

# A Hyperstability Approach to Generalized Functional Equations on Involutive Semigroups

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## Abstract:

In this paper, we examine the hyperstability of a general functional equation involving involutions on semigroups, where  $f : S^2 \rightarrow X$  and  $(S, \cdot)$  is an arbitrary semigroup equipped with involutive mappings  $\sigma$  and  $\tau$ . The functional equation studied in this paper emerges naturally from the theory of additive and quadratic equations, particularly as a hybrid form combining structural properties of both. Such equations typically appear when analyzing mappings that preserve symmetrical or involutive relationships within algebraic structures, especially in semigroups endowed with additional symmetries. By employing a technique inspired by Maksa and Páles, we derive sufficient asymptotic conditions ensuring hyperstability. Additionally, we extend the result to an inhomogeneous variant incorporating a perturbation term  $F : S^4 \rightarrow X$ . These findings contribute to the broader understanding of stability phenomena in functional equations on algebraic structures.

**Keywords:** semigroups, involutions, stability, hyperstability, functional equations.

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## 1. Introduction

One important aspect of functional equations is their stability. In mathematics, the stability of an equation refers to the sensitivity of its solutions to small perturbations in the equation itself. In the context of functional equations, stability refers to the extent to which the solutions of an equation remain close to each other under small changes to the equation. Stability is an essential concept in functional analysis, as it can help ensure the existence and uniqueness of solutions to functional equations. For example, if a functional equation is stable, it may be possible to use fixed-point theorems to show that the equation has a unique solution. The study of the stability of functional equations is an active area of research in mathematics, with many open problems and challenges. Researchers are interested in understanding the stability properties of different classes of functional equations, as well as developing new techniques and methods for analyzing the stability of these equations.

The idea of studying the stability of functional equations emerged in 1940 when the mathematician S. M. Ulam put together a list of unsolved problems that included the following problem:

**Ulam's problem:** [17]

Let  $(G_1, *_1)$  be a group and let  $(G_2, *_2)$  be a metric group with a metric  $d(.,.)$ . Given  $\varepsilon > 0$ , does there exist a  $\delta > 0$  such that if a mapping  $h: G_1 \rightarrow G_2$  satisfies the inequality

$$d(h(x *_1 y), h(x) *_2 h(y)) < \varepsilon$$

for all  $x, y \in G_1$ , then there exists a homomorphism  $H: G_1 \rightarrow G_2$  with

$$d(h(x), H(x)) < \delta$$

for all  $x \in G_1$  ?

This question aroused the attention of many mathematicians. Almost a year later, D. H. Hayes published a paper in which he presented an answer to Ulam's problem in the case where  $G_1$  and  $G_2$  are Banach spaces to investigate the stability of Cauchy functional equation.

**Theorem 1.1.** [14] Let  $E_1$  and  $E_2$  be two Banach spaces and  $f: E_1 \rightarrow E_2$  be a function such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

for some  $\delta > 0$  and for all  $x, y \in E_1$ . Then the limit

$$A(x) := \lim_{n \rightarrow \infty} 2^{-n} f(2^n x)$$

exists for each  $x \in E_1$ , and  $A: E_1 \rightarrow E_2$  is the unique additive function such that

$$\|f(x) - A(x)\| \leq \delta$$

for all  $x \in E_1$ . Moreover, if  $f(tx)$  is continuous in  $t$  for each fixed  $x \in E_1$ , then the function  $A$  is linear.

Overall, Hyers' contribution to the theory of functional equations had significantly impacted mathematics and opened up new areas of research in this field, and it was only the beginning of a long list of necessary studies and results in the stability of functional equations. Starting with D. G. Bourgin [8], [9] and T. Aoki [4] who addressed the stability problem with unbounded Cauchy differences, in the scenario where the relevant inequality is not bounded, Th. M. Rassias [16] generalized Hyers' theorem by proving the existence of singular linear mappings close to approximate additive mappings. These results can be incorporated into the following theorem.

**Theorem 1.2.** Let  $X$  and  $Y$  be normed space and Banach space, respectively. Let  $c$  and  $p$  be two real numbers such that  $c \geq 0$  and  $p \neq 1$ . Consider the operator  $f: X \rightarrow Y$  that fulfills the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq c(\|x\|^p + \|y\|^p)$$

for all  $x, y \in X \setminus \{0\}$ . Then there exists an additive mapping  $T: X \rightarrow Y$  with

$$\|f(x) - T(x)\| \leq \frac{c}{|1 - 2^{p-1}|} \|x\|^p$$

for all  $x \in X \setminus \{0\}$ .

In the spirit of Rassias' approach, G. L. Forti [12] and P. Găvruta [13] generalized all of the aforementioned stability results by swapping out the Cauchy differences for a control function  $\varphi$ . At the same time, a special kind of stability known as hyperstability emerged. The hyperstability of

functional equations refers to the property that small perturbations of a given functional equation lead to solutions that are close to the original one. In other words, if a function satisfies a functional equation, then nearby functions also satisfy the same equation.

The following definition helps us distinguish between three key ideas in the theory of the stability of functional equations.

**Definition 1.3.** Let  $X$  be a nonempty set,  $(Y, d)$  be a metric space, and  $\varphi: X \times X \rightarrow [0, \infty)$  and  $\psi: X \rightarrow [0, \infty)$  be two functions. Let  $g_i: X \times X \rightarrow X$  for  $i \in \{1, \dots, 4\}$  and  $F_1, F_2: Y \times Y \rightarrow Y$ . Then:

(1) The following equation

$$F_1(f(g_1(x, y)), f(g_2(x, y))) = F_2(f(g_3(x, y)), f(g_4(x, y))) \quad (1.1)$$

is called a functional equation that specifies the unknown function  $f: X \rightarrow Y$ .

(2) If for every function  $f: X \rightarrow Y$  satisfying the inequality

$$d(F_1(f(g_1(x, y)), f(g_2(x, y))), F_2(f(g_3(x, y)), f(g_4(x, y)))) \leq \varphi(x, y) \quad (1.2)$$

for all  $x, y \in X$ , there exists a function  $T: X \rightarrow Y$  that satisfies the equation (1.1) such that

$$d(f(x), T(x)) \leq \psi(x) \quad (1.3)$$

for all  $x \in X$ , then we say that the functional equation (1.1) is generalized Hyers-Ulam-Rassias stable on  $(X, Y)$  with control functions  $\varphi$  and  $\psi$ . When  $\varphi(x, y)$  in (1.2) and  $\psi(x)$  in (1.3) are replaced by the reals  $\delta > 0$  and  $\varepsilon > 0$  respectively, then we say that corresponding phenomenon of the functional equation (1.1) is Hyers-Ulam stable on  $(X, Y)$  with controls  $\varepsilon$  and  $\delta$ .

(3) If for every function  $f: X \rightarrow Y$  satisfying the inequality (1.2), and either  $f$  is bounded or it is a solution to Equation (1.1), then we say that the functional equation (1.1) is superstable on  $(X, Y)$  with control function  $\varphi$ .

(4) If for every function  $f: X \rightarrow Y$  satisfying the inequality (1.2) and  $f$  is a solution of Equation (1.1), then we say that the functional equation (1.1) is hyperstable on  $(X, Y)$  with control  $\varphi$ .

The use of hyperstability in functional equations is mainly theoretical, as it allows us to extend the validity of known solutions to nearby cases that may be of interest. For example, if we have a solution to a functional equation that describes a certain phenomenon, we can use hyperstability to conclude that nearby phenomena are likely to exhibit similar behavior.

Hyperstability can also be used in the study of the stability of numerical methods for solving functional equations. If we know that the equation is hyperstable, then small errors in the numerical solution will not significantly affect the accuracy of the result.

Overall, hyperstability is a useful property of functional equations that allows us to extend the range of their applicability and better understand their behavior.

Although the term "hyperstability" was first used in 2001 [15], the first hyperstability finding appears to have been published in 1949 [8]. Let  $X$  be a real normed space and  $(S, \cdot)$  be an arbitrary semigroup. In 2001, one of the important results was shown in stability by Gy. Maksa and Zs. Páles [15]. They investigated the hyperstability of the following particular class of linear functional equations

$$f(x) + f(y) = \frac{1}{n} \sum_{i=1}^n f(x\varphi_i(y)) \quad (1.4)$$

where  $f: S \rightarrow X$  and where  $\varphi_1, \dots, \varphi_n: S \rightarrow S$  are pairwise distinct automorphisms of  $S$ . Let  $\sigma: S \rightarrow S$  be an involution, that is,  $\sigma(\sigma(x)) = x$  and  $\sigma(x \cdot y) = \sigma(x) \cdot \sigma(y)$  for all  $x, y \in S$ . In 2018, Jae-Hyeong Bae and Won-Gil Park [7] introduced the following functional equation

$$f(x_1 \cdot x_2, x_3 \cdot x_4) + f(x_1 \cdot \sigma(x_2), x_3 \cdot \tau(x_4)) = 2f(x_1, x_3) + 2f(x_2, x_4), \quad (1.5)$$

for all  $x_1, x_2, x_3, x_4 \in X$ , where  $f: X \times X \rightarrow Y$  with  $Y$  is a Banach space and where  $\sigma$  and  $\tau$  are involutions on  $X$ .

When  $\sigma(x) = \tau(x) = x^{-1}$ , the following quadratic equation is obtained

$$f(x_1 \cdot x_2, x_3 \cdot x_4) + f(x_1 \cdot x_2^{-1}, x_3 \cdot x_4^{-1}) = 2f(x_1, x_3) + 2f(x_2, x_4) \quad (1.6)$$

The stability of Eq. (1.6) was investigated in [5] and [6]. We observe that the functional equation (1.6), can be reformulated in a more compact and structured form by introducing the mapping

$$\varphi(x_2, x_4) := (\sigma(x_2), \tau(x_4))$$

where  $\sigma$  and  $\tau$  are involutions on a semigroup  $S$ . Let  $\mathcal{S} := S \times S$  denote the direct product semigroup equipped with the componentwise multiplication

$$(x_1, x_3)(x_2, x_4) := (x_1x_2, x_3x_4)$$

For  $X = (x_1, x_3)$  and  $Y = (x_2, x_4)$  in  $\mathcal{S}$ , the original equation can then be expressed as

$$f(XY) + f(X\varphi(Y)) = 2f(X) + 2f(Y), \forall X, Y \in \mathcal{S} \quad (1.7)$$

This equation exhibits a symmetric quadratic structure on the semigroup  $\mathcal{S}$  with an involutive automorphism  $\varphi$ . Such reformulations are useful for highlighting the algebraic structure underlying the equation and for enabling more general approaches to stability analysis. The equation (1.7) is the quadratic type, its central solutions are determined on arbitrary semigroups and its all solutions on abelian semigroups in several works (see for example [1], [2], and [11]).

The primary objective of this work is to investigate the hyperstability of a general functional equation of the form

$$f(x_1 \cdot x_2, x_3 \cdot x_4) + f(x_1 \cdot \sigma(x_2), x_3 \cdot \tau(x_4)) = 2f(x_1, x_3) + 2f(x_2, x_4),$$

where  $f: S^2 \rightarrow X$ , and  $\sigma, \tau$  are involutive mappings on a semigroup  $S$ . Our approach is inspired by the asymptotic technique introduced by Maksa and Páles [15], which has proven effective in establishing hyperstability results for linear-type functional equations. The main contribution of this paper lies in extending this method to a more general setting involving products and involutions on semigroups. Specifically, we provide sufficient asymptotic conditions under which the equation admits hyperstability on arbitrary semigroups. Moreover, we go beyond the homogeneous case by analyzing the hyperstability of an associated inhomogeneous functional equation:

$$f(x_1 \cdot x_2, x_3 \cdot x_4) + f(x_1 \cdot \sigma(x_2), x_3 \cdot \tau(x_4)) = 2f(x_1, x_3) + 2f(x_2, x_4) + F(x_1, x_2, x_3, x_4),$$

where  $F: S^4 \rightarrow X$  is an arbitrary perturbation function. We prove that under suitable assumptions, any approximate solution of this perturbed equation must be an exact solution, thereby establishing its hyperstability as well. These results contribute to the ongoing development of the theory of stability in functional equations by broadening the class of equations and algebraic structures for which hyperstability can be rigorously validated. One of the main goals of this paper is to study the hyperstability of a general functional equation involving involutions on semigroups, using an asymptotic approach inspired by the method of Maksa and Páles[15]. Our analysis focuses on identifying sufficient conditions under which the approximate solutions of the functional equation converge to exact solutions, thereby establishing its hyperstability in the sense of Ulam.

In addition to analyzing the hyperstability of the original functional equation in its standard form, this work also aims to recast the equation within a product semigroup framework. By identifying the underlying structure in terms of the product semigroup  $\mathcal{S} = S \times S$  and an involutive automorphism  $\varphi: \mathcal{S} \rightarrow \mathcal{S}$ , we reformulate the original equation as

$$f(\xi\zeta) + f(\xi\varphi(\zeta)) = 2f(\xi) + 2f(\zeta), \quad \xi, \zeta \in \mathcal{S}$$

This alternative formulation, which reinterprets the functional equation within the Cartesian product semigroup  $S \times S$ , serves a dual purpose. First, it exposes the underlying algebraic symmetry of the equation by aligning it with a structured binary operation on paired elements, thus clarifying the involutive behavior induced by  $\sigma$  and  $\tau$ . Second, this reformulation enables the application of general semigroup techniques to a broader class of functional equations by transforming the problem into an equivalent one on the product structure. Such a representation is particularly advantageous when exploring stability properties, as it allows the equation to be treated in terms of single-variable operations on pairs, thereby facilitating the use of asymptotic and fixed-point methods. This perspective not only streamlines the analytical process but also opens pathways for further generalizations to multi-variable or higher-dimensional functional equations on composite algebraic systems.

## 2. Main results

In what follows, we denote by  $\mathbb{R}_+$  the set of nonnegative real numbers and by  $\mathbb{N}$  the set of positive integers. Let  $X$  be a real normed space, and let  $(S, \cdot)$  be a semigroup. Throughout this section, we consider two mappings  $\sigma, \tau: S \rightarrow S$  that are assumed to be involutive endomorphisms unless specified otherwise.

The analysis presented in this section is motivated by the asymptotic technique introduced by Maksa and Páles [15], which has proven to be a powerful tool in establishing hyperstability results. Our approach begins with a key lemma that forms the basis for the subsequent stability results.

**Lemma 2.1.** Let  $f: S^2 \rightarrow X$  be an arbitrary function. Then the function  $D_f: S^4 \rightarrow X$  that is defined by

$$D_f(x_1, x_2, x_3, x_4) = f(x_1 \cdot x_2, x_3 \cdot x_4) + f(x_1 \cdot \sigma(x_2), x_3 \cdot \tau(x_4)) - 2f(x_1, x_3) - 2f(x_2, x_4), \quad (2.1)$$

for all  $x_1, x_2, x_3, x_4 \in S$ , satisfies the following functional equation

$$\begin{aligned}
 & 2D_f(x_1, x_2, x_3, x_4) + D_f(x_1 \cdot x_2, a, x_3 \cdot x_4, b) + D_f(x_1 \cdot \sigma(x_2), a, x_3 \cdot \tau(x_4), b) \\
 & = D_f(x_1, x_2 \cdot a, x_3, x_4 \cdot b) + D_f(x_1, x_2 \cdot \sigma(a), x_3, x_4 \cdot \tau(b)) \\
 & \quad + 2D_f(x_2, a, x_4, b)
 \end{aligned} \tag{2.2}$$

for all  $x_1, x_2, x_3, x_4, a, b \in S$ .

**Proof.** When we evaluate the left and right sides of (2.2), we obtain

$$\begin{aligned}
 & 2D_f(x_1, x_2, x_3, x_4) + D_f(x_1 \cdot x_2, a, x_3 \cdot x_4, b) + D_f(x_1 \cdot \sigma(x_2), a, x_3 \cdot \tau(x_4), b) \\
 & = 2f(x_1 \cdot x_2, x_3 \cdot x_4) + 2f(x_1 \cdot \sigma(x_2), x_3 \cdot \tau(x_4)) - 4f(x_1, x_3) - 4f(x_2, x_4) \\
 & \quad + f(x_1 \cdot x_2 \cdot a, x_3 \cdot x_4 \cdot b) + f(x_1 \cdot x_2 \cdot \sigma(a), x_3 \cdot x_4 \cdot \tau(b)) \\
 & \quad - 2f(x_1 \cdot x_2, x_3 \cdot x_4) - 2f(a, b) + f(x_1 \cdot \sigma(x_2) \cdot a, x_3 \cdot \tau(x_4) \cdot b) \\
 & \quad + f(x_1 \cdot \sigma(x_2) \cdot \sigma(a), x_3 \cdot \tau(x_4) \cdot \tau(b)) - 2f(x_1 \cdot \sigma(x_2), x_3 \cdot \tau(x_4)) - 2f(a, b) \\
 & = -4f(x_1, x_3) - 4f(x_2, x_4) + f(x_1 \cdot x_2 \cdot a, x_3 \cdot x_4 \cdot b) + f(x_1 \cdot x_2 \cdot \sigma(a), x_3 \cdot x_4 \cdot \tau(b)) \\
 & \quad - 4f(a, b) + f(x_1 \cdot \sigma(x_2) \cdot a, x_3 \cdot \tau(x_4) \cdot b) + f(x_1 \cdot \sigma(x_2) \cdot \sigma(a), x_3 \cdot \tau(x_4) \cdot \tau(b)),
 \end{aligned}$$

for all  $x_1, x_2, x_3, x_4, a, b \in S$ , and

$$\begin{aligned}
 & D_f(x_1, x_2 \cdot a, x_3, x_4 \cdot b) + D_f(x_1, x_2 \cdot \sigma(a), x_3, x_4 \cdot \tau(b)) + 2D_f(x_2, a, x_4, b) \\
 & = f(x_1 \cdot x_2 \cdot a, x_3 \cdot x_4 \cdot b) \\
 & \quad + f(x_1 \cdot \sigma(x_2) \cdot \sigma(a), x_3 \cdot \tau(x_4) \cdot \tau(b)) - 2f(x_1, x_3) - 2f(x_2 \cdot a, x_4 \cdot b) \\
 & \quad + f(x_1 \cdot x_2 \cdot \sigma(a), x_3 \cdot x_4 \cdot \tau(b)) + f(x_1 \cdot \sigma(x_2) \cdot a, x_3 \cdot \tau(x_4) \cdot b) - 2f(x_1, x_3) \\
 & \quad - 2f(x_2 \cdot \sigma(a), x_4 \cdot \tau(b)) + 2f(x_2 \cdot a, x_4 \cdot b) + 2f(x_2 \cdot \sigma(a), x_4 \cdot \tau(b)) - 4f(x_2, x_4) - 4f(a, b) \\
 & = -4f(x_1, x_3) - 4f(x_2, x_4) + f(x_1 \cdot x_2 \cdot a, x_3 \cdot x_4 \cdot b) + f(x_1 \cdot x_2 \cdot \sigma(a), x_3 \cdot x_4 \cdot \tau(b)) \\
 & \quad - 4f(a, b) + f(x_1 \cdot \sigma(x_2) \cdot a, x_3 \cdot \tau(x_4) \cdot b) + f(x_1 \cdot \sigma(x_2) \cdot \sigma(a), x_3 \cdot \tau(x_4) \cdot \tau(b)),
 \end{aligned}$$

for all  $x_1, x_2, x_3, x_4, a, b \in S$ . Therefore, (2.2) holds.

In the next theorem, we study the hyperstability of the equation (1.5). Specifically, we prove that the functional equation (1.5) is hyperstable in the class of functions  $f: S \rightarrow X$  for certain asymptotic properties of the control function  $\varepsilon: S^4 \rightarrow \mathbb{R}_+$ .

**Theorem 2.2.** Let  $\varepsilon: S^4 \rightarrow \mathbb{R}_+$  be a function, and consider that there exists a sequence  $\{u_n\}_{n \in \mathbb{N}}$  in  $S$  that satisfies the following two conditions

$$\lim_{n \rightarrow \infty} \inf \varepsilon(x_1, x_2 \cdot u_n, x_3, x_4 \cdot u_n) = 0 \tag{2.3}$$

and

$$\lim_{n \rightarrow \infty} \inf \varepsilon(x_1, x_2 \cdot \sigma(u_n), x_3, x_4 \cdot \tau(u_n)) = 0, \tag{2.4}$$

for all  $x_1, x_2, x_3, x_4 \in S$ . Assume that  $f: S^2 \rightarrow X$  satisfies the inequality

$$\|f(x_1 \cdot x_2, x_3 \cdot x_4) + f(x_1 \cdot \sigma(x_2), x_3 \cdot \tau(x_4)) - 2f(x_1, x_3) - 2f(x_2, x_4)\| \leq \varepsilon(x_1, x_2, x_3, x_4) \tag{2.5}$$

for all  $x_1, x_2, x_3, x_4 \in S$ . Then the functional equation (1.5) is hyperstable on  $S$ .

**Proof.** In view of the function  $D_f$  defined by (2.1), the inequality (2.5) becomes

$$D_f(x_1, x_2, x_3, x_4) \leq \varepsilon(x_1, x_2, x_3, x_4), \quad x_1, x_2, x_3, x_4 \in S \tag{2.6}$$

Using Lemma 2.1, we get

$$\begin{aligned}
 & 2D_f(x_1, x_2, x_3, x_4) + D_f(x_1 \cdot x_2, a, x_3 \cdot x_4, b) + D_f(x_1 \cdot \sigma(x_2), a, x_3 \cdot \tau(x_4), b) \\
 & = D_f(x_1, x_2 \cdot a, x_3, x_4 \cdot b) + D_f(x_1, x_2 \cdot \sigma(a), x_3, x_4 \cdot \tau(b)) + 2D_f(x_2, a, x_4, b)
 \end{aligned} \tag{2.7}$$

for all  $x_1, x_2, x_3, x_4, a, b \in S$ . Assume that there is a sequence  $\{u_n\}_{n \in \mathbb{N}}$  of elements of  $S$  that satisfying the conditions (2.3) and (2.4). Replacing  $x_2$  by  $x_2 \cdot u_n$  and  $x_4$  by  $x_4 \cdot u_n$  in (2.5), where  $n \in \mathbb{N}$ , we get

$$\|D_f(x_1, x_2 \cdot u_n, x_3, x_4 \cdot u_n)\| \leq \varepsilon(x_1, x_2 \cdot u_n, x_3, x_4 \cdot u_n), \quad x_1, x_2, x_3, x_4 \in S \tag{2.8}$$

In view of (2.3), we obtain

$$\liminf_{n \rightarrow \infty} D_f(x_1, x_2 \cdot u_n, x_3, x_4 \cdot u_n) = 0, \quad x_1, x_2, x_3, x_4 \in S \tag{2.9}$$

Replacing  $x_2$  by  $x_2 \cdot \sigma(u_n)$  and  $x_4$  by  $x_4 \cdot \tau(u_n)$  in (2.5), where  $n \in \mathbb{N}$ , we find

$$\|D_f(x_1, x_2 \cdot \sigma(u_n), x_3, x_4 \cdot \tau(u_n))\| \leq \varepsilon(x_1, x_2 \cdot \sigma(u_n), x_3, x_4 \cdot \tau(u_n)), \quad x_1, x_2, x_3, x_4 \in S \tag{2.10}$$

By applying the condition (2.4), we get

$$\liminf_{n \rightarrow \infty} D_f(x_1, x_2 \cdot \sigma(u_n), x_3, x_4 \cdot \tau(u_n)) = 0, \quad x_1, x_2, x_3, x_4 \in S \tag{2.11}$$

Let  $x_1, x_2, x_3, x_4, \alpha, \beta \in S$  be fixe. By replacing  $a$  by  $\alpha \cdot u_n$  and  $b$  by  $\beta \cdot u_n$  in (2.7), we conclude the following equality

$$\begin{aligned}
 & 2D_f(x_1, x_2, x_3, x_4) + D_f(x_1 \cdot x_2, \alpha \cdot u_n, x_3 \cdot x_4, \beta \cdot u_n) \\
 & + D_f(x_1 \cdot \sigma(x_2), \alpha \cdot u_n, x_3 \cdot \tau(x_4), \beta \cdot u_n) \\
 & = D_f(x_1, x_2 \cdot \alpha \cdot u_n, x_3, x_4 \cdot \beta \cdot u_n) + D_f(x_1, x_2 \cdot \sigma(\alpha \cdot u_n), x_3, x_4 \cdot \tau(\beta \cdot u_n)) \\
 & + 2D_f(x_2, \alpha \cdot u_n, x_4, \beta \cdot u_n)
 \end{aligned} \tag{2.12}$$

Letting  $n \rightarrow \infty$  with using (2.9) and (2.11), we derive from (2.12) that

$$D_f(x_1, x_2, x_3, x_4) = 0, \quad x_1, x_2, x_3, x_4 \in S$$

That is,  $f$  is a solution of equation (1.5) which means that (1.5) is hyperstable on  $S$ . We arrive at the following corollary as a direct result of the Theorem 2.2.

**Corollary 2.3.** Let  $\varepsilon: S^4 \rightarrow \mathbb{R}_+$  be a function and we consider that there exist  $\alpha \in S$ ,  $0 \leq s < 1$  and  $0 \leq t < 1$  such that

$$\begin{cases} \varepsilon(x_1, x_2 \cdot \alpha, x_3, x_4 \cdot \alpha) \leq s\varepsilon(x_1, x_2, x_3, x_4) \\ \varepsilon(x_1, x_2 \cdot \sigma(\alpha), x_3, x_4 \cdot \tau(\alpha)) \leq t\varepsilon(x_1, x_2, x_3, x_4) \end{cases} \tag{2.13}$$

for all  $x_1, x_2, x_3, x_4 \in S$  and all  $n \in \mathbb{N}$ . If  $f: S \rightarrow X$  satisfies the inequality (2.5), then it is a solution to the functional equation (1.5).

**Proof.** It is not hard to demonstrate, using induction on  $n \in \mathbb{N}$ , that

$$\begin{cases} \varepsilon(x_1, x_2 \cdot \alpha^n, x_3, x_4 \cdot \alpha^n) \leq s^n \varepsilon(x_1, x_2, x_3, x_4) \\ \varepsilon(x_1, x_2 \cdot \sigma(\alpha^n), x_3, x_4 \cdot \tau(\alpha^n)) \leq t^n \varepsilon(x_1, x_2, x_3, x_4) \end{cases} \tag{2.14}$$

for all  $x_1, x_2, x_3, x_4 \in S$ . Therefore, the conditions (2.3) and (2.4) hold with  $u_n = \alpha^n$  for all  $n \in \mathbb{N}$ . We conclude that the functional equation (1.5) is hyperstable on  $S$  based on Theorem 2.2.

In the next theorem, we discuss the hyperstability of the following inhomogeneous functional equation:

$$f(x_1 \cdot x_2, x_3 \cdot x_4) + f(x_1 \cdot \sigma(x_2), x_3 \cdot \tau(x_4)) = 2f(x_1, x_3) + 2f(x_2, x_4) + F(x_1, x_2, x_3, x_4) \quad (2.15)$$

for all  $x_1, x_2, x_3, x_4 \in S$ , where  $F: S^4 \rightarrow X$ .

**Theorem 2.4.** Let  $X, S, \sigma$ , and  $\tau$  remain unchanged from Theorem 2.2. Let  $f: S^2 \rightarrow X$  and  $F: S^4 \rightarrow X$  be two functions satisfy the inequality

$$\|f(x_1 \cdot x_2, x_3 \cdot x_4) + f(x_1 \cdot \sigma(x_2), x_3 \cdot \tau(x_4)) - 2f(x_1, x_3) - 2f(x_2, x_4) - F(x_1, x_2, x_3, x_4)\| \leq \varepsilon(x_1, x_2, x_3, x_4) \quad (2.16)$$

for all  $x_1, x_2, x_3, x_4 \in S$ . If the functional equation (2.15) admits a solution  $f_0: S^2 \rightarrow X$ , then (2.15) is hyperstable on  $S$ .

**Proof.** Let  $g: S^2 \rightarrow X$  be a function defined by  $g(x_1, x_2) := f(x_1, x_2) - f_0(x_1, x_2)$  for all  $x_1, x_2 \in S$ . Then

$$\begin{aligned} & \|g(x_1 \cdot x_2, x_3 \cdot x_4) + g(x_1 \cdot \sigma(x_2), x_3 \cdot \tau(x_4)) - 2g(x_1, x_3) - 2g(x_2, x_4)\| \\ &= \|f(x_1 \cdot x_2, x_3 \cdot x_4) + f(x_1 \cdot \sigma(x_2), x_3 \cdot \tau(x_4)) - (f_0(x_1 \cdot x_2, x_3 \cdot x_4) \\ &+ f_0(x_1 \cdot \sigma(x_2), x_3 \cdot \tau(x_4))) - 2f(x_1, x_3) - 2f(x_2, x_4) - F(x_1, x_2, x_3, x_4) \\ &- (2f_0(x_1, x_3) + 2f_0(x_2, x_4) + F(x_1, x_2, x_3, x_4))\| \\ &\leq \|f(x_1 \cdot x_2, x_3 \cdot x_4) + f(x_1 \cdot \sigma(x_2), x_3 \cdot \tau(x_4)) - 2f(x_1, x_3) - 2f(x_2, x_4) \\ &- F(x_1, x_2, x_3, x_4)\| \\ &+ \|f_0(x_1 \cdot x_2, x_3 \cdot x_4) + f_0(x_1 \cdot \sigma(x_2), x_3 \cdot \tau(x_4)) - 2f_0(x_1, x_3) - 2f_0(x_2, x_4) \\ &- F(x_1, x_2, x_3, x_4)\| \\ &= \|f(x_1 \cdot x_2, x_3 \cdot x_4) + f(x_1 \cdot \sigma(x_2), x_3 \cdot \tau(x_4)) - 2f(x_1, x_3) - 2f(x_2, x_4) \\ &- F(x_1, x_2, x_3, x_4)\| \\ &\leq \varepsilon(x_1, x_2, x_3, x_4), \quad x_1, x_2, x_3, x_4 \in S. \end{aligned}$$

According to Theorem 2.2, the function  $g = f - f_0$  satisfies the homogeneous equation (1.5). Therefore, by linearity, we have:

$$\begin{aligned} & f(x_1 \cdot x_2, x_3 \cdot x_4) + f(x_1 \cdot \sigma(x_2), x_3 \cdot \tau(x_4)) - 2f(x_1, x_3) - 2f(x_2, x_4) - F(x_1, x_2, x_3, x_4) \\ &= g(x_1 \cdot x_2, x_3 \cdot x_4) + g(x_1 \cdot \sigma(x_2), x_3 \cdot \tau(x_4)) - 2g(x_1, x_3) - 2g(x_2, x_4) \\ &+ f_0(x_1 \cdot x_2, x_3 \cdot x_4) + f_0(x_1 \cdot \sigma(x_2), x_3 \cdot \tau(x_4)) - 2f_0(x_1, x_3) - 2f_0(x_2, x_4) - F(x_1, x_2, x_3, x_4) \\ &= 0 \end{aligned}$$

for all  $x_1, x_2, x_3, x_4 \in S$ . Since both  $g$  and  $f_0$  satisfy their respective equations, each component vanishes, and we conclude that  $f$  is also a solution of (2.15).

Before closing this section, we reformulate the main functional equation within the framework of the Cartesian product semigroup  $S \times S$ . This structural reinterpretation allows us to express the equation in a more symmetric and compact form, which not only reflects the underlying algebraic properties induced by the involutions  $\sigma$  and  $\tau$ , but also enables broader generalizations. By identifying elements of  $S^2$  with ordered pairs and defining the multiplication and involutive mappings accordingly, we

derive a natural equivalent of the original equation on the product semigroup. The following corollary confirms that the hyperstability property persists in this extended setting. **Corollary 2.5.** Let  $\mathcal{S} := S \times S$  be the direct product of a semigroup  $S$  with itself, equipped with the product operation

$$(x_1, x_3)(x_2, x_4) := (x_1x_2, x_3x_4)$$

and let  $\varphi: \mathcal{S} \rightarrow \mathcal{S}$  be defined by  $\varphi(x_2, x_4) := (\sigma(x_2), \tau(x_4))$ , where  $\sigma$  and  $\tau$  are involutions on  $S$ .

Suppose that  $f: \mathcal{S} \rightarrow X$  satisfies the inequality

$$\|f(\xi\zeta) + f(\xi\varphi(\zeta)) - 2f(\xi) - 2f(\zeta)\| \leq \varepsilon(\xi, \zeta)$$

for all  $\xi, \zeta \in \mathcal{S}$ , where  $\varepsilon: \mathcal{S}^2 \rightarrow \mathbb{R}_+$  satisfies the asymptotic conditions analogous to those in Theorem 2.2. Then the functional equation

$$f(\xi\zeta) + f(\xi\varphi(\zeta)) = 2f(\xi) + 2f(\zeta), \forall \xi, \zeta \in \mathcal{S} \tag{2.17}$$

is hyperstable on  $\mathcal{S}$ .

**Proof.** Let  $\xi = (x_1, x_3)$  and  $\zeta = (x_2, x_4)$  be arbitrary elements in  $\mathcal{S}$ . Then:

$$\begin{aligned} \xi\zeta &= (x_1x_2, x_3x_4) \\ \xi\varphi(\zeta) &= (x_1\sigma(x_2), x_3\tau(x_4)) \end{aligned}$$

Hence, the equation

$$f(\xi\zeta) + f(\xi\varphi(\zeta)) = 2f(\xi) + 2f(\zeta)$$

is equivalent to the original functional equation:

$$f(x_1x_2, x_3x_4) + f(x_1\sigma(x_2), x_3\tau(x_4)) = 2f(x_1, x_3) + 2f(x_2, x_4)$$

which was shown in Theorem 2.2 to be hyperstable under the given asymptotic condition on  $\varepsilon$ .

Moreover, we can define a sequence  $\{U_n\}_{n \in \mathbb{N}} \subset \mathcal{S}$  by  $U_n = (\alpha^n, \alpha^n)$  for some fixed  $\alpha \in S$  satisfying the decay condition:

$$\varepsilon(\xi \cdot U_n, \zeta \cdot U_n) \rightarrow 0 \quad \text{and} \quad \varepsilon(\xi \cdot \varphi(U_n), \zeta \cdot \varphi(U_n)) \rightarrow 0$$

as  $n \rightarrow \infty$ , for all  $\xi, \zeta \in \mathcal{S}$ .

Therefore, by direct application of the same reasoning in Theorem 2.2, we conclude that

$$f(\xi\zeta) + f(\xi\varphi(\zeta)) = 2f(\xi) + 2f(\zeta)$$

holds for all  $\xi, \zeta \in \mathcal{S}$ , i.e., the equation (2.17) is hyperstable on  $\mathcal{S}$ .

### Conclusion

In this paper, we have investigated the hyperstability of a general functional equation involving two involutive mappings defined on an arbitrary semigroup. Using an asymptotic approach based on the method of Maksa and Páles [15], we established sufficient conditions under which approximate solutions to the equation

$$f(x_1x_2, x_3x_4) + f(x_1\sigma(x_2), x_3\tau(x_4)) = 2f(x_1, x_3) + 2f(x_2, x_4)$$

must in fact be exact solutions. These results extend prior stability theorems and contribute to the ongoing development of the theory of functional equations in algebraic settings involving involutions.

Furthermore, we provided a reformulation of the equation over the product semigroup

$S = S \times S$ , where we demonstrated via Corollary 2.5 that the hyperstability property is preserved under this transformation. This reformulation not only highlights the underlying algebraic symmetry of the problem but also broadens the structural scope of the stability analysis.

Future work may explore analogous phenomena in more complex algebraic systems, including inverse and topological semigroups or frameworks involving multiple interacting involutions. Additional investigations may also target non-quadratic functional equations and their inhomogeneous counterparts under similar asymptotic stability conditions.

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