ & $v \in S$ then we have an obvious contradiction because S is an independent set.
If $u \notin S$ & $v \notin S$ then S is an isolate dominating set in $G - e$.
Then $\gamma_0(G - e) \leq |S| = \gamma_0(G)$.
Which contradicts the hypothesis.
Therefore, $u \in S$ & $v \notin S$ or $\in S$ & $u \notin S$.

Suppose $\in S$ & $v \notin S$.
Now suppose $v \notin P_{extn}[u, S]$ then S is an isolate dominating set in $G - e$, which implies that $\gamma_0(G - e) \leq \gamma_0(G)$.
Which is a contradiction.
Therefore, $v \in P_{extn}[u, S]$.
Similarly, if $v \in S$ & $u \notin S$ then $u \in P_{extn}[v, S]$.

(2) Suppose S is a total dominating set.
If $u \in S$ & $v \in S$ then S is an isolate dominating set in $-e$.
Therefore, $\gamma_0(G - e) \leq |S| \leq \gamma_0(G)$.
Which is a contradiction.
Therefore, $u \notin S$ or $v \notin S$.

Conversely, suppose $\gamma_0(G - e) > \gamma_0(G)$.
Let $T \subset V(G)$ be such that $T \neq \emptyset$ & $|T| \leq \gamma_0(G)$.
Suppose $|T| = 1$.
Let $T = \{z\}$.
Suppose T is an isolate dominating set in $G - e$ then $z \neq u$ & $z \neq v$.
Then $\{z\}$ is an isolate dominating set in G which implies that $\gamma_0(G) = 1$.
Which is not true.
Therefore, any set T with $|T| = 1$ can not be an isolate dominating set of $G - e$.
Suppose $T \subset V(G)$ be such that $|T| = 2$ & T is an isolate dominating set of $G - e$.

Let $u \in T$ & $v \in T$.
Then T is a minimum dominating set of G.
Which is a total dominating set and $u, v \in T$.
This contradicts (2).

Suppose $u \in T$ & $v \notin T$.

Now v is an adjacent to some vertex x of T and of course $x \neq u$ because u and v are not adjacent in $G - e$.
Now v is an adjacent two distinct vertex u & x of T in the graph G.
Therefore, $v \notin P_{extn}[u, S]$ in G.

Similarly, If $v \in T$ & $u \notin T$ then $u \notin P_{extn}[v, T]$ in G.
Thus, if $u \in T$ & $v \notin T$ or $v \in T$ & $u \notin T$ gives rise to a contradiction.

Therefore, $\notin T$ & $v \notin T$.
Therefore, T is a minimum isolate dominating set of G such that $u \notin T$ & $v \notin T$.
Which is contradict (1).

Therefore, any set T with $|T| = 2$ can not be an isolate dominating set of $G - e$.
Thus, we conclude that any set T of vertices of $G - e$ with $|T| \leq 2$ can not be an isolate dominating set of $G - e$.
Therefore $\gamma_0(G - e) > 2 = \gamma_0(G)$. ▌

Conclusion:

If G is a graph , $v \in V(G)$ and suppose G has a minimum isolate dominating set S which has an isolate different from v then v gives rises to two distinct vertices v_1 & v_2 ∋ $\gamma_0(G - v_i) = \gamma_0(G)$ or $i = 1,2$. Thus, it follows that for any graph G which contains a minimum isolate dominating set of S with isolate u and if there is a vertex v inV_0^+ then $V_0^0 \geq 2|V_0^+|$.

References

1. B. H. Arriola, Isolate domination in the join and corona of graphs, Appl. Math. Sci. 9, (2015), 1543-1549.
2. C. Berge, Theory of Graphs and its Applications, Methuen, London (1962).
3. D. B. West, Introduction to Graph Theory, Prentice Hall of India, New Delhi (2006).
4. D. K. Thakkar and J. C. Bosamiya, Graph critical with respect to independent domination, Journal of Discrete Mathematical Sciences and Cryptography, 16, (2013), 179-186.
5. E. J. Cockayne, B. Gamble and B. Shepherd, An upper bound for the k-domination number of a graph, J. Graph Theory, 9, (1985), 533-534.
6. E. J. Cockayne, R. M. Dawes and S. T. Hedetniemi, Total Domination in Graphs, Networks, 10, (1980), 211-219.
7. F. Harary, Graph Theory, Addison Wesley, Reading Mass., (1969).
8. G. Chartrand and L. Lesniak, Graphs and Digraphs, 4th ed., Chapman and Hall/CRC (2005)
9. I. S. Hamid, S. Balamurugan and A. Navaneethakrishnan, Electronic Journal of Graph Theory and Applications 4(1), (2016), 94-100.

10. I. S. Hamid and S. Balamurugan, Isolate Domination in Unicyclic Graphs, International Journal of Mathematics and soft computing Vol 3, No.3, (2013), 79-83.
11. I. Sahul Hamid and S. Balamurugan, ”Isolate Domination Number and Maximum Degree” bulletin of the international mathematical virtual institute Vol.3(2013),127-133.
12. I. Sahul Hamid and S. Balamurugan, “Isolate Domination in Graphs”, Arab Journal of Mathematical Sciences Math Sci 22(2016), 232-241.
13. J. A. Bondy and U. S. R. Murty, Graph Theory, Springer (2008).
14. J. Clerk and D. A. Holton, A First look at Graph Theory, World Scientic (1991).
15. J. F. Fink and M. S. Jacobson, n-domination in Graphs, In Y. Alavi and A. J. Schwenk editiors, Graph Theory with Applications to Algorithms and Computer Science, pages 23-300, (Kalamazoo, MI 1984), (1985) Wiley.
16. J. L. Gross and J. Yellen, Graph Theory and its Applications, CRC Press (1998).
17. Michael A. Henning., Anders Yeo “Total Domination in Graphs”, Springer, New York, (2013).
18. M. A. Henning and A. Yeo, Total Domination in Graphs, Springer, New York (2013).
19. M. A. Henning, O. R. Oellermann and H. C. Swart, Bounds on distance Domination Parameters, J. Combin Inform. System Sci., 16, (1991), 11-18.
20. R. Balakrishnan and K. Ranganathan, A Textbook of Graph Theory, Springer, New York (2012).
21. R. J. Wilson and J. J. Watkins, An Introductory Approach, John Wiley & Sons, Inc. New York (1990).
22. S. Balamurugan, Changing and unchanging isolate domination: Edge removal Discrete mathematics, Algorithms and Applications Vol. 9, No.1, (2017).
23. S. T. Hedetniemi and R. Laskar, Eds. Topics in Domination in Graphs, Discrete mathematics, 86 (1990).
24. T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Domination in Graphs Advanced Topics, Marcel Dekker, Inc., New York (1998).
25. T. W. Haynes, S. T. Hedetniemi, P. J. Slater, “Fundamental of Domination In graphs”, Marcel Dekker, New York, (1998).