

Cost Analysis Approach-the $M^x/M/1$ Queues with Working Breakdowns and Interruptions

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Abstract

In this paper, we have studied the steady-state analysis of single-server Markovian bulk arrival queueing systems with working breakdowns and interruptions. For this model, the probability-generating functions of the system size distributions in steady state are obtained. Various important characteristics of the model, like the expected number of customers in the system, the expected waiting time of a customer in the system, etc., are derived. In steady state, we have discussed some important performance measures of the system and provided some numerical examples to show that our results are computationally tractable. The suitability of the model is tested in a cost optimization issue where the mean number of customers is optimized by reducing the average cost per unit hour using a direct search technique.

Keywords: Poisson arrival, Markovian Queue, vacation, single server, working breakdown, maintenance.

1 INTRODUCTION

Markovian queues are a major subfield of applied probability. In fact, Markovian queues are crucial to the theory and practice of continuous-time Markov chains as well as to the creation of broader queueing models. The current analysis considers a Markovian queue in which the server experiences failures while serving customers. Instead of stopping all service delivery during a breakdown phase, the server reduces the service pace.

Good references are just one of several that [1] V. Jacob, S.R. Chakravarthy, and A. Krishnamoorthy discussed the interpretation of interruptions caused by customers and the retrying of those consumers. N.K. Jaiswal conducted research on the preemptive resume priority queue in [3]. [4] Jau-Chuan Ke used server breakdown and startup/closedown times to model the batch arrival queues under vacation policies. [5] Kim BK and Lee DH investigated and examined the M/G/1 queue that had malfunctions and catastrophic events. The M/G/1/1 queue modeled by B. Krishna Kumar, D. Arivuadainambi, and A. Vijayakumar has an unreliable server and no waiting capacity. [6]. [7] A study conducted by Krishnamoorthy A, Pramod PK, and Chakravarthy examined interrupted lines. [8] In stochastic models, Keilson, J., and Servi, L.D. investigated the queue as a distributional form of matrix-geometric solutions for Little's law. [9] Khalaf R. F., Madan K. C., and Lukas C. A. were taken into consideration as a queue model that included Bernoulli schedules for random breakdowns, general repair times, general vacation periods, and general prolonged vacation times. The cost optimization of a single-server queue with working breakdowns under the N policy was examined by Chen, J.-Y., Yen, T.-C., and Wang, K.-H. [10].

The article has a structure as follows: The model is illustrated in Section 2. In Section 3, we derive the stability condition through the matrix geometric method. For the model, the probability-generating functions of the system size distributions in steady state are obtained in Section 4. Stochastic decomposition is provided in Section 5. Various important characteristics for the model, like the expected number of customers in the system, the expected waiting time of a customer in the system, etc., are derived from Section 6. Section 7 discusses the reliability measures of the model. Practical applications is provided in section 8. Numerical examples are exhibited to study the parameters and their effects on the model in Section 9. In Section 10, we have discussed the cost model and its optimization through a direct search method.

2 THE MODEL

The inter-arrival stream of customers constitutes a compound Poisson process with rate λ . Services are provided to the customer who is at the head of the queue and the service is single at a time according to an exponential distribution with rate μ . The arrival stream of customers joins the system in batches/bulks. The size of batches is a random variable X of nonnegative values with a probability distribution $b_k = Pr \{X = k\}, k \geq 1$. The server is possible to partial failures while providing service to the customer. At the time of failure, the server continues the service process with slower service rate $\mu_b (\mu_b < \mu)$. The time between failures and the time between repairs is assumed to be an exponential random variable with rate α and γ respectively. After service completion, the server is sent to repair. Once the repair process completed, the server starts to provide service in regular service rate μ . The arrival process, service, breakdown and maintenance times are assumed to be mutually independent.

Let $C(t)$ be the state of the server at time t . Then

$$C(t) = \begin{cases} 0, & \text{the server is in maintenance state,} \\ 1, & \text{the server is in normal state,} \\ 2, & \text{the server is in working breakdown state} \end{cases}$$

Let $X(t)$ be the number of customers present in the system at time t . Then $\{(C(t), X(t)), t \geq 0\}$ is a CTMC.

$$P_{i,n}(t) = Prob\{C(t) = i, X(t) = n\}, i = 0,1,2, \text{ and } n \geq 0.$$

3 STABILITY CONDITION

A matrix geometric method is applied to analyze the problem and to derive the stability condition of the model. Using the lexicographical order for the states, the infinitesimal generator of the process $\{(X(t), C(t)), t \geq 0\}$ can be written as a block Jacobi matrix and is denoted as

$$Q = \begin{bmatrix} A_{00} & A_1 & A_2 & A_3 & A_4 & A_5 & \dots & \\ B_{00} & B_0 & B_1 & B_2 & B_3 & B_4 & \dots & \\ \begin{matrix} \square \\ \square \end{matrix} & C_0 & B_0 & B_1 & B_2 & B_3 & \dots & \dots \\ \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & C_0 & B_0 & B_1 & B_2 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \text{-----(1)}$$

Where C_0, B_0 and $B_i (i = 1, 2, \dots)$ are matrices of order m (the number of environmental states, that is $m=4$ in this study). B_0 is the generator of the environmental process, $B_i (i = 1, 2, \dots)$ are diagonal matrices. Each element of the matrix Q is listed in the following.

$$A_{00} = \begin{bmatrix} -\lambda + \gamma & \gamma \\ 0 & -\lambda \end{bmatrix},$$

$$A_i = \begin{bmatrix} \lambda_{g_i} & 0 & 0 \\ 0 & \lambda_{g_i} & 0 \end{bmatrix},$$

$$B_{00} = \begin{bmatrix} 0 & 0 \\ 0 & \mu \\ \mu_b & 0 \end{bmatrix},$$

,

$$B_0 = \begin{bmatrix} -\lambda + \gamma & \gamma & 0 \\ 0 & -\lambda + \alpha + \mu & \gamma \\ 0 & 0 & -\lambda + \mu_b \end{bmatrix}$$

$$B_i = \begin{bmatrix} \lambda_{g_i} & 0 & 0 \\ 0 & \lambda_{g_i} & 0 \\ 0 & 0 & \lambda_{g_i} \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu & 0 \\ \mu_b & 0 & 0 \end{bmatrix},$$

Consider

$$b = \sum_{i=1}^{\infty} iB_i = \begin{bmatrix} \lambda_g & 0 & 0 \\ 0 & \lambda_g & 0 \\ 0 & 0 & \lambda_g \end{bmatrix}$$

where g is the expectation.

From the Theorem of Neuts (1981), we know that the steady-state probability vector exists if and only

if

$$Xbe < XC_0e \text{ -----(2)}$$

X is the invariant probability of the matrix $D = C_0 + B_0 + B_1 + \dots$

$$D = \begin{bmatrix} -\gamma & \gamma & 0 \\ 0 & -\alpha & \alpha \\ \mu_b & 0 & -\mu_b \end{bmatrix},$$

The vector X satisfies $XD=0$ and $Xe=1$, where $e = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{3 \times 1}$. After some calculations, the vector

$$X = \frac{1}{\mu_b\alpha + \mu_b\gamma + \alpha\gamma} [\mu_b\alpha \quad \mu_b\gamma \quad \mu_b\gamma]_{1 \times 3} \text{ -----(3)}$$

Can be obtained and substituting in the condition (2) and doing some algebraic manipulation leads to

$$\rho = \frac{\lambda g[\mu_b(\alpha+\gamma)+\alpha\gamma]}{\mu_b\gamma[\mu+\alpha]} < 1 \quad \text{-----(4)}$$

4 STEADY STATE ANALYSIS

The steady state probabilities equations governing the model are derived as follows:

$$(\lambda + \gamma)P_{0,0} = \mu_b P_{2,1} \quad , \quad n = 0 \quad \text{-----(5)}$$

$$(\lambda + \gamma)P_{0,n} = \lambda \sum_{i=1}^n g_i P_{0,n-i} + \mu_b P_{2,n+1} \quad , \quad n \geq 1 \quad \text{-----(6)}$$

$$\lambda P_{1,0} = \mu P_{1,1} + \gamma P_{0,0} \quad , \quad n = 0 \quad \text{-----(7)}$$

$$(\lambda + \mu + \alpha)P_{1,n} = \mu P_{1,n+1} + \gamma P_{0,n} + \lambda \sum_{i=1}^n g_i P_{1,n-i} \quad , \quad n \geq 1 \quad \text{(8)}$$

$$(\lambda + \mu_b)P_{2,1} = \alpha P_{1,1} \quad , \quad n = 1 \quad \text{-----(9)}$$

$$(\lambda + \mu_b)P_{2,n} = \alpha P_{1,n} + \lambda \sum_{i=1}^{n-1} g_i P_{2,n-i} \quad , \quad n \geq 2 \quad \text{-----(10)}$$

Define the following partial probability generating functions:

$$P_0(Z) = \sum_{n=0}^{\infty} P_{0,n} Z^n \quad , \quad P_1(Z) = \sum_{n=1}^{\infty} P_{1,n} Z^n \quad ,$$

$$P_2(Z) = \sum_{n=1}^{\infty} P_{2,n} Z^n \quad , \quad G(z) = \sum_{n=1}^{\infty} g_i z^n \quad , \quad i \geq 1$$

Solving the above equations along with the corresponding probability generating functions we get:

$$[\lambda Z(1 - G(Z)) + \gamma Z]P_0(Z) - \mu_b P_2(Z) = 0 \quad \text{-----(11)}$$

$$-\gamma P_0(Z) + [\lambda(1 - G(Z)) + \alpha + \mu(1 - 1/Z)]P_1(Z) = [\alpha + \mu(1 - 1/Z)]P_{1,0} \quad \text{-----(12)}$$

$$[\lambda Z(1 - G(Z)) + \mu_b]P_2(Z) - \alpha P_1(Z) = -\alpha P_{1,0} \quad \text{-----(13)}$$

Which is turn yields

$$P_0(Z) = \frac{A_0(Z)}{\Delta'(Z)} \quad \text{-----(14)}$$

$$P_1(Z) = \frac{A_1(Z)}{\Delta'(Z)} \quad \text{-----(15)}$$

$$P_2(Z) = \frac{A_2(Z)}{\Delta'(Z)} \quad \text{-----(16)}$$

$$P(Z) = P_0(Z) + P_1(Z) + P_2(Z) + P_3(Z) \quad \text{-----(17)}$$

Where, $A_0(Z) = \lambda\mu_b\alpha G'(Z)P_{1,0}$

$$A_1(Z) = [\lambda^2\alpha[-2Z(1 - G(Z))G'(Z) + (1 - G(Z))]^2 + [\lambda\alpha\mu_b + \gamma\alpha\lambda][-ZG'(Z) + (1 - G(Z))] + [\lambda\mu\mu_b + \lambda\mu\gamma]x[(1 - Z)G'(Z) + (1 - G(Z))] + \lambda^2\mu(1 - Z)2(1 - G(Z))G'(Z) + (1 - G(Z))]^2 + \gamma\alpha\mu_b + \gamma\mu\mu_b]P_{1,0}$$

$$A_2(Z) = [2\lambda^2\alpha Z(1 - G(Z))G'(Z) - \alpha\lambda^2(1 - G(Z))]^2 - \gamma\lambda\alpha[(1 - G(Z) - ZG'(Z))]P_{1,0}$$

$$\Delta'(Z) = \lambda^3(1 - G(Z))^3 + (1 - G(Z))^2[-3\lambda^3ZG'(Z) + \lambda^2(\mu_b + \alpha + \mu + \gamma) + \lambda(\alpha\mu_b + \mu\mu_b + \gamma\mu_b + \gamma\alpha + \gamma\mu)] + G'(Z)\lambda Z[-\gamma\alpha - \gamma\mu + \frac{\gamma\mu}{Z} - \gamma\mu_b - \alpha\mu_b - \mu\mu_b + \frac{\mu\mu_b}{Z}] + \gamma\mu_b[\alpha + \mu]$$

Now letting Z=1 then the above equations can be rewritten as follows:

$$P_0(1) = \frac{G'(1)\mu_b\lambda\alpha P_{1,0}}{-\lambda G''(1)[\gamma\alpha + \gamma\mu_b + \alpha\mu_b] + \gamma\mu_b[\alpha + \mu]} \text{-----(18)}$$

$$P_1(1) = \frac{-\lambda\alpha G'(1)[\mu_b + \gamma] + \gamma\mu_b[\mu + \alpha]P_{1,0}}{-\lambda G'(1)[\mu_b\gamma + \alpha\gamma + \alpha\mu_b] + \mu_b\gamma[\mu + \alpha]} \text{-----(19)}$$

$$P_2(1) = \frac{\lambda\alpha\gamma G'(1)P_{1,0}}{-\lambda G'(1)[\mu_b\gamma + \alpha\gamma + \alpha\mu_b] + \mu_b\gamma[\mu + \alpha]} \text{-----(20)}$$

Using the normalization condition and solving the above equations:

$$P_0(1) + P_1(1) + P_2(1) = 1$$

We obtain

$$P_{1,0} = \frac{-\lambda G'(1)[\mu_b\gamma + \alpha\gamma + \alpha\mu_b] + \mu_b\gamma[\mu + \alpha]}{\gamma\mu_b[\mu + \alpha]} \text{-----(21)}$$

From equation (5),(7) and (9) the other probabilities are obtained as follows:

$$P_{0,0} = \frac{\lambda\alpha\mu_b P_{1,0}}{[\mu(\lambda + \gamma)(\lambda + \mu_b) + \gamma\alpha\mu_b]} \text{-----(22)}$$

$$P_{2,1} = \frac{\lambda(\lambda + \gamma)(\mu_b\alpha)P_{1,0}}{[\mu(\lambda + \gamma)(\lambda + \mu_b) + \gamma\alpha\mu_b]} \text{-----(23)}$$

5 STOCHASTIC DECOMPOSITION APPROACH

This section concerns the analysis of the stochastic decomposition property of the system size distribution. The literature on vacations models recognizes this property as one of the most interesting features on this matter. Yang and Templeton [11] studied the stochastic decomposition of retrial queues, whose applications were discussed later by Artalejo and Falin [12] and Yang [13]. The classical interpretation of the stochastic decomposition property shows that the system size distribution decomposes into two random variables one of which corresponds to the system size of the ordinary queue without working breakdowns and the other random variable gives the increase in the system size due to the presence of working breakdowns in the system.

In particular, in the context of our system, we observe that the probability generating function of the system size P(z) can be written as

$$P(Z) = P_c(Z) \times P_{wb}(Z)$$

Where

$$P_{wb}(Z) = \frac{N(Z) \times U(Z)}{\Delta'(Z)\gamma\mu_b[\mu + \alpha][\lambda_g - \mu][1 - Z]}$$

$$P_c(Z) = \frac{(\lambda_g - \mu)(1 - Z)}{(\lambda + \mu)Z - \mu - \lambda ZG(Z)}$$

Where,

$$N(Z) = \lambda^2\mu(1 - G(Z))^2 + \lambda(1 - G(Z))[\alpha\mu_b + \mu\mu_b + \mu\gamma + 2\lambda\mu(1 - Z)G'(Z)] + G'(Z)\lambda(1 - Z)[\mu_b\alpha + \mu\mu_b + \mu\gamma] + \gamma\mu_b[\alpha + \mu]$$

$$U(Z) = -\mu\gamma\mu_b[\alpha + \mu] + \lambda^2ZG'(1)G(Z)[\alpha\gamma + \gamma\mu_b + \alpha\mu_b] - \lambda ZG(Z)\gamma\mu_b[\alpha + \mu] - \lambda G'(1)Z(\lambda + \mu)[\alpha\gamma + \gamma\mu_b + \alpha\mu_b] + (\lambda + \mu)Z\gamma\mu_b[\alpha + \mu] + \lambda G'(1)\mu[\alpha\gamma + \gamma\mu_b + \alpha\mu_b]$$

$P_c(Z)$ is the probability generating function of the random variable N which is the number in the system in the classical $M^x/M/1$ queueing model. $P_{wb}(Z)$ is the probability generating function of the extra load in the system due to the presence of working break downs.

6 PERFORMANCE MEASURES

According to the distribution of the steady state, various system performance measures can be developed.

1.Expected number of customers in the system: $E(X) = P_0'(1) + P_1'(1) + P_2'(1)$

$$\left[\frac{\theta_1\theta_4 - \theta_3\theta_2}{2\theta_1^2} + \frac{\theta_1\theta_6 - \theta_5\theta_2}{2\theta_1^2} + \frac{\theta_1\theta_8 - \theta_7\theta_2}{2\theta_1^2} \right]$$

Where,

$$\theta_1 = -\lambda G'(1)[\mu_b\alpha + \gamma\alpha + \gamma\mu_b] + \mu_b\gamma[\mu + \alpha]$$

$$\theta_2 = -\lambda G''(1)[\mu_b\alpha + \gamma\alpha + \gamma\mu_b] + G'(1)^2 2\lambda^2[\gamma + \mu_b + \alpha] - 2\lambda G'(1)[\gamma\mu + \gamma\alpha + \mu_b\mu + \gamma\mu_b + \alpha\mu_b]$$

$$\theta_3 = \mu_b\lambda\alpha G'(1)P_{1,0}$$

$$\theta_4 = \mu_b\lambda\alpha G''(1)P_{1,0}$$

$$\theta_5 = [-\lambda G'(1)[\alpha\mu_b + \alpha\gamma] + \gamma\mu_b[\mu + \alpha]]P_{1,0}$$

$$\theta_6 = P_{1,0}[-2\lambda G'(1)[\alpha\mu_b + \alpha\gamma + \mu\mu_b + \mu\gamma] + 2\lambda^2\alpha(G'(1))^2 - \lambda G''(1)[\gamma\mu_b + \gamma\alpha]]$$

$$\theta_7 = \gamma\lambda\alpha G'(1)P_{1,0}$$

$$\theta_8 = [-2\alpha\lambda^2(G'(1))^2 + 2\gamma\lambda\alpha G'(1) + \gamma\lambda\alpha G''(1)]P_{1,0}$$

2.Expected number of customer customers in the queue

$$E(Q) = \sum_{n=1}^{\infty} (n - 1)P_n$$

3.Expected waiting time in the system $E(W_s) = \frac{E(X)}{\lambda_g}$

4.Expected waiting time in the queue $E(W_q) = \frac{E(Q)}{\lambda_g}$

5. The proportion of time, the server is busy = $P_1(1) - P_{1,0}$

6. The proportion of time, the server being in maintenance $Q_0 = P_0(1)$

7. The proportion of time, the server being normal

$$Q_1 = P_1(1)$$

8. The proportion of time, the server being in non-reliable state $Q_2 = P_2(1)$

7 RELIABILITY MEASURES

In queuing theory literature, some studies are devoted to reliability measures under different queuing situations because of its importance. Availability is defined as the probability that the system is operating properly when it is requested for use. On the other hand, failure frequency of the server is the probability when the system is not operating in normal condition. Thus, we obtain the following two important reliability measures,

(i) The availability of the server is

$$P_1(1) = \frac{-\lambda\alpha G'(1)[\mu_b + \gamma] + \gamma\mu_b[\mu + \alpha]}{-\lambda G'(1)[\gamma\alpha + \gamma\mu_b + \alpha\mu_b] + \gamma\mu_b[\alpha + \mu]}$$

(ii) The failure frequency of the server is $P_0(1) + P_2(1) = P_{1,0} \frac{G'(1)\mu_b\lambda\alpha + \lambda\alpha\gamma G'(1)}{-\lambda G'(1)[\gamma\alpha + \gamma\mu_b + \alpha\mu_b] + \gamma\mu_b[\alpha + \mu]}$

8 PRACTICAL APPLICATION

In queueing theory, a breakdown time refers to a period when a system, such as a server or a machine, becomes unavailable due to maintenance, repair, or unexpected failure. During breakdown times, the service provided to customers may be affected, leading to delays and disruptions in the queueing system. One strategy for managing breakdown times in queueing systems is to continue providing service at a reduced rate rather than stopping work immediately. This approach can help in several ways: In a restaurant setting, if a piece of equipment breaks down, like a dishwasher or oven, the staff might continue serving customers but at a slower pace, using alternative methods or equipment until the issue is resolved. This way, they can still provide some level of service to customers, rather than abruptly stopping operations.

Customer Service Centers: In call centers or customer service departments, agents may continue to work but at a reduced capacity during breakdowns, such as when their computer systems experience technical issues or when they face disruptions in internet connectivity. This can lead to longer call handling times and increased queue lengths, affecting customer satisfaction levels.

Public Transportation: During breakdowns or delays in public transportation systems like buses or trains, operators may still provide service, but at a slower pace. For example, a bus experiencing mechanical problems might continue its route but at reduced speed, leading to longer waiting times at bus stops and increased passenger frustration.

Health care Facilities: In hospitals or clinics, medical staff may continue to provide care during equipment breakdowns or technical issues, but at a slower pace or with limited functionality. For

instance, a medical imaging machine experiencing technical difficulties may still be operational but produce scans at a slower rate, affecting patient throughput and waiting times.

Manufacturing Plants: In manufacturing settings, breakdowns of machinery or equipment can result in slower production rates. Workers may attempt to continue production using alternative methods or manual processes, but output levels are typically reduced, leading to bottlenecks in the production line and delays in fulfilling orders.

Information Technology Services: In IT departments or data centers, slow work during breakdown times can occur when servers or network infrastructure experience performance degradation or outages. IT staff may implement temporary fixes or workarounds to maintain essential services, but at a reduced capacity, resulting in slower response times for user requests and increased downtime.

However, it's important to communicate with customers about the situation transparently. Letting them know about the issue, the steps being taken to resolve it, and any potential delays can help manage expectations and maintain customer satisfaction.

9 NUMERICAL RESULTS

In this section we discuss a few illustrative examples to show the qualitative aspects of the queuing model under study. We consider rated fifteen examples including 4 on reliability measures. The purpose of the following four examples are to study the reliability measures, varying some selected parameters (the failure rate and repair rate). We fix $\lambda = 5, \mu = 15, \mu_b = 10$.

Example 1 Figure 1 illustrate the availability of the server against the failure rate α . We observe that the availability of the server under study state decreases with increase in the failure rate α for the different repair rates $\gamma = 2, 2.5$ and 3 as expected.

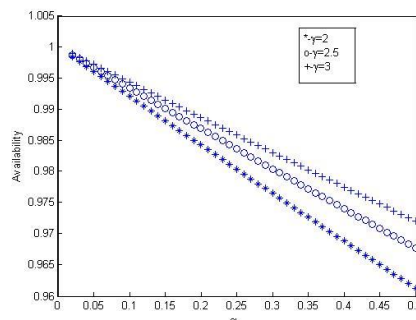


Figure 1: The effect of α on the server availability

Example 2 Figure 2 tells the effect of the repair rate γ for three different values of the failure rate $\alpha = 1.2, 1.4$ and 1.6 on the availability of the server. It is seen that the availability of the server increases with increasing γ . This is because the increasing repair rate brings the server to available.

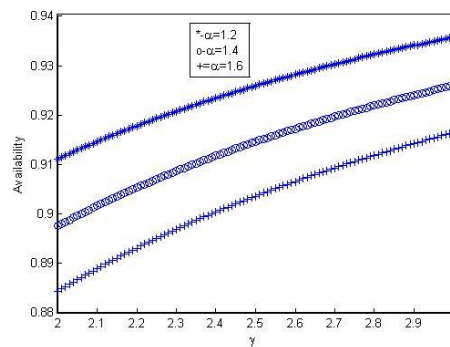


Figure 2: The effect of γ on the server availability

Example 3 Figure 3. depicts variation of the failure frequency of the server against the failure rate α for three different values of the arrival rate $\lambda = 4, 5$ and 6 . As we expect, the failure frequency of the server increases with increasing in the failure rate α .

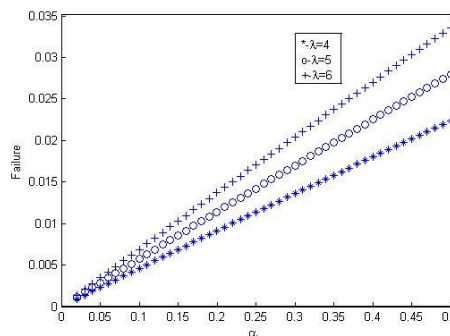


Figure 3: The effect of α on the server failure

Example 4 Figure 4 displays the effect of the repair rate γ on the failure frequency of the server for three different values of λ . The failure frequency of the server decreases with increase in the repair rate γ as it is quite natural.

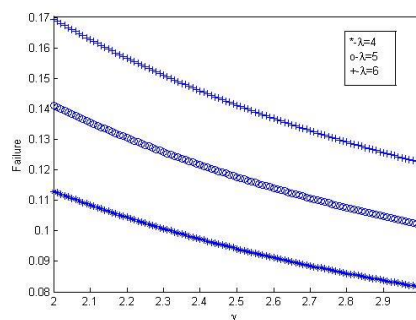


Figure 4: The effect of γ on the server failure

Example 5 In this example, we study the effect of varying λ and μ on the probability $P_{1,0}$. We fix $\alpha = 2$, $\mu_b = 50$, $\gamma = 3$, $G'(1) = 0.4$ and $G''(1) = 0.03$. As we expected, in figure 5 we notice that the probability $P_{1,0}$ appears to decrease as a function λ for fixed μ and for fixed λ the probability measure $P_{1,0}$ increases as a function of the service rate μ .

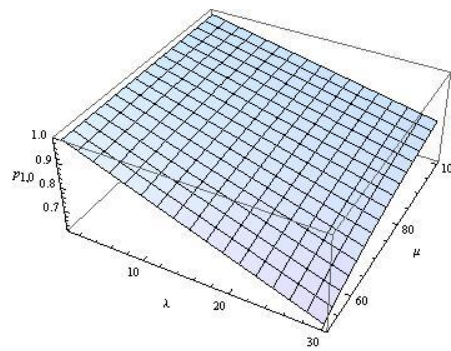


Figure 5: The effect of λ and μ on $P_{1,0}$

Example 6 In this example, we study the effect of varying λ and μ on the probability $P_{0,0}$. As we expected, figure 6 shows that the probability measure $P_{0,0}$ decreases as a function of μ for fixed λ . For fixed μ the probability measure $P_{0,0}$ decreases as function of the arrival rate λ .

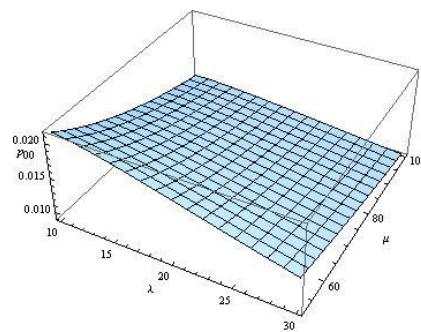


Figure 6: The effect of λ and μ on $P_{0,0}$

Example 7 In this example, we study the effect of service rate μ and the arrival rate λ on the probability measure $P_0(1)$ and the results are plotted in the figure 7. It is seen that the probability measure $P_0(1)$ increases as a function of the arrival rate λ for fixed μ . Also, we observe that the probability measure $P_0(1)$ decreases as a function of the service rate μ for fixed λ .

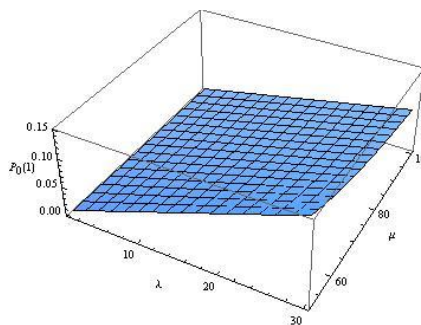


Figure 7: The effect of λ and μ on $P_0(1)$

Example 8 This example, clearly shows the effect of varying arrival rate λ and the service rate μ on the probability measure $P_1(1)$. Here figure 8 shows that the probability measure $P_1(1)$ decreases as a function of the arrival rate λ for fixed μ and for fixed λ we observe that the probability measure $P_1(1)$ decreases as a function of the service rate μ for fixed λ .

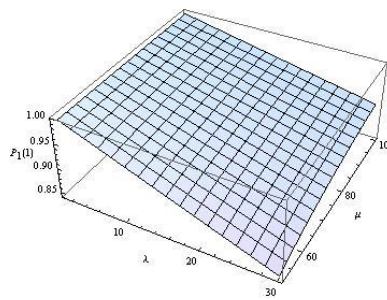


Figure 8: The effect of λ and μ on $P_1(1)$

Example 9 In this example, we study the effect of varying λ and μ on the probability measure $P_2(1)$. As we expected, in figure 9 we notice that the probability measure $P_2(1)$ appears to increase as a function of λ for fixed μ and it decreases as a function of the service rate μ for fixed λ .

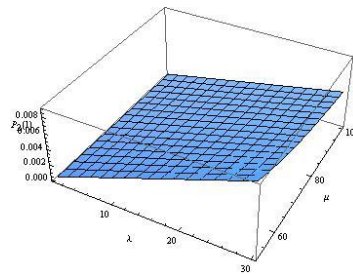


Figure 9: The effect of λ and μ on $P_2(1)$

Example 10 In this example, we investigate the impact of the normal service rate μ and partial service rate μ_b on the study state mean number of customers in the system under the ergodicity condition. In figure 10, we notice that the study state mean number of customer $E(X)$ increases as the partial service rate μ_b for fixed normal service rate μ and it also increases for the normal service rate μ for the fixed partial service rate μ_b .

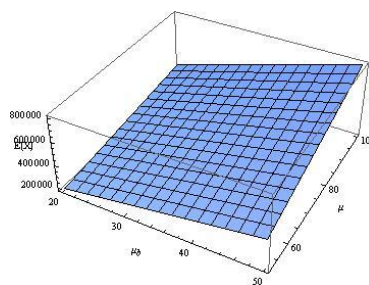


Figure 10: The effect of μ and μ_b on $E(X)$

Example 11 we study the effect of the varying arrival rate λ and the repair rate γ on the probability measure $P_0(1)$ and the results are plotted in the figure 11. It is seen that the probability measure $P_0(1)$ increases as the function of the arrival rate λ for fixed repair rate γ and we also observe that the probability measure $P_0(1)$ decreases as the function of the repair rate γ for fixed arrival rate λ .

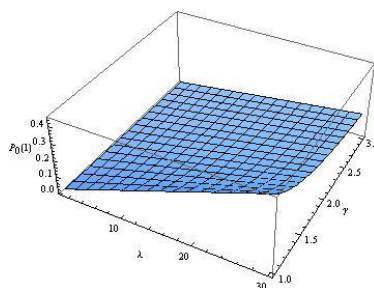


Figure 11: The effect of λ and γ on $P_0(1)$

Example 12 This example, clearly shows the effect of varying λ and the failure rate α on the probability measure $P_1(1)$. As we expected, in figure 12 we notice that the probability measure $P_1(1)$ appears to decrease as a function of arrival rate λ for fixed failure rate α . For fixed λ the probability measure $P_1(1)$ decreases as a function of the failure rate α .

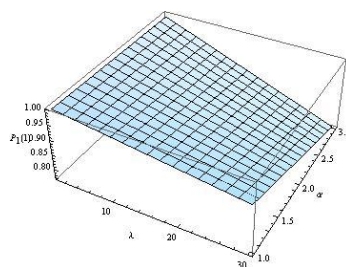


Figure 12: The effect of λ and α on $P_1(1)$

Example 13 In this example, we study the effect of the varying service rate μ and the failure rate α on the probability measure $P_1(1)$. It is clearly noticed in figure 13 that the probability measure $P_1(1)$ increases as a function μ for fixed failure rate α and the same probability measure $P_1(1)$ decreases as a function of the failure rate α for fixed service rate μ .

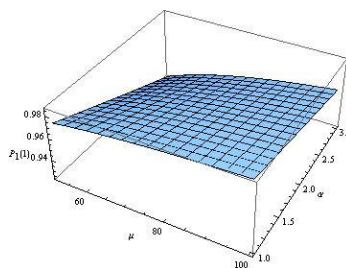


Figure 13: The effect of μ and α on $P_1(1)$

Example 14 we study the effect of varying arrival rate λ and the failure rate α on the probability measure $P_2(1)$ and the results are plotted in figure 14. It is seen that the probability measure $P_2(1)$ increases as the function of the arrival rate λ for fixed failure rate α . Also observed that the probability measure $P_2(1)$ increases as the function of the failure rate α for fixed λ .

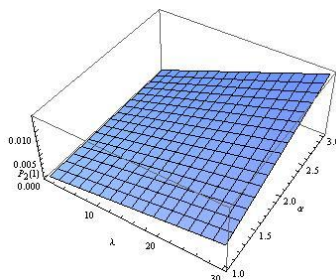


Figure 14: The effect of λ and α on $P_2(1)$

10 THE COST MODEL AND ITS OPTIMIZATION

In this section, we study the total expected cost function with decision variable μ and μ_b for the above discussed model. Our main aim is to obtain the optimal values for the continuous random variables say (μ^*, μ_b^*) in order to minimize the cost.

Cost Function: Let us define the following cost elements:

C_0 : cost per unit time when the server is in maintenance state.

C_1 : cost per unit time for when the server is busy during the normal service period.

C_2 : break down cost per unit time for a broken server.

C_3 : fixed cost for fast service rate and

C_4 : fixed cost for slow service rate.

Using the definition of these cost elements listed above, the expected cost function per unit time is given by

$$F(\mu, \mu_b) = C_0 P_0'(1) + C_1 P_1'(1) + C_2 P_2'(1) + C_3 \mu + C_4 \mu_b \quad \text{-----}(24)$$

The cost minimization problem can be formulated as $F(\mu^*, \mu_b^*) = \text{Minimize } F(\mu, \mu_b)$, Subject to $\mu > \mu_b$ and $\rho < 1$ -----(25)

The complexity of expressions $P_1'(1)$, $P_2'(1)$ and $P_0'(1)$ complicate the cost function in Equation. Unfortunately, it is impossible to derive the analytic solutions for the optimal service rates at the minimum expected cost. Thus, we sought to compute the numerical results of the optimal service rates μ^* and μ_b^* using Direct search method.

Direct Search Method:

We assume the following cost parameters as $C_0 = 200, C_1 = 500, C_2 = 300, C_3 = 20, C_4 = 10$

Numerical examples are presented to determine the optimal value μ by means of the direct search method. Also, we fix $\lambda=2, \alpha =0.02, \gamma =1, G'(1)=0.4, G''(1)=0.03$ Varying μ from 4 to 14, and choosing different values of μ_b , the minimum expected cost $F(\mu, \mu_b)$ are shown in Table 1 and figure 15 for

$\mu_b=2.5,3,3.5$ the minimum expected cost Rs.222.3553 is achieved at $\mu =5$ for $\mu_b=2.5$, Rs. 227.1716 is achieved at $\mu =5$ for $\mu_b=3$ and Rs. Rs.232.0458 is achieved at $\mu =5$ for $\mu_b=3.5$.

It is seen that initially the total cost decreases and starts increasing with the grant of μ for fixed values of μ_b . The convex nature of the cost function with respect to μ show the trend for the optimum cost by increasing the normal service domain of the customers. From Table2, we concluded that the minimum expected cost Rs.204.1258 is attained at $\mu =5$ for $\lambda =1.5$, Rs.232.0458 is attained at $\mu =5$, for $\lambda = 2$ and Rs.2517.3021 is attained at $\mu =6$ for $\lambda =4.2$ for fixed $\lambda =2.5$, $\alpha =0.002$, $\mu_b=3.5$, $\gamma =1$, $G'(1)=0.4$, $G''(1)=0.03$.

From figure16 it is evident that total cost function decreases first and then increases consequently the convex nature arises in the total cost function. This confirms the possibility of obtaining the optimum service rates. From Table 3, we concluded that the minimum expected cost Rs.236.9736 is attained at $\mu =5$ for $\alpha =0.1$, Rs.242.3192 is attained at $\mu =6$ for $\alpha =0.2$ and Rs.247.2811 is attained at $\mu =6$ for $\alpha =0.3$ for fixed $\lambda =2$, $\gamma =1$, $\mu_b=3.5$, $G'(1)=0.4$, $G''(1)=0.03$. As we hoped, the figure 17 shows the convexity in the total cost function. From Table 4, we conclude that the minimum expected cost Rs.367.8189 is attained at $\mu =9$ for $\gamma =0.2$ Rs.275.5773 is attained at $\mu =7$ for $\gamma =0.45$ and Rs.251.891 is attained at $\mu =6$ for $\gamma =0.6$ for fixed $\lambda =2$, $\mu_b=3$, $\gamma =0.6$, $\alpha =0.2$, $G'(1)=0.4$, $G''(1)=0.03$. As we hoped, the figure 18 shows the convexity in the total cost function.

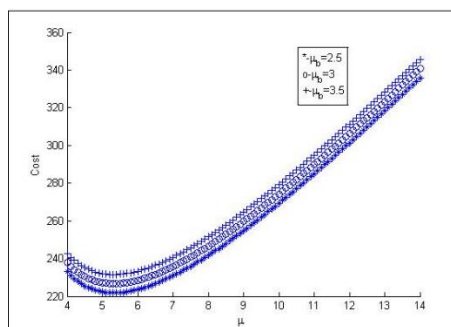


Figure 15: The effect of μ on the total cost function

μ	$\mu_b = 2.5$		$\mu_b = 3$		$\mu_b = 3.5$	
	$P_{1,0}$	cost	$P_{1,0}$	cost	$P_{1,0}$	cost
4	0.7954	232.9611	0.7957	237.7114	0.7959	242.5405
5	0.8362	222.3553	0.8364	227.1716	0.8365	232.0458
6	0.8634	223.5591	0.8636	228.4138	0.8637	233.3144
7	0.8828	230.8441	0.883	235.724	0.8831	240.6418
8	0.8975	241.6706	0.8976	246.5683	0.8977	251.4983

Table 1: The effect of μ on the cost function for various values of μ_b

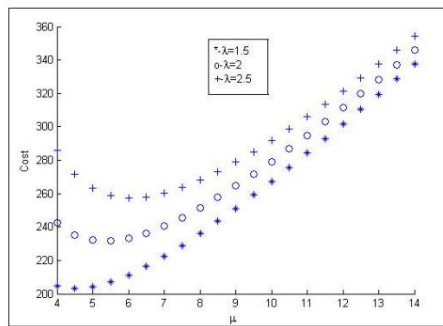


Figure 16: The effect of μ on the total cost function

μ	$\lambda = 1.5$		$\lambda = 2$		$\lambda = 2.5$	
	$P_{1,0}$	cost	$P_{1,0}$	cost	$P_{1,0}$	cost
4	0.8469	204.5071	0.7959	242.5405	0.7448	286.1305
5	0.8774	204.1258	0.8365	232.0458	0.7957	263.0633
6	0.8978	211.303	0.8637	233.3144	0.8296	257.3021
7	0.9123	222.4923	0.8831	240.6418	0.8539	20.1645
8	0.9233	236.0656	0.8977	251.4983	0.8721	267.9425

Table 2: The effect of μ on the cost function for various values of λ

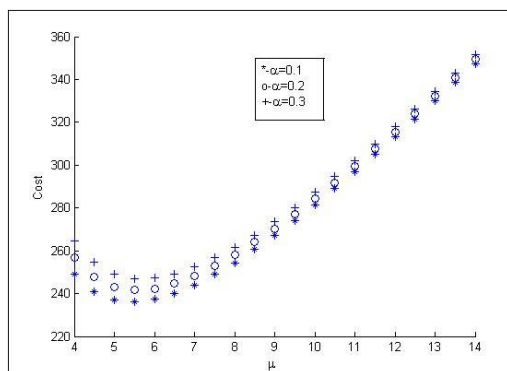


Figure 17: The effect of μ on the total cost function

μ	$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.3$	
	$P_{1,0}$	cost	$P_{1,0}$	cost	$P_{1,0}$	cost
4	0.7798	248.9268	0.7605	256.8546	0.7422	264.7262
5	0.823	236.9736	0.8066	243.0957	0.7908	249.1787
6	0.852	237.3286	0.8378	242.3192	0.824	247.2811
7	0.8728	244.0292	0.8603	248.243	0.8481	252.435
8	0.8885	254.4287	0.8774	258.0757	0.8664	261.7056
9	0.9008	267.1723	0.8907	270.3873	0.8808	273.5886

Table 3: The effect of μ on the cost function for various values of α

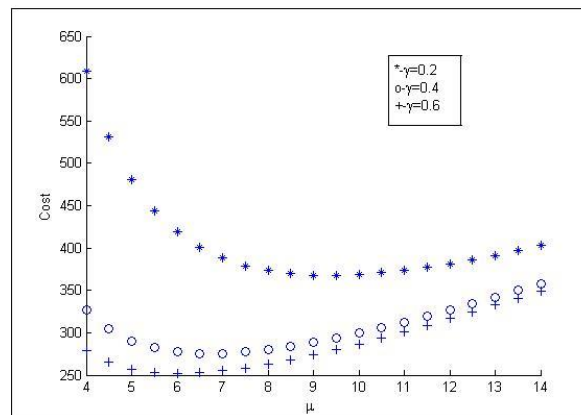


Figure 18: The effect of μ on the total cost function

μ	$\gamma = 0.2$		$\gamma = 0.45$		$\gamma = 0.6$	
	$P_{1,0}$	cost	$P_{1,0}$	cost	$P_{1,0}$	cost
4	0.6063	608.7468	0.7016	326.7336	0.7333	278.7352
5	0.6821	480.2275	0.759	290.554	0.7846	257.072
6	0.7333	419.026	0.7978	277.4138	0.8194	251.891
7	0.7704	388.0538	0.8259	275.5773	0.8444	255.0319
8	0.7984	373.1639	0.8472	280.1133	0.8634	262.9564
9	0.8203	367.8189	0.8638	288.5918	0.8783	273.8824

Table 4: The effect of μ on the cost function for various values of γ

Conclusion

We have studied the steady-state analysis of single-server Markovian bulk arrival queueing systems with working breakdowns and interruptions. For this model, the probability-generating functions of the system size distributions in steady state are obtained. Various important characteristics of the model, like the expected number of customers in the system, the expected waiting time of a customer in the system, etc., are derived. Numerical examples are presented to study the parameters and their effects on the model. Optimizing the cost function is analyzed using MATLAB. Mathematica is used for presenting three-dimensional figures.

References

- [1] V.Jacob, S.R.Chakravarthy, A.Krishnamoorthy, (2012). On a customer-induced interruption in a service system. *Stochastic Analysis and Applications* 30(6),949-962.
- [2] V.Jacob, A.Krishnamoorthy, (2015). Analysis of customer-Induced interruption and retrieval of interrupted customers. *American Journal of Mathematical and Management Sciences* 34(4),34-.
- [3] Jaiswal, N.K. (1961). Preemptive resume priority queue. *Operations Research*, 732-770.
- [4] Jau-Chuan Ke, (2007) Batch arrival queues under vacation policies with server breakdown and startup/closedown times, *Applied Mathematical modeling*, Vol.31, 1282-1292.
- [5] Kim BK. Lee DH. (2014) The M/G/1 queue with disasters and working breakdowns, *Applied Mathematical Modelling* 38, 1788-1798.
- [6] B.Krishna Kumar.,D.Arivuadainambi and A.Vijayakumar,(2002). An M/G/1/1 queue with unreliable server and no waiting capacity, *Information and Management Science*, 13(2) 35-50.
- [7] Krishnamoorthy A, Pramod PK, Chakravarthy SR.9(2014) Queues with interruptions: a survey. *Top*,22:290-320.
- [8] Keilson, J., and Servi, L.D. (1988): *A Distributional form of Little’s law*. M.F. (1994): *Matrix-Geometric Solutions in Stochastic Models - An Algorithmic Approach*, 2nd ed., Dover Publications, Inc., NewYork.

- [9] Khalaf R. F., Madan K. C. and Lukas C. A. (2011). An $M^x/G/1$ Queue with Bernoulli Schedule General Vacation Times, General Extended Vacations, Random Breakdowns, General Delay Times for Repairs to Start and General Repair Times. *Journal of Mathematics Research*, 3, 8 -20.
- [10] Chen, J.-Y., Yen, T.-C., and Wang, K.-H., (2016). "Cost Optimization of a Single- Server Queue with Working Breakdowns Under the N Policy," *Journal of Testing and Evaluation*, Vol 44, No.5, 2059-2067.
- [11] Neuts, M. (1981). *Matrix geometric solutions in stochastic models*, Baltimore: John Hopkins University Press.