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Symbolic Computational Algorithm for Hirota Bilinear Form to Higher-dimensional Nonlinear Partial Differential Equations in Nonlinear Sciences

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Abstract:

In the physical world, many real systems are governed by nonlinear partial differential equations from fluid dynamics and plasma phyasics to shallow-water waves and oceanographic systems. There is no uniform approach for solving nonlinear partial differential equations; consequently, we consider each equation as a separate problem. The most effective technique for building multi-soliton solutions of an integrable nonlinear PDE is the Hirota direct method from the several methods used to explore nonlinear PDEs to obtain solitons, lumps, rogue waves, breathers, and kink waves. To derive the multi-solitons of the non-linear PDEs, this method requires first transforming the equation into the bilinear form proposed by Hirota, and then applying the dependent variable transformation. Converting nonlinear evolution equations into Hirota bilinear form is the primary goal of the research study. This work investigated several well-known nonlinear PDEs, including the KdV Equation, Boussinesq Equation, GS Equation, KP equation, and other equations in order to comprehend and apply the concept. We design the algorithm using symbolic software *Mathematica*. These equations are widely recognized for multi-solitons and their integrability, which having applications in diverse fileds oceanography, fluid dynamics, plasma physics, mechanics, and other nonlinear sciences.

Keyword: Symbolic Computation, Bilinear Form, Nonlinear PDE, Symbolic Algorithm

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1 Introduction

In mathematics and science, a partial differential equation that has nonlinear components is called a nonlinear evolution equation or partial differential equation (PDE). They describe distinct physical systems from water dynamics to gravity, and have been used in mathematics to solve important problems such as the Poincaré and Calabi conjectures. They are challenging to analyze since there aren't many general techniques that work for all of these equations, and usually, each one needs to be looked at separately. The Hirota method, which Ryogo Hirota introduced in 1971 [1–5], is a popular and powerful mathematical tool for identifying soliton solutions after changing the PDE to bilinear form. In several disciplines, such as oceanography, fluid dynamics, optics, plasma physics, and engineering sciences, soliton is a highly regular solution that has wave-like properties. Therefore bilinearization of nonlinear PDEs is the most necessary step.

In physics and mathematics, a soliton is a confined wave packet that is extremely stable, nonlinear, and self-reinforcing. After colliding with other localized wave packets, it maintains its form even when moving freely at a constant speed. Its exceptional stability results from the medium's dispersive and nonlinear effects being balancedly cancelled. Weakly nonlinear dispersive PDEs have wide group to describe nonlinear systems was eventually demonstrated to have stable solutions in soliton theory [6].

For integrable non-linear PDEs, precise N-soliton or multisoliton solutions may be found using Hirota's bilinear approach. However, Hirota asserts that converting a non-linear PDE into its bilinear form is the most crucial step in this process. Even when the correct transformation of the dependant variable is known, the process of converting a nonlinear evolution equation into a bilinear equation becomes time-consuming. As a result, creating a method to determine a nonlinear PDE's bilinear form is crucial. To do these calculations, system software like *Maple Matlab*, and *Mathematica,* are useful. Hirota first applied this method to the KdV equation

$$\chi_t + 6\chi\chi_x + \chi_{xxx} = 0,$$

converting it into the bilinear form

$$(D_x^4 + D_x D_t)\, h \cdot h = 0,$$

by using the transformation $\chi = 2(\ln h)_{xx}$, which resulted in a simple soliton solution development [3]. Many scholars have been interested in the nonlinearity of PDEs and have used a number of systematic techniques to obtain multiple lump, breather, and soliton solutions. There are several different methods that are to be used such as Bäcklund transformation [7, 8], Darboux transformation [9–12], Lie symmetry analysis [13, 14], simplified Hirota method [15, 16], Pfaffian technique [17, 18], and other methods.

In this work, we investigate how several nonlinear PDEs, including the Korteweg-de Vries (KdV) [4], Kadomtsev–Petviashvili (KP) equations [5], and other equations that are transformed into Hirota bilinear form. The most effective technique for building multi-soliton solutions of an integrable nonlinear PDE is the Hirota direct approach. Solitons are crucial to the analysis of shallow water waves because they are created when the nonlinearity and dispersion effect are ignored. Moreover, they are found in a number of disciplines, including fluid dynamics, dusty plasma, oceanography, marine engineering, and plasma physics.

The following section 2 explores the general structure of the bilinear form in D-operator for a general nonlinear PDE. It describe the Cole-Hopf transformation, Hirota's D-operator and algorithm for bilinear form. In section 3, we show the application of algorithm to several well-known equations to obtain the bilinear forms. Section 4 discusses the results of the bilinear form for the studied equations, and the ending section concludes the research work.

2 Hirota Bilinear Form of a nonlinear PDE

The algorithm for obtaining the Hirota Bilinear Form and its application to differential equation solving will be the main topics of the next part. Hirota provided an algebraic method for finding precise soliton solutions, provided that the nonlinear PDE could be transformed into bilinear form. Initially, dependent variable transformation was used to convert the PDE to a bilinear equation for an auxiliary function [1]. As we will see, the concept is based on the properties of a certain bilinear differential operator known as the Hirota bilinear operator (D-operator) [19].

2.1 Cole–Hopf transformations

A mathematical method for investigating partial differential equations (PDEs), especially nonlinear PDEs, is the Cole–Hopf transformation. Cole and Hopf [20, 21] created the transformation in the 1950s to simplify and sometimes linearize certain types of nonlinear PDEs.The Cole–Hopf transformation has been found to be a great mathematical tool to explore solitons and the integrable equations. It helps scientists to investigate the wave structures of nonlinear equations to explore the existence of solitary waves, lump solutions, and several others that can persist in specific nonlinear systems. We have Cole-Hopf transformation

$$u = K\left(\log h\right)_{x^p},$$

where p is the order of partial derivative in x based on the balance between the PDE's higher-order and nonlinear terms. The phase variable must be used to obtain the dispersion in order to create the mentioned transformation [22], which we'll discuss through examples in a further section.

2.2 The Hirota Bilinear Operator

Hirota gave the D-operator, a binary form that outputs a new function after receiving two functions as input. It has numerous qualities that make it helpful for differential equation analysis. In particular, it enables us to identify soliton solutions, which are analytic solutions to them. A pair of functions provide the operator's definition as (h, k) of a real variable a [19], is

$$D_a(h,k) = \lim_{a' \to a}\left(\frac{\partial}{\partial a} - \frac{\partial}{\partial a'}\right) h(a)k(a'). \tag{1}$$

When the operator is applied repeatedly and to distinct variables (a and b in this case), but it is obviously generalizable to any number of real variables, we expand the definition [4] to

$$D_a^p D_b^q(h,k) = \lim_{a' \to a,\, b' \to b}\left(\frac{\partial}{\partial a} - \frac{\partial}{\partial a'}\right)^p \left(\frac{\partial}{\partial b} - \frac{\partial}{\partial b'}\right)^q h(a,b)k(a',b').$$

Below are some brief outcomes to help the D-operator get some intuition.

$$D_y(h \cdot k) = h_y k - h k_y,$$

$$D_x^4(h \cdot k) = h_{xxxx}k - 4.h_{xxx}k_x + 6.h_{xx}k_{xx} - 4.h_x k_{xxx} + hk_{xxxx},$$
$$D_y^2(h.k) = h_{yy}k + hk_{yy} + 2h_y k_y$$

Actually, we may observe the following formula, which is created by applying binomial expansion to powers of (1), for the expansion of the nth power of the operator.,

$$D_x^n f_1 \cdot f_2 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \partial_x^{n-k} f_1(x)\, \partial_x^k f_2(x). \tag{2}$$

2.3 Algorithm for bilinear Form

We begin with a nonlinear PDE of the form in $(n + 1)$ dimensions as

$$P(u, u_{x_1}, u_{x_2}, \ldots) = 0, \tag{3}$$

where the function $u = u(x_1, x_2, \ldots, x_n, t)$ depends on the spatial variables $x_1, x_2, \ldots, x_n$ and the temporal variable t, and the operator P involves u together with its partial derivatives.

Step 1: *Analysis of the dispersion relation*
We introduce the phase variable ξ_i as

$$\xi_i = k_{1_i}x_1 + k_{2_i}x_2 + k_{3_i}x_3 + \cdots + k_{n_i}x_n + \omega_i t, \tag{4}$$

where ω_i represents the dispersion relation and the coefficients k_{N_i} $(1 \leq N \leq n)$ are constant wave numbers. The above form of the phase variable is a standard choice, illustrating the adaptability of the method to different classes of nonlinear PDEs. Depending on the structure of the PDE, however, this form may vary. We substitute the trial solution $u = e^{\xi_i}$ in linear terms of Eq. (3) and solve for dispersion ω_i.

Step 2: *Determining the transformation constant R*
We now apply the logarithmic transformation

$$u = K\frac{\partial^\eta}{\partial \chi^\eta}(\ln h), \tag{5}$$

where η is chosen so that the order of the highest derivative in Eq. (3) balances with the nonlinear terms. For testing purposes, we consider $h = 1 + e^{\xi_1}$. Substituting this into Eq. (5) provides different possible values of the parameter K, from which a suitable one can be selected.

Step 3: *Deriving a quadratic or quartic equation in* Φ
Substituting the transformation (5) into the original PDE (3) and integrating in x (at least once) leads to an algebraic relation in terms of h. By setting the integration constant to zero at the lowest admissible order, we obtain either a quadratic or a quartic form,

$$Q(h, h_{x_1}, h_{x_2}, \ldots) = 0, \tag{6}$$

where Q involves h together with its derivatives with respect to t and the spatial variables $x_1, x_2, \ldots, x_n$.

Step 4: *Reformulation into Hirota's bilinear form*
The quadratic structure obtained in Step 3 can be expressed in terms of Hirota's bilinear operators. The Hirota derivative is defined by

$$D_a^p D_b^q(h,k) = \lim_{a'\to a,\, b'\to b}\left(\frac{\partial}{\partial a} - \frac{\partial}{\partial a'}\right)^p \left(\frac{\partial}{\partial b} - \frac{\partial}{\partial b'}\right)^q h(a,b)k(a',b').$$

Using this representation, Eq. (6) can be rewritten in bilinear form, thereby yielding the Hirota bilinear expression corresponding to the original nonlinear PDE (3).

3 Applications of Hirota Bilinear form

The process for obtaining Hirota bilinear forms is often illustrated using classical examples of integrable equations, such as the KdV and the KP equation. These equations are essential to the theory of solitons and provide excellent examples of how well Hirota's approach works.

3.1 KdV equation in (1+1)-dimensions

The nonlinear KdV equation models the evolution of long, weak nonlinear waves in one spatial dimension [22]. Originally, it was described in hydro-dynamics to discuss the wave motion in shallow water. This equation is

$$u_t + 6uu_x + u_{xxx} = 0, \tag{7}$$

where x denotes the spatial coordinate, t represents time, and u as dependent variable for wave amplitude.

A remarkable property of Eq. (7) is the existence of *solitons*. The persistence of solitons is a direct consequence of the balance between the nonlinear and dispersive contributions in the equation. Because of this, the KdV equation has become a central model in the study of nonlinear wave dynamics, with applications ranging across oceanography, plasma physics, and optics.

To explore the dispersion relation in the KdV equation (7), we introduce the phase variable

$$\xi_i = \mu_i x + d_i t,$$

where $\mu_i\,(i = 1, 2, \ldots)$ are constants and d_i is the dispersion parameter. Substituting the trial solution $u = e^{\xi_i}$ into the linear part of Eq. (7) gives

$$u = e^{\xi_i}, \quad u_t = d_i e^{\xi_i}, \quad u_{xxx} = \mu_i^3 e^{\xi_i}.$$

Thus, from the relation

$$u_t + u_{xxx} = 0,$$

we obtain

$$d_i e^{\xi_i} + \mu_i^3 e^{\xi_i} = e^{\xi_i}\left(\mu_i^3 + d_i\right) = 0,$$

which implies

$$d_i = -\mu_i^3.$$

Hence, the phase variable becomes

$$\xi_i = \mu_i x - \mu_i^3 t.$$

Next, we construct solutions of Eq. (7) using the Cole–Hopf transformation, which introduces a logarithmic representation of the dependent variable:

$$u = K(\log h)_{xx}, \tag{8}$$

where $h = h(x,t)$. It may be written as

$$u = w_{xx}, \quad \text{with} \quad w = K(\log h). \tag{9}$$

To determine the value of K, we choose the auxiliary function in the Cole–Hopf transformation as

$$h(x,t) = 1 + e^{\xi_1} = 1 + e^{\mu_1 x + d_1 t} = 1 + e^{\mu_1 x - \mu_1^3 t}. \tag{10}$$

Substituting Eq. (10) into Eq. (7) and get the K as

$$K = 2.$$

Therefore, the logarithmic transformation reduces to

$$u = 2(\log K)_{xx}. \tag{11}$$

Now, from (9), we have

$$u_t = w_{xxt}, \quad u_x = w_{xxx} \quad and \quad u_{xxx} = w_{xxxxx}$$

putting the above expressions into Eq.(7), we get

$$w_{xxt} + 6\, w_{xx} w_{xxx} + w_{xxxxx} = 0 \tag{12}$$

on integrating w.r.t. x

$$w_{xt} + 6\int w_{xx}w_{xxx}\,\partial x + w_{xxxx} = 0. \tag{13}$$

We compute integral term in equation (13) with constant of integration as zero

$$I = \int w_{xx}w_{xxx}\,\partial x = \frac{1}{2}\int 2w_{xx}w_{xxx}\,\partial x = \frac{1}{2}w_{xx}^2, \tag{14}$$

substituting the value of I in equation (13), we get

$$w_{xt} + 3w_{xx}^2 + w_{xxxx} = 0. \tag{15}$$

As we have $w = 2(\log f)$, we can get the followings:

$$w_x = 2\frac{h_x}{h},$$

$$w_{xt} = 2\frac{hh_{xt} - h_xh_t}{h^2},$$

$$w_{xx} = 2\frac{hh_{xx} - h_x^2}{h^2},$$

$$w_{xxx} = 2\frac{2h_x^3 - 3hh_xh_{xx} + h^2h_{xxx}}{h^3},$$

$$w_{xxxx} = 2\frac{-6h_x^4 + 12hh_x^2h_{xx} - 3h^2h_{xx}^2 - 4h_xh^2h_{xxx}}{h^4},$$

putting all the above values in equation (15), we get a quadratic equation in h as

$$-2\frac{h_xh_t}{h^2} + 2\frac{h_{xt}}{h} + 6\frac{h_{xx}^2}{h^2} - 8\frac{h_xh_{xxx}}{h^2} + 2\frac{h_{xxxx}}{h} = 0,$$

or

$$hh_{xt} - h_xh_t + 3h_{xx}^2 - 4h_xh_{xxx} + hh_{xxxx} = 0, \tag{16}$$

Since Hirota bilinear operator is defined as

$$D_a^pD_b^q(h,k) = \lim_{a'\to a,\, b'\to b}\left(\frac{\partial}{\partial a} - \frac{\partial}{\partial a'}\right)^p\left(\frac{\partial}{\partial b} - \frac{\partial}{\partial b'}\right)^q h(a,b)k(a',b').$$

let $p = 1$ and $q = 1$, then

$$D_xD_t(h.k) = h_{xt}k - h_tk_x - h_xk_t + hk_{xt},$$

$$D_xD_t(h.h) = 2\left(hh_{xt} - h_xh_t\right)$$

let $p = 4$ and $q = 0$, then

$$D_x^4(h.k) = h_{xxxx}k - 4.h_{xxx}k_x + 6.h_{xx}k_{xx} - 4.h_xk_{xxx} + hk_{xxxx}$$

$$D_x^4(h.h) = 2\left(3h_{xx}^2 - 4h_xh_{xxx} + hh_{xxxx}\right)$$

Equation (16) can be rewritten in terms of the Hirota D operator as

$$(D_xD_t + D_x^4)h.h = 0 \tag{17}$$

This equation (17) is called the Hirota bilinear form for KDV equation (7).

3.2 Boussinesq Equation in (1+1)-dimensions

The integrable Boussinesq equation [23] is structured as

$$u_{tt} - 3(u^2)_{xx} - u_{xx} - u_{xxxx} = 0. \tag{18}$$

To investigate the dispersion relation for Eq. (18), we introduce the phase variable

$$\xi_i = \mu_i x + d_i t, \tag{19}$$

where $\mu_i\,(i = 1, 2, \ldots)$ are constants and d_i is the dispersion parameter. Substituting the trial solution $u = e^{\xi_i}$ into the linear part of Eq. (18), namely

$$u_{tt} - u_{xx} - u_{xxxx} = 0,$$

leads to the relation

$$d_i = \sqrt{\mu_i^2 + \mu_i^4}.$$

Thus, the phase variable becomes

$$\xi_i = \mu_i x + \sqrt{\mu_i^4 + \mu_i^2}\, t.$$

Next, we apply a logarithmic-type transformation of the dependent variable:

$$u = K(\ln h)_{xx}, \tag{20}$$

which may be equivalently expressed as

$$u = w_{xx}, \quad \text{where} \quad w = K(\ln h). \tag{21}$$

Choosing the function h as

$$h = 1 + e^{\xi_1}, \quad \text{with} \quad \xi_1 = \mu_1 x + \sqrt{\mu_1^2 + \mu_1^4}\, t,$$

and substituting into Eq. (18), we obtain $K = 2$.
Hence, the transformation (20) simplifies to

$$u(x,t) = 2(\ln h)_{xx}.$$

Now, from (21), we have

$$u_{tt} = w_{xxtt}, \quad u_{xx} = w_{xxxx} \quad \textit{and} \quad u_{xxx} = w_{xxxxx}$$

putting the above expressions into Eq.(18), we get

$$w_{xxtt} + w_{xxxx} - 3\,(w_{xx}^2)_{xx} - w_{xxxxxx} = 0 \tag{22}$$

on integrating w.r.t. x

$$w_{xtt} + w_{xxx} - 3\,(w_{xx}^2)_{x} - w_{xxxxx} = 0 \tag{23}$$

on again integrating w.r.t. x

$$w_{tt} + w_{xx} - 3\,(w_{xx}^2) - w_{xxx} = 0 \tag{24}$$

As we have $w = 2(\log h)$, we can get the followings:

$$w_x = 2\frac{h_x}{h},$$

$$w_{xx} = 2\frac{hh_{xx} - h_x^2}{h^2},$$
$$w_{xxx} = 2\frac{2h_x^3 - 3hh_xh_{xx} + h^2h_{xxx}}{h^3},$$
$$w_{tt} = 2\frac{hh_{tt} - h_t^2}{h^2},$$

which changes Eq.(18) into a bilinear equation in h as

$$hh_{tt} - h_t^2 - hh_{xx} + h_x^2 - hh_{xxxx} + 4h_xh_{xxx} - 3h_{xx}^2 = 0 \tag{25}$$

Since Hirota bilinear operator is defined as

$$D_a^pD_b^q(h,k) = \lim_{a'\to a,\, b'\to b}\left(\frac{\partial}{\partial a} - \frac{\partial}{\partial a'}\right)^p\left(\frac{\partial}{\partial b} - \frac{\partial}{\partial b'}\right)^q h(a,b)k(a',b').$$

let $p = 0$ and $q = 2$, then

$$D_t^2(h.k) = h_{tt}k - 2h_tk_t + hk_{tt},$$
$$D_t^2(h.h) = 2\left(hh_{tt} - h_t^2\right)$$

let $p = 2$ and $q = 0$, then

$$D_x^2(h.k) = h_{xx}k - 2h_xk_x + hk_{xx},$$
$$D_x^2(h.h) = 2\left(hh_{xx} - h_x^2\right)$$

let $p = 4$ and $q = 0$, then

$$D_x^4(h.k) = h_{xxxx}k - 4.h_{xxx}k_x + 6.h_{xx}k_{xx} - 4.h_xk_{xxx} + hk_{xxxx}$$
$$D_x^4(h.h) = 2\left(3.h_{xx}^2 - 4.h_xh_{xxx} + hh_{xxxx}\right)$$

Equation (25) can be written in terms of the operator D as

$$(D_t^2 - D_x^2 - D_x^4)h.h = 0 \tag{26}$$

This equation (26) is called the Hirota bilinear form for Boussinesq equation (18).

3.3 KP equation in (2+1)-dimensions

The KP equation extends the KdV model, which was originally formulated in two dimensions [24]. Its mathematical form is

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0, \tag{27}$$

where x and y and t are spatial and temporal variable, and u represents the wave amplitude.
The KP equation is a fundamental model in the theory of solitons and integrable systems, as it admits solutions that describe nonlinear wave interactions in two dimensions. Similar to the KdV equation, it supports soliton solutions; however, owing to its higher dimensionality, it captures more intricate behaviors, such as two-soliton interactions with nontrivial structures. Applications of the KP equation arise in diverse areas, including oceanography, plasma physics, and fluid mechanics, where it provides insight into multidimensional nonlinear wave dynamics.
For the KP equation (27), we define the phase variable as

$$\xi_i = \mu_ix + v_iy - d_it, \tag{28}$$

where $\mu_i, v_i\,(i = 1, 2, \ldots)$ are constants, and d_i denotes the dispersion parameter. Substituting the
trial solution $u = e^{\xi_i}$ in linear part of Eq. (27), we obtain the dispersion as

$$d_i = \frac{\mu_i^4 - v_i^2}{\mu_i}. \tag{29}$$

Next, we introduce the transformation

$$u(x, y, t) = K(\ln h)_{xx}, \tag{30}$$

which may also be written as

$$u = w_{xx}, \quad \text{with} \quad w = K(\ln h). \tag{31}$$

Choosing the function h as

$$h = 1 + e^{\xi_1}, \quad \text{where} \quad \xi_1 = \mu_1 x + v_1 y - \left(\frac{\mu_1^4 - v_1^2}{\mu_1}\right) t,$$

and substituting into Eq. (27), the value of K is determined to be $K = 2$.
Thus, the logarithmic transformation (30) reduces to

$$u = 2(\ln h)_{xx}. \tag{32}$$

Now, from (31), we have

$$u_t = w_{xxt}, \quad u_x = w_{xxx}, \quad u_{xxx} = w_{xxxxx} \quad \mathit{and} \quad u_{yy} = w_{xxyy}$$

putting the above expressions into Eq.(27), we get

$$(w_{xxt} + 6\,w_{xx}w_{xxx} + w_{xxxxx})_x - w_{xxyy} = 0 \tag{33}$$

on integrating w.r.t. x

$$w_{xxt} + 6w_{xx}w_{xxx} + w_{xxxxx} - w_{xyy} = 0. \tag{34}$$

on again integrating w.r.t. x

$$w_{xt} + 6\int w_{xx}w_{xxx}\,\partial x + w_{xxxx} - w_{yy} = 0. \tag{35}$$

We compute integral term in equation (35) with constant of integration as zero

$$I = \int w_{xx}w_{xxx}\,\partial x = \frac{1}{2}\int 2w_{xx}w_{xxx}\,\partial x = \frac{1}{2}w_{xx}^2, \tag{36}$$

substituting the value of I in equation (35), we get

$$w_{xt} + 3w_{xx}^2 + w_{xxxx} - w_{yy} = 0. \tag{37}$$

As we have $w = 2(\log h)$, we can get the followings:

$$w_x = 2\frac{h_x}{h},$$

$$w_{xt} = 2\frac{hh_{xt} - h_x h_t}{h^2},$$

$$w_{xx} = 2\frac{hh_{xx} - h_x^2}{h^2},$$
$$w_{xxx} = 2\frac{2h_x^3 - 3hh_xh_{xx} + h^2h_{xxx}}{h^3},$$
$$w_{xxxx} = 2\frac{-6h_x^4 + 12hh_x^2h_{xx} - 3h^2h_{xx}^2 - 4h_xh^2h_{xxx}}{h^4},$$
$$w_{yy} = 2\frac{hh_{yy} - h_y^2}{h^2},$$

which converts Eq.(27) into a bilinear equation in h as

$$hh_{xt} - h_xh_t + 3h_{xx}^2 - 4h_xh_{xxx} + hh_{xxxx} - hh_{yy} + h_y^2 = 0. \tag{38}$$

Since Hirota bilinear operator is defined as

$$D_a^pD_b^qD_c^r(h,k) = \lim_{a'\to a,\, b'\to b,\, c'\to c}\left(\frac{\partial}{\partial a} - \frac{\partial}{\partial a'}\right)^p\left(\frac{\partial}{\partial b} - \frac{\partial}{\partial b'}\right)^q\left(\frac{\partial}{\partial c} - \frac{\partial}{\partial c'}\right)^r h(a,b,c)k(a',b',c').$$

let $p = 1$,$q = 0$ and $r = 1$, then

$$D_xD_t(h.k) = h_{xt}k - h_tk_x - h_xk_t + hk_{xt},$$
$$D_xD_t(h.h) = 2\left(hh_{xt} - h_xh_t\right)$$

let $p = 4$, $q = 0$ and $r = 0$, then

$$D_x^4(h.k) = h_{xxxx}k - 4h_{xxx}k_x + 6h_{xx}k_{xx} - 4h_xk_{xxx} + hk_{xxxx}$$
$$D_x^4(h.h) = 2\left(3h_{xx}^2 - 4h_xh_{xxx} + hh_{xxxx}\right)$$

let $p = 0$, $q = 2$ and $r = 0$, then

$$D_y^2(h.k) = h_{yy}k + hk_{yy} + 2h_yk_y$$
$$D_y^2(h.h) = 2\left(hh_{yy} - h_y^2\right)$$

Therefore, using bilinear differentials D, the bilinear Eq. (38) can be expressed in Hirota's bilinear form as

$$\left[D_xD_t + D_x^4 - D_y^2\right]h.h = 0 \tag{39}$$

This equation (39) is called the Hirota bilinear form for KP equation (27).

3.4 KP equation with variable coefficients in (2+1)-dimensions

The nonlinear KP equation with a variable coefficient represents a generalization of the standard KP equation [25]. Its mathematical form is

$$\left(u_t + uu_x + u_{xxx}\right)_x + g(t)u_{xy} + 3u_{yy} = 0, \tag{40}$$

where u denotes the wave and x, y, t are spatial and temporal coordinates, and $g(t)$ is a time-dependent coefficient that modulates the coupling between the x and y directions.
To determine the dispersion relation, we introduce the phase variable

$$\xi_i = \mu_ix + v_iy - d_i(t), \tag{41}$$

where $\mu_i, v_i\,(i = 1, 2, \ldots)$ are constants and $d_i(t)$ is the time-dependent dispersion function.
Substituting
$u = e^{\xi_i}$ into the linear terms of Eq. (40) yields

$$(u_t + u_{xxx})_x + g(t)u_{xy} + 3u_{yy} = 0.$$

The relevant partial derivatives are

$$\begin{aligned} u_t &= -d_i'(t)e^{\xi_i}, \\ u_x &= \mu_i e^{\xi_i}, \quad u_{xxx} = \mu_i^3 e^{\xi_i}, \\ u_{yy} &= v_i^2 e^{\xi_i}, \\ u_{xy} &= \mu_i v_i e^{\xi_i}. \end{aligned}$$

Substituting these expressions into the linearized equation gives

$$\left(-d_i'(t)e^{\xi_i} + \mu_i^3 e^{\xi_i}\right)_x + 3v_i^2 e^{\xi_i} + g(t)\mu_i v_i e^{\xi_i} = 0,$$

which simplifies to

$$(-d_i'(t) + \mu_i^3)\mu_i + 3v_i^2 + g(t)\mu_i v_i = 0.$$

Solving for $d_i'(t)$ leads to

$$d_i'(t) = \mu_i^3 + \frac{3v_i^2}{\mu_i} + g(t)v_i,$$

and integrating over time gives the dispersion relation

$$d_i(t) = \int \left(\mu_i^3 + g(t)v_i + \frac{3v_i^2}{\mu_i}\right) dt. \tag{42}$$

To construct solutions, we apply the transformation

$$u(x, y, t) = K(\ln h)_{xx}, \tag{43}$$

which can equivalently be expressed as

$$u = w_{xx}, \quad \text{where} \quad w = K(\ln h). \tag{44}$$

Choosing the function h as

$$h = 1 + e^{\xi_1}, \quad \text{with} \quad \xi_1 = \mu_1 x + v_1 y - \int \left(\mu_1^3 + g(t)v_1 + \frac{3v_1^2}{\mu_1}\right) dt,$$

and substituting into Eq. (40), we find that $K = 12$. Therefore, the transformation (43) reduces to

$$u = 12(\ln h)_{xx}. \tag{45}$$

Now, from (44), we have

$$u_t = w_{xxt}, \quad u_x = w_{xxx}, \quad u_{xxx} = w_{xxxxx}, \quad u_{yy} = w_{xxyy} \quad and \quad u_{xy} = w_{xxxy}$$

putting the above expressions into Eq.(40), we get

$$(w_{xxt} + w_{xx}w_{xxx} + w_{xxxxx})_x + 3w_{xxyy} + g(t)w_{xxxy} = 0 \tag{46}$$

on integrating w.r.t. x

$$w_{xxt} + w_{xx}w_{xxx} + w_{xxxxx} + 3w_{xyy} + g(t)w_{xxy} = 0. \tag{47}$$

on again integrating w.r.t. x

$$w_{xt} + \int w_{xx}w_{xxx}\,\partial x + w_{xxxx} + 3w_{yy} + g(t)w_{xy} = 0. \tag{48}$$

We compute integral term in equation (48) with constant of integration as zero

$$I = \int w_{xx}w_{xxx}\,\partial x = \frac{1}{2}\int 2w_{xx}w_{xxx}\,\partial x = \frac{1}{2}w_{xx}^2, \tag{49}$$

substituting the value of I in equation (48), we get

$$w_{xt} + \frac{w_{xx}^2}{2} + w_{xxxx} + 3w_{yy} + g(t)w_{xy} = 0. \tag{50}$$

As we have $w = 12(\log h)$, we can get the followings:

$$w_x = 12\frac{h_x}{h},$$

$$w_{xt} = 12\frac{hh_{xt} - h_xh_t}{h^2},$$

$$w_{xx} = 12\frac{hh_{xx} - h_x^2}{h^2},$$

$$w_{xxx} = 12\frac{2h_x^3 - 3hh_xh_{xx} + h^2h_{xxx}}{h^3},$$

$$w_{xxxx} = 12\frac{-6h_x^4 + 12hh_x^2h_{xx} - 3h^2h_{xx}^2 - 4h_xh^2h_{xxx}}{h^4},$$

$$w_{yy} = 12\frac{hh_{yy} - h_y^2}{h^2},$$

$$w_{xy} = 12\frac{hh_{xy} - h_xh_y}{h^2},$$

which converts Eq.(40) into a bilinear equation in h as

$$hh_{xt} - h_xh_t + 3h_{xx}^2 - 4h_xh_{xxx} + hh_{xxxx} + 3hh_{yy} - 3h_y^2 - g(t)h_yh_x + g(t)hh_{xy} = 0. \tag{51}$$

Since Hirota bilinear operator is defined as

$$D_a^pD_b^qD_c^r(h,k) = \lim_{a'\to a,\, b'\to b,\, c'\to c}\left(\frac{\partial}{\partial a} - \frac{\partial}{\partial a'}\right)^p\left(\frac{\partial}{\partial b} - \frac{\partial}{\partial b'}\right)^q\left(\frac{\partial}{\partial c} - \frac{\partial}{\partial c'}\right)^r h(a,b,c)k(a',b',c').$$

let $p = 1$,$q = 0$ and $r = 1$, then

$$D_xD_t(h.k) = h_{xt}k - h_tk_x - h_xk_t + hk_{xt},$$

$$D_xD_t(h.h) = 2\left(hh_{xt} - h_xh_t\right)$$

let $p = 4$, $q = 0$ and $r = 0$, then

$$D_x^4(h.k) = h_{xxxx}k - 4h_{xxx}k_x + 6h_{xx}k_{xx} - 4h_xk_{xxx} + hk_{xxxx}$$

$$D_x^4(h.h) = 2\left(3h_{xx}^2 - 4h_x h_{xxx} + hh_{xxxx}\right)$$

let $p = 0$, $q = 2$ and $r = 0$, then

$$D_y^2(h.k) = h_{yy}k + hk_{yy} + 2h_y k_y$$

$$D_y^2(h.h) = 2\left(hh_{yy} - h_y^2\right)$$

let $p = 1$,$q = 1$ and $r = 0$, then

$$D_x D_y(h.k) = h_{xy}k - h_y k_x - h_x k_y + hk_{xy},$$

$$D_x D_y(h.h) = 2\left(hh_{xy} - h_x h_y\right)$$

Therefore, using bilinear differentials D, the Eq. (51) gives the Hirota's bilinear form as

$$\left[D_x D_t + D_x^4 + 3D_y^2 + g(t)D_x D_y\right] h.h = 0 \tag{52}$$

This equation (52) is called the Hirota bilinear form for KP equation with variable coefficient (40).

3.5 Graphene-Sheets Equation in (2+1)-dimensions

Let us consider a variable-coefficient equation that models the thermophoretic waves in graphene sheets [26]. The equation is expressed as

$$u_{xt} + \left(uu_x + u_{xxx} + (\alpha(t) + \beta)u_x\right)_x + \gamma(t)u_{yy} = 0, \tag{53}$$

where $u = u(x, y, t)$ represents the thermophoretic displacement, x and y are the longitudinal and lateral coordinates, t is time, $\alpha(t)$ and β denote coefficient of thermal conductivity and $\gamma(t)$ as lateral dispersion coefficient.
We take the phase variable as

$$\xi_i = \mu_i x + v_i y - d_i(t), \tag{54}$$

where $\mu_i, v_i\,(i = 1, 2, \ldots)$ are constants and $d_i(t)$ is a time-dependent dispersion function. Substituting $u = e^{\xi_i}$ into the linear part of Eq. (53) gives

$$u_{xt} + u_{xxxx} + (\alpha(t) + \beta)u_{xx} + \gamma(t)u_{yy} = 0.$$

The relevant derivatives are

$$\begin{aligned} u_t &= -d_i'(t)u, \\ u_{xxxx} &= \mu_i^4 u, \\ u_{xx} &= \mu_i^2 u, \\ u_{yy} &= v_i^2 u. \end{aligned}$$

Substituting these into the linearized equation leads to

$$-d_i'(t)u + \mu_i^4 u + (\alpha(t) + \beta)\mu_i^2 u + \gamma(t)v_i^2 u = 0,$$

which simplifies to

$$-d_i'(t) + \mu_i^4 + \mu_i^2\alpha(t) + \mu_i^2\beta + \gamma(t)v_i^2 = 0.$$

Solving for $d_i'(t)$ gives

$$d_i'(t) = \mu_i^4 + \mu_i^2\alpha(t) + \mu_i^2\beta + \gamma(t)v_i^2,$$

and integrating over time yields the dispersion relation

$$d_i(t) = \int \frac{\mu_i^4 + \mu_i^2\alpha(t) + \gamma(t)v_i^2 + \mu_i^2\beta}{\mu_i}\,dt. \tag{55}$$

To construct solutions, we employ thetransformation

$$u(x, y, t) = K(\ln h)_{xx}, \tag{56}$$

which can equivalently be written as

$$u = w_{xx}, \quad \text{where} \quad w = K(\ln h). \tag{57}$$

Choosing the auxiliary function

$$h(x, y, t) = 1 + e^{\xi_1}, \quad \text{with} \quad \xi_1 = \mu_1 x + v_1 y - \int \frac{\mu_1^4 + \mu_1^2\alpha(t) + \gamma(t)v_1^2 + \mu_1^2\beta}{\mu_1}\,dt,$$

and substituting into Eq. (53), we find $K = 12$. Therefore, the transformation (56) reduces to

$$u = 12(\ln h)_{xx}. \tag{58}$$

Now, from (57), we have

$$u_{xt} = w_{xxxt}, \quad u_x = w_{xxx}, \quad u_{xxx} = w_{xxxxx} \quad \textit{and} \quad u_{yy} = w_{xxyy}$$

putting the above expressions into Eq.(53), we get

$$w_{xxxt} + (w_{xx}w_{xxx} + w_{xxxxx} + (\alpha(t) + \beta)w_{xxx})_x + \gamma(t)w_{xxyy} = 0 \tag{59}$$

on integrating w.r.t. x

$$w_{xxt} + w_{xx}w_{xxx} + w_{xxxxx} + (\alpha(t) + \beta)w_{xxx} + \gamma(t)w_{xyy} = 0 \tag{60}$$

on again integrating w.r.t. x

$$w_{xt} + \int w_{xx}w_{xxx}\,dx + w_{xxxx} + (\alpha(t) + \beta)w_{xx} + \gamma(t)w_{yy} = 0. \tag{61}$$

We compute integral term in equation (61) with constant of integration as zero

$$I = \int w_{xx}w_{xxx}\,\partial x = \frac{1}{2}\int 2w_{xx}w_{xxx}\,\partial x = \frac{1}{2}w_{xx}^2, \tag{62}$$

substituting the value of I in equation (61), we get

$$w_{xt} + \frac{w_{xx}^2}{2} + w_{xxxx} + (\alpha(t) + \beta)w_{xx} + \gamma(t)w_{yy} = 0. \tag{63}$$

As we have $w = 12(\log h)$, we can get the followings:

$$w_x = 12\frac{h_x}{h},$$

$$w_{xt} = 12\frac{hh_{xt} - h_xh_t}{h^2},$$

$$w_{xx} = 12\frac{hh_{xx} - h_x^2}{h^2},$$

$$w_{xxx} = 12\frac{2h_x^3 - 3hh_xh_{xx} + h^2h_{xxx}}{h^3},$$

$$w_{xxxx} = 12\frac{-6h_x^4 + 12hh_x^2h_{xx} - 3h^2h_{xx}^2 - 4h_xh^2h_{xxx}}{h^4},$$

$$w_{yy} = 12\frac{hh_{yy} - h_y^2}{h^2},$$

which converts Eq.(53) into a bilinear equation in h as

$$(hh_{xt} - h_xh_t) + (\beta + \alpha(t))(hh_{xx} + h_x^2) + \gamma(t)(hh_{yy} - h_y^2) + (hh_{xxxx} - 4h_{xxx}h_x + 3h_{xx}^2) = 0. \tag{64}$$

Since Hirota bilinear operator is defined as

$$D_a^pD_b^qD_c^r(h,k) = \lim_{a'\to a,\, b'\to b,\, c'\to c}\left(\frac{\partial}{\partial a} - \frac{\partial}{\partial a'}\right)^p\left(\frac{\partial}{\partial b} - \frac{\partial}{\partial b'}\right)^q\left(\frac{\partial}{\partial c} - \frac{\partial}{\partial c'}\right)^r h(a,b,c)k(a',b',c').$$

let $p = r = 1$ and $q = 0$ then

$$D_xD_t(h.k) = h_{xt}k - h_tk_x - h_xk_t + hk_{xt},$$

$$D_xD_t(h.h) = 2\left(hh_{xt} - h_xh_t\right)$$

let $p = 2$ and $q = r = 0$then

$$D_x^2(h.k) = h_{xx}k + hk_{xx} + 2h_xk_x$$

$$D_x^2(h.h) = 2\left(hh_{xx} - h_x^2\right)$$

let $p = r = 0$ and $q = 2$ then

$$D_y^2(h.k) = h_{yy}k + hk_{yy} + 2h_yk_y$$

$$D_y^2(h.h) = 2\left(hh_{yy} - h_y^2\right)$$

let $p = 4$ and $q = r = 0$ then

$$D_x^4(h.k) = h_{xxxx}k - 4.h_{xxx}k_x + 6.h_{xx}k_{xx} - 4.h_xk_{xxx} + hk_{xxxx}$$

$$D_x^4(h.h) = 2\left(hh_{xxxx} - 4.h_xh_{xxx} + 3.h_{xx}^2\right)$$

Therefore, using bilinear differentials D, the bilinear Eq. (64) can be expressed in Hirota's bilinear form as

$$\left[D_xD_t + (\beta + \alpha(t))D_x^2 + \gamma(t)D_y^2 + D_x^4\right]h.h = 0 \tag{65}$$

This equation (65) is called the Hirota bilinear form for Graphene sheets equation (53).

3.6 Bogoyavlenskii–Kadomtsev–Petviashvili (BKP) Equation in (3+1)-dimensions

The integrable BKP equation is given by [27]

$$u_{yt} + 3u_{xz} - 3u_x u_{xy} - 3u_{xx} u_y - u_{xxxy} = 0, \tag{66}$$

where x, y, z, t are spatial and temporal coordinates, and u represents the wave amplitude.
To analyze the dispersion properties, we define the phase variable

$$\xi_i = \mu_i x + v_i y + w_i z - d_i t, \tag{67}$$

where $\mu_i, v_i, w_i\, (i = 1, 2, \ldots)$ are constants, and d_i is the dispersion coefficient. Substituting $u = e^{\xi_i}$ into the linear terms of Eq. (66), namely

$$u_{yt} + 3u_{xz} - u_{xxxy} = 0,$$

yields the dispersion relation

$$d_i = \frac{-3\mu_i w_i + \mu_i^3 v_i}{v_i}. \tag{68}$$

Next, we employ the transformation

$$u(x, y, z, t) = K(\ln h)_x, \tag{69}$$

and select the h function as

$$h = 1 + e^{\xi_1}, \quad \text{with} \quad \xi_1 = \mu_1 x + v_1 y + w_1 z - \left(\frac{-3\mu_1 w_1 + \mu_1^3 v_1}{v_1}\right) t.$$

Substituting into Eq. (66) and simplifying gives $K = 2$. Therefore, the transformation (69) reduces to

$$u = 2(\ln h)_x. \tag{70}$$

Now, from (70), we have

$$u_x = 2\left(\frac{h_{xx}h - h_x^2}{h^2}\right), \quad u_y = 2\left(\frac{h_{xy}h - h_x h_y}{h^2}\right),$$
$$u_z = 2\left(\frac{h_{xz}h - h_x h_z}{h^2}\right), \quad u_t = 2\left(\frac{h_{xt}h - h_x h_t}{h^2}\right).$$

The higher-order derivatives become

$$u_{yt} = 2\frac{(h_{xyt}h + h_{xy}h_t - h_{xt}h_y - h_x h_{yt})h^2 - 2hh_t(h_{xy}h - h_x h_y)}{h^4},$$

$$u_{xz} = 2\frac{(h_{xxz}h + h_{xx}h_z - 2h_x h_{xz})h^2 - 2hh_z(h_{xx}h - h_x^2)}{h^4},$$

$$u_{xx} = 2\frac{h_{xxx}h^3 - 3h_x h_{xx}h^2 + 2h_x^3 h}{h^4}, \quad u_{xy} = 2\frac{(h_{xxy}h + h_{xx}h_y - 2h_x h_{xy})h^2 - 2hh_y(h_{xx}h - h_x^2)}{h^4},$$

$$\begin{aligned} u_{xxxy} = {} & 2\frac{h_{xxxy}}{h} - 2\frac{h_{xxxx}h_y}{h^2} - 8\frac{h_{xxxy}h_x}{h^2} - 8\frac{h_{xxx}h_{xy}}{h^2} + 16\frac{h_{xxx}h_x h_y}{h^3} \\ & - 12\frac{h_{xx}h_{xxy}}{h^2} + 12\frac{h_{xx}^2 h_y}{h^3} + 36\frac{h_x h_{xy} h_{xx}}{h^3} + 18\frac{h_x^2 h_{xxy}}{h^3} \\ & - 54\frac{h_x^2 h_{xx} h_y}{h^4} - 16\frac{h_x^3 h_{xy}}{h^4} + 16\frac{h_x^4 h_y}{h^5}. \end{aligned}$$

Substituting these expressions into Eq. (66), we get

$$hh_{yt} - h_y h_t + 3\left(hh_{xz} - h_x h_z\right) - h_{xxxy}h + 3h_{xxy}h_x - 3h_{xy}h_{xx} + h_y h_{xxx} = 0. \tag{71}$$

Since the Hirota bilinear operator is defined as

$$\begin{aligned} D_a^p D_b^q D_c^r D_d^s(h,k) = \lim_{a'\to a,\, b'\to b,\, c'\to c,\, d'\to d} & \left(\frac{\partial}{\partial a} - \frac{\partial}{\partial a'}\right)^p \left(\frac{\partial}{\partial b} - \frac{\partial}{\partial b'}\right)^q \\ & \times \left(\frac{\partial}{\partial c} - \frac{\partial}{\partial c'}\right)^r \left(\frac{\partial}{\partial d} - \frac{\partial}{\partial d'}\right)^s h(a,b,c,d)k(a',b',c',d'), \end{aligned},$$

we evaluate the following cases:

$$D_y D_t(h.h) = 2\left(hh_{yt} - h_y h_t\right), \quad D_x D_z(h.h) = 2\left(hh_{xz} - h_x h_z\right),$$

$$(D_x^3 D_y)(h.h) = 2\left(h_{xxxy}h - 3h_{xxy}h_x + 3h_{xy}h_{xx} - h_y h_{xxx}\right).$$

Thus, Eq. (71) gives Hirota bilinear form as

$$\left(D_y D_t + 3D_x D_z - D_x^3 D_y\right) h \cdot h = 0. \tag{72}$$

This equation (72) is called the Hirota bilinear form for the BKP equation (66).

3.7 Boiti–Leon–Manna–Pempinelli equation in (3+1)-dimensions

The integrable BLMP equation is expressed as [28]

$$u_{yt} + u_{zt} + u_{xxxy} + u_{xxxz} - 3u_x u_{xy} - 3u_x u_{xz} - 3u_{xx}u_y - 3u_{xx}u_z = 0, \tag{73}$$

where x, y, z, t are spatial and temporal coordinates, and u represents the wave amplitude.
To determine the dispersion relation, we introduce the phase variable

$$\xi_i = \mu_i x + v_i y + w_i z - d_i t, \tag{74}$$

where $\mu_i, v_i, w_i\, (i = 1, 2, \ldots)$ are constants and d_i is the dispersion coefficient. Substituting $u = e^{\xi_i}$ into the linear part of Eq. (73), namely

$$u_{yt} + u_{zt} + u_{xxxy} + u_{xxxz} = 0,$$

the relevant derivatives are

$$\begin{aligned} u_{yt} &= v_i(-d_i)e^{\xi_i}, \\ u_{zt} &= w_i(-d_i)e^{\xi_i}, \\ u_{xxxy} &= \mu_i^3 v_i e^{\xi_i}, \\ u_{xxxz} &= \mu_i^3 w_i e^{\xi_i}. \end{aligned}$$

Substituting these expressions into the linearized equation yields

$$-(v_i + w_i)d_i + \mu_i^3(v_i + w_i) = 0,$$

which leads to the dispersion relation

$$d_i = \mu_i^3. \tag{75}$$

Next, we employ the transformation

$$u(x,y,z,t) = K(\ln h)_x, \tag{76}$$

and take h function as

$$h = 1 + e^{\xi_1}, \quad \text{with} \quad \xi_1 = \mu_1 x + v_1 y + w_1 z - \mu_1^3 t.$$

Substituting into Eq. (73) and simplifying, we find $K = -2$. Therefore, the transformation (76) reduces to

$$u = -2(\ln h)_x. \tag{77}$$

Now, from (77), we have

$$u_x = 2\frac{\partial}{\partial x}\left(\frac{h_x}{h}\right) = 2\left(\frac{h_{xx}h - h_x^2}{h^2}\right)$$

$$u_y = 2\frac{\partial}{\partial y}\left(\frac{h_y}{h}\right) = 2\left(\frac{h_{xy}h - h_x h_y}{h^2}\right)$$

$$u_z = 2\frac{\partial}{\partial z}\left(\frac{h_z}{h}\right) = 2\left(\frac{h_{xz}h - h_x h_z}{h^2}\right)$$

$$u_t = 2\frac{\partial}{\partial t}\left(\frac{h_t}{h}\right) = 2\left(\frac{h_{xt}h - h_x h_t}{h^2}\right)$$

$$u_{yt} = 2\frac{(h_{xyt}h + h_{xy}h_t - h_{xt}h_y - h_x h_{yt})h^2 - 2hh_t(h_{xy}h - h_x h_y)}{h^4}$$

$$u_{xz} = 2\frac{(h_{xxz}h + h_{xx}h_z - 2h_x h_{xz})h^2 - 2hh_z(h_{xx}h - h_x^2)}{h^4}$$

$$u_{xx} = 2\frac{h_{xxx}h^3 - 3h_x h_{xx}h^2 + 2h_x^3 h}{h^4}$$

$$u_{xy} = 2\frac{(h_{xxy}h + h_{xx}h_y - 2h_x h_{xy})h^2 - 2hh_y(h_{xx}h - h_x^2)}{h^4}$$

$$\begin{aligned} u_{xxxy} = &-2\frac{h_{xxxxy}}{h} + 2\frac{h_{xxxx}h_y}{h^2} + 8\frac{h_{xxxy}h_x}{h^2} + 8\frac{h_{xxx}h_{xy}}{h^2} - 16\frac{h_{xxx}h_x h_y}{h^3} \\ &+12\frac{h_{xx}h_{xxy}}{h^2} - 12\frac{h_{xx}^2 h_y}{h^3} - 36\frac{h_x h_{xy} h_{xx}}{h^3} - 18\frac{h_x^2 h_{xxy}}{h^3} + 54\frac{h_x^2 h_{xx} h_y}{h^4} \\ &+16\frac{h_x^3 h_{xy}}{h^4} - 16\frac{h_x^4 h_y}{h^5} \end{aligned}$$

$$\begin{aligned} u_{xxxz} = &-2\frac{h_{xxxxz}}{h} + 2\frac{h_{xxxx}h_z}{h^2} + 8\frac{h_{xxxz}h_x}{h^2} + 8\frac{h_{xxx}h_{xz}}{h^2} - 16\frac{h_{xxx}h_x h_z}{h^3} \\ &+12\frac{h_{xx}h_{xxz}}{h^2} - 12\frac{h_{xx}^2 h_z}{h^3} - 36\frac{h_x h_{xz} h_{xx}}{h^3} - 18\frac{h_x^2 h_{xxz}}{h^3} + 54\frac{h_x^2 h_{xx} h_z}{h^4} \\ &+16\frac{h_x^3 h_{xz}}{h^4} - 16\frac{h_x^4 h_z}{h^5} \end{aligned}$$

putting the above expressions into Eq.(73), we get

$$hh_{yt} - h_y h_t + hh_{zt} - h_z h_t + hh_{xxxy} - 3h_x h_{xxy} + 3h_{xy}h_{xx} - h_y h_{xxx} + hh_{xxxz} - 3h_x h_{xxz} + 3h_{xz}h_{xx} - h_z h_{xxx} = 0 \tag{78}$$

Since Hirota bilinear operator is defined as

$$D_a^p D_b^q D_c^r D_d^s(h,k) = \lim_{a'\to a,\, b'\to b,\, c'\to c,\, d'\to d} \left(\frac{\partial}{\partial a} - \frac{\partial}{\partial a'}\right)^p \left(\frac{\partial}{\partial b} - \frac{\partial}{\partial b'}\right)^q \left(\frac{\partial}{\partial c} - \frac{\partial}{\partial c'}\right)^r \left(\frac{\partial}{\partial d} - \frac{\partial}{\partial d'}\right)^s h(a,b,c,d)k(a',b',c',d')$$

Let $p = 0$,$q = 1$,$r = 0$ and $s = 1$, then

$$D_y D_t(h.k) = h_{yt}k - h_t k_y - h_y k_t + hk_{yt},$$
$$D_y D_t(h.h) = 2\left(hh_{yt} - h_y h_t\right)$$

let $p = 0$,$q = 0$,$r = 1$ and $s = 1$, then

$$D_z D_t(h.k) = h_{zt}k - h_t k_z - h_z k_t + hk_{yz},$$
$$D_z D_t(h.h) = 2\left(hh_{zt} - h_z h_t\right)$$

let $p = 3$,$q = 1$,$r = 0$ and $s = 0$, then

$$(D_x^3 D_y)(hk) = h_{xxxy}k - 3.h_{xxy}k_x + 3.h_{xy}k_{xx} - h_y k_{xxx} - h_{xxx}k_y + 3.h_{xx}k_{xy} - 3.h_x k_{xxy} + hk_{xxxy}$$
$$(D_x^3 D_y)(hh) = 2\left(h_{xxxy}h - 3.h_{xxy}h_x + 3.h_{xy}h_{xx} - h_y h_{xxx}\right)$$

let $p = 3$,$q = 0$,$r = 1$ and $s = 0$, then

$$(D_x^3 D_z)(hk) = h_{xxxz}k - 3.h_{xxz}k_x + 3.h_{xz}k_{xx} - h_z k_{xxx} - h_{xxx}k_z + 3.h_{xx}k_{xz} - 3.h_x k_{xxz} + hk_{xxxz}$$
$$(D_x^3 D_z)(hh) = 2\left(h_{xxxz}h - 3.h_{xxz}h_x + 3.h_{xz}h_{xx} - h_z h_{xxx}\right)$$

Equation (78) makes D-operator form as

$$(D_y D_t + D_z D_t + D_x^3 D_y + D_x^3 D_z)h.h = 0 \tag{79}$$

This equation (79) is called the Hirota bilinear form for BLMP equation (73).

4 Results and Analysis

In above examples, each nonlinear equation was analyzed for Hirota's bilinear equation and its bilinear form. First, the original equation was transformed using a Cole-Hopf transformation that converts the equation to a simplified bilinear equation. Thereafter, Hirota's D-operators were applied to express the equation in bilinear operator notation. This approach works for equations with multiple dimensions, variable coefficients, or higher-order derivatives. The bilinear forms provide a convenient framework for obtaining exact solutions with the symbolic computation.

4.1 KdV Equation

The equation in $u(x,t)$ as

$$u_t + 6uu_x + u_{xxx} = 0.$$

Using the Cole-Hopf transformation, the solution is expressed as

$$u = 2(\ln h)_{xx}.$$

that gives a bilinear equation as

$$hh_{xt} - h_x h_t + 3h_{xx}^2 - 4h_x h_{xxx} + hh_{xxxx} = 0,$$

which gives Hirota's bilinear operator form as

$$(D_x D_t + D_x^4)h \cdot h = 0.$$

4.2 Boussinesq Equation

The Boussinesq equation is

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0.$$

Using the Cole-Hopf transformation, the solution is expressed as

$$u = 2(\ln h)_{xx},$$

that gives a bilinear equation as

$$hh_{tt} - h_t^2 - hh_{xx} + h_x^2 - hh_{xxxx} + 4h_x h_{xxx} - 3h_{xx}^2 = 0,$$

which gives Hirota's bilinear operator form as

$$(D_t^2 - D_x^2 - D_x^4)h \cdot h = 0.$$

4.3 KP Equation

The equation is

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0.$$

Using the Cole-Hopf transformation, the solution is expressed as

$$u = 2(\ln h)_{xx},$$

that gives a bilinear equation as

$$hh_{xt} - h_x h_t + 3h_{xx}^2 - 4h_x h_{xxx} + hh_{xxxx} - hh_{yy} + h_y^2 = 0.$$

which gives Hirota's bilinear operator form as

$$(D_x D_t + D_x^4 - D_y^2)h \cdot h = 0.$$

4.4 KP Equation with Variable Coefficient

The vc-KP equation is

$$(u_t + uu_x + u_{xxx})_x + g(t)u_{xy} + 3u_{yy} = 0.$$

Using the transformation, the solution is expressed as

$$u = 12(\ln h)_{xx},$$

that gives a bilinear equation as

$$hh_{xt} - h_x h_t + 3h_{xx}^2 - 4h_x h_{xxx} + hh_{xxxx} + 3hh_{yy} - 3h_y^2 - g(t)h_y h_x + g(t)hh_{xy} = 0,$$

which gives Hirota's bilinear operator form as

$$(D_x D_t + D_x^4 + 3D_y^2 + g(t)D_x D_y)h \cdot h = 0.$$

4.5 Graphene-Sheets Equation

The Graphene-Sheets equation is

$$u_{xt} + (uu_x + u_{xxx} + (\alpha(t) + \beta)u_x)_x + \gamma(t)u_{yy} = 0.$$

Using the transformation, the solution is expressed as

$$u = 12(\ln h)_{xx},$$

that gives a bilinear equation as

$$(hh_{xt} - h_x h_t) + (\beta + \alpha(t))(hh_{xx} + h_x^2) + \gamma(t)(hh_{yy} - h_y^2) + (hh_{xxxx} - 4h_x h_{xxx} + 3h_{xx}^2) = 0,$$

which gives Hirota's bilinear operator form as

$$(D_x D_t + (\beta + \alpha(t))D_x^2 + \gamma(t)D_y^2 + D_x^4)h \cdot h = 0.$$

4.6 BKP Equation

The BKP equation is

$$u_{yt} + 3u_{xz} - 3u_x u_{xy} - 3u_{xx}u_y - u_{xxxy} = 0.$$

Using the transformation, the solution is expressed as

$$u = 2(\ln h)_x,$$

that gives a bilinear equation as

$$hh_{yt} - h_y h_t + 3(hh_{xz} - h_x h_z) - hh_{xxxy} + 3h_{xxy}h_x - 3h_{xy}h_{xx} + h_y h_{xxx} = 0,$$

which gives Hirota's bilinear operator form as

$$(D_y D_t + 3D_x D_z - D_x^3 D_y)h \cdot h = 0.$$

4.7 BLMP Equation

The equation is

$$u_{yt} + u_{zt} + u_{xxxy} + u_{xxxz} - 3u_x u_{xy} - 3u_x u_{xz} - 3u_{xx}u_y - 3u_{xx}u_z = 0.$$

Using the transformation, the solution is expressed as

$$u = -2(\ln h)_x,$$

that gives a bilinear equation as

$$hh_{yt} - h_y h_t + hh_{zt} - h_z h_t + hh_{xxxy} - 3h_x h_{xxy} + 3h_{xy}h_{xx} - h_y h_{xxx} + hh_{xxxz} - 3h_x h_{xxz} + 3h_{xz}h_{xx} - h_z h_{xxx} = 0,$$

which gives Hirota's bilinear operator form as

$$(D_y D_t + D_z D_t + D_x^3 D_y + D_x^3 D_z)h \cdot h = 0.$$

5 Conclusions

In this manuscript, we explored the symbolic algorithm for creating the Hirota'a bilinear form for nonlinear PDEs in higher-dimensions. Algorithm found the Cole-Hopf transformation by balancing nonlinear and higher order terms. It constructed the bilinear equations using the obtained transformations. Next, it constructed the Hirota's D-operators so that it could convert the bilinear equation to bilinear D-operator form. After converting nonlinear PDEs into bilinear forms,we can obtain exact solutions such as lumps, breathers, and solitons. We successfully used the algorithm to obtain the Hirota bilinear form for several well-known equations such as the KdV, Boussinesq, KP, and BLMP equations and other nonlinear equations. The resulting bilinear forms serve as a basis for additional research, such as the development of multi-soliton solutions and the investigation of interaction phenomena. The algorithm can be further streamlined and made accessible for more complex systems by utilizing symbolic computation tools such as Maple or Matlab.

Declarations

Competing interests

There is no conflict of interest, according to the authors.

Authors' contributions

Each author made an equal contribution to the final draft of the work. The authors would have consented and approved the final work.

Data availability statement

Not applicable to this research as no data were analyzed and created in this work.

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