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A Unified Numerical Framework for the Inversion of Integral Transforms: Laplace and Fourier

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Abstract:
The analytical inversion of Laplace and Fourier transforms often presents significant challenges for functions commonly encountered in engineering and scientific fields. Although numerous numerical inversion methods are available, these techniques tend to be highly specialized, algorithmically complex, and sensitive to parameter variations. This paper introduces an innovative, unified methodological framework for the numerical computation of these inverse transforms. By leveraging the intrinsic relationship between the two transforms through analytic continuation and implementing a robust regularization strategy within a Fourier inversion framework, we derive a generalized inversion formula. This method effectively converts the problem of inverting a Laplace transform into a specially designed Fourier inversion, which is then solved using a computationally efficient and stable algorithm based on the Fast Fourier Transform (FFT) with a smoothing kernel. We validate the effectiveness and precision of this universal method through several standard examples, comparing its performance with traditional techniques such as the Bromwich integral and Stehfest's algorithm for Laplace inversion. The proposed approach offers a streamlined, powerful, and versatile computational tool for researchers and practitioners.

1. Introduction:

Integral transforms, especially the Laplace and Fourier transforms, are fundamental tools for addressing differential and integral equations encountered in physics, engineering, and applied mathematics [1]. The Laplace transform, expressed as $L\{f(v)\} = \int_0^{\infty} f(v)e^{-\gamma v}\, dv = F(\gamma)$ is crucial for examining linear time-invariant systems and tackling initial-value problems. Meanwhile, the Fourier transform, $F(\eta) = f\{f(v)\} = \int_{-\infty}^{\infty} f(v)e^{-i\eta v}\, dv$ is essential for frequency domain analysis and resolving issues on infinite domains. Although converting a function f(v) into its transform F(γ) or F(η) is generally straightforward, the reverse process—retrieving the original function f(t) from its transform—often presents analytical difficulties or is impossible to achieve in a closed form. The analytical inversion involves evaluating a complex contour integral (Bromwich integral for Laplace) or a principal value integral (for Fourier), which is only practical for a limited range of simple functions [2]. As a result, robust numerical methods become vital. Current techniques for numerical inversion, such as the Fourier series approximation [3], Talbot's method [4], and the Weeks method [5] for Laplace transforms, or discrete inverse FFT for Fourier transforms, are well-established but may experience instability, slow convergence, or sensitivity to parameters specific to the transform. This paper introduces a unified approach that frames both inversion challenges within a single, stable computational framework. We present a core theorem that formally connects the two inversions and describe a practical FFT-based algorithm for implementation.

This article introduces a method that offers a unified formula for addressing both Fourier and Laplace Transforms, along with their inverses. The inspiration for this work came from a partial differential equation (PDE) course one of the authors attended 38 years ago, particularly focusing on the use of Inverse Symbolic Operators to solve PDEs [6]. Since then, various authors have proposed methods for solving Laplace Transforms, but none are quite like the one detailed in this article [7].

2. **Keywords:** Inverse Laplace Transform, Inverse Fourier Transform, Numerical Methods, Analytic Continuation, Fast Fourier Transform (FFT), Bromwich Integral, Regularization, Computational Mathematics.

3. Methods:

4.1. Symbolic operators

The inversion of transforms can be expressed using inverse symbolic operators.

For a given transform pair $f(v) \leftrightarrow F(\gamma)$, the inversion is formally defined by the operator.

$$f(v) = L^{-1}\{F(\gamma)\}, \quad f(v) = F^{-1}\{F(\eta)\}.$$

The Laplace inverse is typically written as the Bromwich integral:

$$f(v) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(\gamma)e^{\gamma v}d\gamma,$$

Where σ is a real number to the right of all singularities of $F(\gamma)$.

Similarly, the Fourier inverse is given given by:

$$f(v) = \frac{1}{2\pi}\int_{0}^{\infty} F(\eta)e^{i\eta v}d\eta.$$

4.2. Core theorem: Unified Laplace-Fourier Inversion

Theorem(Unified Inverse Transform):

Let $F(\gamma)$ be analytic in the Half-plane $\Re(\gamma) > 0$. Then the inverse Laplace transform can be expressed Fourier-type integral:

$$f(v) = \frac{e^{\sigma v}}{2\pi}\int_{-\infty}^{\infty} F(\sigma + i\eta)\, e^{i\eta v}d\eta,$$

Where $\sigma > 0$ ensures convergence.

Proof: By definition of the inverse Laplace transform(Bromwich integral), for $v > 0$ and any real c to the right of every singularity of $F(\gamma)$,

$$f(v) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(\gamma)e^{\gamma v}d\gamma.$$

Since F is analytic in the half-plane $\Re(\gamma) > 0$, choose any $\sigma > 0$ lying to the right of all singularities of F.

By analytically and standard contour-deformation arguments the integral along $\Re\gamma = c$ equals the integral along $\Re\gamma = \sigma$.

Put $\gamma = \sigma + i\eta$ with $\eta \in (-\infty, \infty)$.

Then $d\gamma = -id\eta$.

Substituting into the Bromwich integral gives

$$f(v) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(\sigma + i\eta)e^{(\sigma+i\eta)v}i\, d\eta.$$

The factor i cancels the $\frac{1}{2\pi i}$ prefactor, so

$$f(v) = \frac{1}{2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} F(\sigma + i\eta)e^{\sigma v}e^{i\eta v}d\eta.$$

Pulling the constant $e^{\sigma v}$ outside the integral yields the Fourier-type representation

$$f(v) = \frac{e^{\sigma v}}{2\pi} \int_{-\infty}^{\infty} F(\sigma + i\eta)\, e^{i\eta v} d\eta.$$

Which is the desired formula.

4.3. Computational approach

From the theorem, the Laplace inversion reduces to:

$$f(v) \approx \frac{e^{\sigma v}}{2\pi} \sum_{k=-N}^{N} F(\sigma + i\eta_k) e^{i\eta_k v} \Delta\eta,$$

Where $\eta_k = k\Delta\eta$.

This is essentially a discrete Fourier transform(DFT) and can be evaluated efficiently using

Algorithm:

Choose $\sigma > 0$ large enough to ensure $F(\sigma + i\eta)$ is analytic and decays sufficiently, but not so large that $e^{\sigma v}$ causes numerical instability.

Sample $F(\sigma + i\eta)$ at equispaced frequencies

$$\eta_k = k\Delta\eta, \qquad k = -N, \dots\dots, N.$$

Recognize the sum as a discrete Fourier transform.

Use the Fast Fourier Transform to compute efficiently.

$$\hat{f}(v_j) = \sum_{k=-N}^{N} F(\sigma + i\eta_k) e^{i\eta_k v_j}.$$

Here $v_j = j\Delta v$, where $\Delta v = \frac{2\pi}{(2N+1)\Delta\eta}$ by the Nyquist relation.

Multiply the Fast Fourier Transform output by the prefactor:

$$f(v_j) \approx \frac{e^{\sigma v_j}}{2\pi} \Delta\eta \hat{f}(v_j).$$

4.4. Numerical Considerations: Error and Stability

While the unified inversion formula is elegant, practical implementation requires careful treatment of Stability, truncation and Disecretization errors.

a) Choice of Shift Parameter σ

i. For Laplace inversion, the factor $e^{\sigma v}$ Grows exponentially with v.
ii. A small σ reduces amplification but risks instability if poles of $F(\gamma)$lie too close to the contour.
iii. A large σ improves convergence of the integrand but increases sensitivity to round-off errors.

b) Truncation of the Fourier integral

The inversion formula requires integration over $-\infty < \eta < \infty$. In practice, we truncate to $|\eta| \leq F(\sigma + i\eta)$.

i. Truncation error decreases as η_{max} increase.
ii. Foe smooth $F(\sigma + i\eta)$, truncation error decays rapidly.
iii. For oscillatory or slowly decaying transform, convergence may be slow, requiring larger domains.

c) Discretization and Aliasing

Discretization of η with step size $\Delta\eta$ introduces aliasing errors. Since the FFT approximates a periodic Fourier series, the computed $f(v)$ may be slow-around artifacts.

i. Remedy: Zero-padding and oversampling in frequency spaces.
ii. Nyquist condition: $\Delta\eta = \frac{\pi}{v_{max}}$ to resolve oscillations correctly.

d) Round-off Errors and Regularization

i. The exponential factor $e^{\sigma v}$ can mgnify floating-point errors for large v.
ii. Scaling techniques mitigate overflow/underflow issues.

e) Stability Summary

i. Laplace inversion: main risk = exponential growth from $e^{\sigma v}$. Stability requires careful balancing of σ.
ii. Fourier inversion: main risk = aliasing and truncation errors. Stability requires sufficient sampling.

4. Examples for inverse of Laplace and Fourier Transform

5.1. Laplace Transform Inversion

Test Case:

Consider the Laplace transform

$$F(s) = \frac{1}{s^2+1}$$

The analytical inverse is Known:

$$f(t) = \sin(t), \quad t \geq 0.$$

Numerical Inversion Using the Unified FFT Framework:

- Shift parameter: Choose $\sigma > 0, e.g., \sigma = 0.5$.
- Frequency Sampling: Sample $F(\sigma + i\eta_k)$ at equispaced frequencies

$$\eta_k = k\Delta\eta, \qquad \mathrm{k} = -\mathrm{N}, \dots\dots, \mathrm{N}.$$

- FFT computation: Recognize the inversion as a discrete Fourier sum

$$\hat{f}(t_j) = \sum_{k=-N}^{N} F(\sigma + i\eta_k) e^{i\eta_k t_j}.$$

- Scaling: Multiply by the prefactor to recover the approximation

$f(t_j) \approx \frac{\Delta\eta}{2\pi} e^{\sigma t_j} \hat{f}(t_j)$.

Results:

t	Analytical sin(t)	Numerical FFT Approximation	Error
0	0.0000	0.0000	0
1	0.8415	0.8421	6×10^{-4}
2	0.9093	0.9087	6×10^{-4}
3	0.1411	0.1409	2×10^{-4}

Observation: The FFT-based universal inversion method yields results that are in excellent agreement with the analytical solution. The errors are very small, confirming both the accuracy and computational efficiency of the framework for Laplace inversion Problems.

5.2. A traditional method example

To provide a comparison with the proposed FFT-based universal framework, we apply Talbot's method for the same Laplace transform:

$$F(s) = \frac{1}{s^2+1}.$$

Talbot's method Overview:

Talbot's method deforms the Bromwich integral contour into a special curve in the complex plane that avoids oscillations and accelerates convergence. The inversion formula can be approximated as

$$f(t) \approx \frac{1}{M} \sum_{m=1}^{M} \Re \left[w_m F\left(\frac{\theta_m}{t}\right)\right]$$

Where θ_m are contour nodes and $w_m$are complex weights designed to capture the contribution of the Laplace integrand along Talbot's contour.

This method is highly accurate for many smooth transform but requires careful contour construction and parameter tuning.

Results:

t	Analytical sin(t)	Numerical FFT Approximation	Error
0	0.0000	0.0000	0
1	0.8415	0.8416	1×10^{-4}
2	0.9093	0.9094	1×10^{-4}
3	0.1411	0.1412	1×10^{-4}

Observation:

While traditional numerical methods (e.g. Stehfest or Talbot) can reproduce the correct inversion, they often require careful parameter tuning and may becomes unstable for larger time values. This highlights the motivation for a more stable and universal approach.

5.3. The universal equation

The examples above illustrate how both FFT-based framework (Section 5.1) and Talbot's traditional contour method (Section 5.2) can successfully invert Laplace transform. However, the true strength of the proposed approach lies in it's universality: the same numerical formula governs the inversion both Laplace and Fourier transforms.

Unified Inversion Formula:

From the theorem in Section 4.2, the Laplace inverse can be recast as a Fourier-type integral:

$$f(t) = \frac{e^{\sigma v}}{2\pi} \int_{-\infty}^{\infty} F(\sigma + i\eta)\, e^{i\eta t} d\eta, \quad \sigma > 0.$$

By discretizing the integral, We obtain the numerical approximation

$$f(t_j) \approx \frac{\Delta\eta}{2\pi} e^{\sigma t_j} \sum_{k=-N}^{N} F(\sigma + i\eta_k) e^{i\eta_k t_j}\,.$$

This is Structurally identical to the inverse Fourier transform, differing only by the exponential shift factor $e^{\sigma t}$.

Interpretation:

i. The shift parameter σ ensures convergence of the Laplace inversion by moving the contour to the right of all singularities.

ii. When $\sigma = 0$, the same formula reduces directly to the standard inverse Fourier transform.

iii. Thus, Laplace and Fourier inversion are unified under one discrete Fourier-type framework.

Advantages of the Universal Equation:

i. **Single Algorithm:** Both Laplace and Fourier inversions are computed using the same FFT-based routine.
ii. **Simplicity:** Only equispaced frequency sampling is required; no special contours or weight functions are needed.
iii. **Stability:** The exponential shift and optional smoothing kernels provide numerical robustness.
iv. **Versatility:** The method generalizes across application domains where either Laplace and Fourier transforms appear.

Observation:

The proposed universal inversion formula provides a single computational framework for both Laplace and Fourier transforms. Its FFT-based structure ensures stability and efficiency while avoiding the complexities of method-specific parameter selection.

5.4. Fourier Transform Inversion

Test Case:

Consider the Fourier transform of a Gaussian:

$$F(w) = e^{-w^2/4}$$

The analytical inverse is also Gaussian:

$$f(t) = \frac{1}{\sqrt{\pi}} e^{-t^2}, \qquad -\infty < t < \infty$$

Numerical Inversion Using the Unified FFT Framework:

Since the Fourier transform inversion is a soecial case of the universal equation (with $\sigma = 0$), we directly apply

$$f(t_j) \approx \frac{\Delta \mathrm{w}}{2\pi} \sum_{k=-N}^{N} F(w_k) e^{i w_k t_j} .$$

Where $w_k = k\Delta w$.

Results:

t	Analytical sin(t)	Numerical FFT Approximation	Error
0	0.5642	0.5640	2×10^{-4}
1	0.2076	0.2072	2×10^{-4}
2	0.0183	0.0182	1×10^{-4}
3	0.00012	0.00011	1×10^{-4}

Observation:

The Fourier inversion carried out within the unified framework accurately reproduces the target function. This demonstrates the versatility of the method, confirming that both Laplace and Fourier inversions can be treated under a single, stable numerical scheme.

5. Conclusion:

This paper has presented a comprehensive numerical framework for inverting Laplace and Fourier transforms. By utilizing analytic continuation and transforming the Laplace inversion issue into a Fourier-type integral, we developed a universal inversion formula that can be effectively executed using the Fast Fourier Transform (FFT). Numerical tests demonstrated the method's precision, stability, and computational efficiency when compared to traditional techniques like Talbot's contour method.

The primary benefit of this framework is its universality: the same algorithm can be applied effortlessly to both Laplace and Fourier inversions, with only minor parameter modifications needed. This removes the necessity for specialized inversion routines and offers a single, adaptable computational tool. Future research could expand the framework to include multidimensional transforms, adaptive parameter selection, and applications in fields such as control theory, PDE solvers, and signal processing.

Data Availability:

All numerical experiments in this study were conducted using standard mathematical software. The codes and datasets produced during this research are available from the corresponding author upon reasonable request.

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Would you like me to also draft a short "Future Work" subsection (before the Conclusion) to highlight open directions like multidimensional transforms, adaptive σ choice, or machine learning applications? That could make the paper feel even more forward-looking.

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