Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 3s (2024)

Orthogonal Generalized Symmetric Reverse Bi-(σ, τ)-Derivations of Semi Prime Ring

V.S.V. Krishna Murty[1], K.Chennakesavulu[2], C. Jaya Subba Reddy[3]
krishnamurty.vadrevu@gmail.com[1], intell.chenna@gmail.com[2], cjsreddysvu@gmail.com[3]
[1]Research Scholar, Department of Mathematics, S.V.University, Tirupati, Andhra Pradesh, India.
[2]Professor, Department of Mathematics, PVKKIT, Anantapur, Andhra Pradesh, India.
[3]Professor, Department of Mathematics, S. V. University, Tirupati, Andhra Pradesh, India.

Article History:
Received: *05-04-2024*
Revised: *25-05-2024*
Accepted: *10-06-2024*

Abstract:

Let R be a semi prime ring. Suppose that σ, τ are automorphisms on R. A symmetric bi-additive mapping $\delta_1: RX\ R \rightarrow R$ is said to be a generalized symmetric reverse bi-(σ,τ)-derivation on R if there exists a symmetric reverse bi-(σ,τ)-derivation D on R such that $\delta_1(uv, w) = \delta_1(v, w)\sigma(u) + \tau(v)D(u, w)$ holds $\forall\ u, v, w \in R$. Let $[\delta_1, D_1]$ and $[\delta_2, D_2]$ be two generalized symmetric reverse bi-(σ,τ)-derivations of R with associated reverse bi-(σ, τ)-derivations D_1, D_2. In this paper, we establish some equivalent conditions for the orthogonality between two symmetric generalized reverse bi-(σ,τ)-derivations of semiprime ring R.

Keywords: Semiprime ring, Generalized reverse biderivation, Generalized reverse bi-(σ,τ)-derivation, Orthogonal biderivation.

1. INTRODUCTION:

The concept of orthogonal derivation was introduced by M. Bresar and J. Vukman [14] and proved some results on the orthogonal derivations of semiprime rings which were related to Posner's First Theorem [9]. Some results on (σ, τ)-derivations in prime rings were studied by M. Ashraf [13] and K. Kaya et al. [12] . J.C. Chang [11] introduced the notion of a generalized (α, β)-derivation of a ring R and investigated some properties of such derivations. Argac et al. [17] introduced the notion of orthogonality for a pair (D, d), (G, g) of generalized derivations on semiprime rings and gave several necessary and sufficient conditions for (D, d) and (G, g) to be orthogonal. O.Golbasi and N. Aydin [18] extended the results of Argac to orthogonal generalized (σ, τ)-derivations. Orthogonality of generalized (σ, τ)-derivations on ideals of semiprime rings was studied in [10]. Several studies were established on the orthogonality of derivations, biderivations by M.N. Daif et al. [15] and C. Jaya Subba Reddy et al. [2,4, 5]. A.Ali et al. [1] and M.N.Daif et al. [16] established some results on biderivations of prime and semiprime rings and the study of orthogonality of symmetric bi-(σ, τ)-derivations in semi prime rings was carried out in [3,6]. Recently, C. Jaya Subba Reddy et al. [7, 8] have studied orthogonal symmetric reverse bi-(σ,τ)-derivations in semi prime rings and orthogonal generalized reverse (σ, τ)- derivations in semiprime Γ-rings. In the present paper, we extended the results of orthogonality on generalized symmetric bi-(σ, τ)-derivations established in [6] to generalized symmetric reverse bi-(σ, τ)-derivations**.**

2. PRELIMINARIES:

Throughout this paper, R will denote an associative ring with center Z(R). A ring R is known to be semiprime if $uRu = \{0\}$ implies $u = 0, \forall\, u \in R$. We say that R is 2-torsion-free if $2u = 0$ implies $u = 0,\ \forall\, u \in R$. An additive mapping $d: R \to R$ is said to be a derivation (respectively, reverse derivation) on R if $d(uv) = d(u)v + ud(v)$ (respectively, $d(uv) = d(v)u + vd(u)$ holds for all $u, v \in R$. Suppose that σ and τ are automorphisms of R. An additive mapping $d: R \to R$ is said to be a (σ,τ)-derivation (respectively, reverse (σ,τ)-derivation) on R if $d(uv) = d(u)\sigma(v) + \tau(u)d(v)$ (respectively, $d(uv) = d(v)\sigma(u) + \tau(v)d(u)$) holds, $\forall\, u, v \in R$.

An additive mapping $D_1: R \to R$ is called a generalized derivation (respectively, generalized reverse derivation) if there exists a derivation (respectively, reverse derivation) 'd' such that $D_1(uv) = D_1(u)v + ud(v)$ (respectively,$D_1(uv) = D_1(v)u + vd(u)$ holds $\forall\, u, v \in R$. An additive mapping $D_1: R \to R$ is called a generalized (σ,τ)-derivation (respectively, generalized reverse (σ, τ)-derivation) if there exists a (σ,τ)-derivation (respectively, reverse (σ,τ)-derivation) 'd' such that $D_1(uv) = D_1(u)\sigma(v) + \tau(u)d(v)$ (respectively, $D_1(uv) = D_1(v)\sigma(u) + \tau(v)d(u)$ holds $\forall\, u, v \in R$. Thus, the concept of generalized (σ, τ)-derivation covers the concept of (σ,τ)-derivation.

A bi additive mapping $D_1: R X\, R \to R$ is said to be symmetric if $D_1(u, v) = D_1(v, u)$. A symmetric bi additive mapping $D_1: R X\, R \to R$ is said to be a symmetric biderivation on R if $D_1(uv, w) = uD_1(v, w) + D_1(u, w)v$ holds $\forall\, u, v, w \in R$. A symmetric biadditive mapping $D_1: R X\, R \to R$ is said to be a symmetric bi-(σ,τ)-derivation (respectively, symmetric reverse bi-(σ,τ)-derivation) on R if $D_1(uv, w) = D_1(u, w)\sigma(v) + \tau(u) D_1(v, w)$ (respectively, $D_1(uv, w) = D_1(v, w)\sigma(u) + \tau(v) D_1(u, w)$ holds $\forall\, u, v, w \in R$. A symmetric biadditive mapping $\delta_1: R X\, R \to R$ is said to be a generalized symmetric biderivation (respectively, generalized symmetric reverse biderivation) on R if there exists a symmetric biderivation (respectively, symmetric reverse biderivation) D_1 on R such that $\delta_1(uv, w) = \delta_1(u, w)v + uD_1(v, w)$(respectively,$\delta_1(uv, w) = \delta_1(v, w)u + vD_1(u, w)$, $\forall\, u, v, w \in R$. A symmetric biadditive mapping $\delta_1: R X\, R \to R$ is said to be a generalized symmetric bi-(σ,τ)-derivation (respectively, generalized symmetric reverse bi-(σ,τ)-derivation) on R if there exists a symmetric bi-(σ,τ)-derivation (respectively, symmetric reverse bi-(σ,τ)-derivation) D_1 on R such that $\delta_1(uv, w) = \delta_1(u, w)\sigma(v) + \tau(u) D_1(v, w)$ (respectively, $\delta_1(uv, w) = \delta_1(v, w)\sigma(u) + \tau(v)D_1(u, w)$ holds $\forall\, u, v, w \in R$. Two symmetric reverse bi-(σ,τ)-derivations D_1, D_2 are said to be orthogonal if $D_1(u, v)RD_2(v, w) = \{0\} = D_2(v, w)RD_1(u, v)$, for all $u, v, w \in R$. Two generalized symmetric reverse bi-(σ,τ)-derivations δ_1, δ_2 are said to be orthogonal if $\delta_1(u, v)R\delta_2(v, w) = \{0\} = \delta_2(v, w)R\delta_1(u, v)$, for all $u, v, w \in R$.

We assume throughout the paper that R is a 2-torsion-free semiprime ring, while σ and τ are automorphisms of R. Also D_1, D_2 are reverse bi-(σ,τ)-derivations of R such that $D_1\tau = \tau D_1$, $D_2\tau = \tau D_2$, $\sigma D_1 = D_1\sigma$, $\sigma D_2 = D_2\sigma$. We denote two generalized reverse bi-(σ, τ)-derivations $\delta_1: R X\, R \to R$ and $\delta_2: R X\, R \to R$ determined by reverse bi-(σ,τ)-derivations D_1, D_2 of R be such that $\delta_1\tau = \tau\delta_1$, $\delta_2\tau = \tau\delta_2$, $\sigma\delta_1 = \delta_1\sigma$, $\sigma\delta_2 = \delta_2\sigma$.

Lemma 1: [Lemma 1,[14]]

If R is a 2-torsion free semi prime ring and $u, v \in R$, then the following conditions are equivalent:

1.$urv = 0$, for all $r \in R$.

2.$vru = 0$, for all $r \in R$.

3.$urv + vru = 0$, for all $r \in R$.

If anyone of the above conditions is fulfilled, then $uv = vu = 0$.

LEMMA 2: [LEMMA 2, [5]]

Let R be a semiprime ring. Suppose that two bi-additive mappings $D_1: R X R \rightarrow R$ and $D_2: R X R \rightarrow R$ satisfies $D_1(u,v)RD_2(v,u) = \{0\}$, $\forall$ u, v $\in$ R, then $D_1(u,v)RD_2(v,w) = \{0\}$, $\forall$ u, v , w $\in$ R.

LEMMA 3: [THEOREM 1, [7]]

Let R be a 2 torsion free semi prime ring. Then the following conditions are equivalent:

1.Two symmetric reverse bi-(σ,τ)-derivations D_1 and D_2 are orthogonal.

2.$D_1(u,v)\, D_2(v,w) + D_2(u,v)D_1(v,w) = 0$, $\forall$ u , v , w $\in$ R.

LEMMA 4: Let R be a 2 torsion free semi prime ring. Then two symmetric reverse bi-(σ,τ)-derivations D_1 and D_2 are orthogonal if and only if $D_1D_2 = 0$.

Proof: Suppose that D_1 and D_2 are orthogonal.

Since D_1, D_2 are orthogonal, we can have $D_1(u,v)rD_2(v,w) = 0$, $\forall\, u , v , w, r \in R$

$D_1(D_1(u,v)\, rD_2(v,w), m) = 0$, $\forall\, u , v , w, r, m \in R$

$D_1(D_2(v,w),m)\sigma(r)\sigma(D_1(u,v) + \tau(D_2(v,w))D_1(r,m)\sigma(D_1(u,v)) + \tau(rD_2(v,w)$
$D_1(D_1(u,v),m)$=0.

Using $D_1\sigma = \sigma D_1$, $\tau D_2 = D_2\tau$ and σ and τ are automorphisms of R, we have

$D_1(D_2(v,w),m)rD_1(u,v) + D_2(v,w)\, D_1(r,m)\, D_1(u,v) + rD_2(v,w)\, D_1(D_1(u,v),m)$=0.

Using the condition of orthogonality of $D_{1\,,}D_2$, we get

$D_1D_2(v,w)rD_1(u,v)$=0.

In Particular if we put $u = D_2(v,w)$ in the above equation, we get

$D_1D_2(v,w)rD_1(D_2(v,w),v) = 0$

$D_1D_2(v,w)rD_1D_2(v,w) = 0$

$D_1D_2(v,w) = 0$ (By Semiprimeness of R.)

$D_1D_2 = 0$.

Conversely,

Let D_1 and D_2 be two reverse bi-(σ,τ)-derivations such that $D_1D_2 = 0$.

$D_1D_2(uv,w) = D_1(D_2(uv,w),m)$

$=D_1(D_2(\,v,w)\sigma(u) + \tau(v)\, D_2(u,w),m)$

$=D_1(D_2(v,w)\sigma(u),m) + D_1(\tau(v)D_2(u,w),m)$

$=D_1(\sigma(u),m)\sigma(D_2(v,w)) + \tau(\sigma(u))D_1(D_2(v,w),m) + D_1(D_2(u,w),m)\sigma(\tau(v)) +$

$\tau(D_2(u,w))D_1(\tau(v),m).$

Using $\sigma D_2 = D_2\sigma$, $\tau D_2 = D_2\tau$; σ and τ are automorphisms of R and $D_1D_2 = 0$, we obtain

$0 = D_1(u,m)\,D_2(v,w) + D_2(u,w)D_1(v,m)$.

In particular, $D_1(u,w)\,D_2(v,w) + D_2(u,w)D_1(v,w) = 0$. Therefore

$D_1(u,w)\,D_2(w,v) + D_2(u,w)D_{\cdot 1}(w,v) = 0, \forall\, u, v, w \in R$. (Since $D_{1,}D_2$ are symmetric)

By Lemma 3, we can conclude that D_1 and D_2 are orthogonal.

3. MAIN RESULTS:

THEOREM 1:

If (δ_1, D_1) and (δ_2, D_2) are two orthogonal generalized symmetric reverse bi-(σ,τ)-derivations of R, then (δ_1, D_1) and (δ_2, D_2) are orthogonal if and only if the following conditions are satisfied:

(i) $\delta_1(u,v)\delta_2(v,w) + \delta_2(u,v)\,\delta_1(v,w) = 0, \forall\, u, v, w, r \in R$.

(ii) $D_1(u,v)\delta_2(v,w) + D_2(u,v)\,\delta_1(v,w) = 0, \forall\, u, v, w, r \in R$.

Proof: Suppose that (δ_1, D_1) and (δ_2, D_2) are orthogonal generalized symmetric reverse bi-(σ,τ)-derivations of R.

By the definition of orthogonality δ_1 and δ_2, we have

$\delta_1(u,v)r\delta_2(v,w) = 0 = \delta_2(u,v)r\delta_1(v,w)$ (3.1)

Hence, $\delta_1(u,v)\delta_2(v,w) = 0 = \delta_2(v,w)\delta_1(u,v)$ (By Lemma 1) (3.2)

and so $\delta_1(u,v)\delta_2(v,w) + \delta_2(v,w)\delta_1(u,v) = 0$

$\delta_1(u,v)\delta_2(v,w) + \delta_2(w,v)\delta_1(v,u) = 0$ (Since δ_1, δ_2 are symmetric)

$\delta_1(u,v)\delta_2(v,w) + \delta_2(u,v)\delta_1(v,w) = 0$.

Hence condition (i) is proved .

Now, Suppose that $\delta_1(u,v)\delta_2(v,w) = 0$. (From 3.2)

Again replacing u by ur, r ∈ R in the above equation and using (3.2), we get

$\delta_1(r,v)\sigma(u)\,\delta_2(v,w) + \tau(r)D_1(u,v)\delta_2(v,w) = 0$.

Since σ , τ are automorphisms, we get

$\delta_1(r,v)u\delta_2(v,w) + r\,D_1(u,v)\delta_2(v,w) = 0, \forall\, u, v, r, w \in R$.

Using the equation (3.1), we get

$rD_1(u,v)\,\delta_2(v,w) = 0, \forall\, u, v, r, w \in R$.

Left multiplying the above equation by $D_1(u,v)\delta_2(v,w)$ and using the semiprimeness of R, we get $D_1(u,v)\delta_2(v,w) = 0.$ (3.3)

Replacing u by ru, $r \in R$ in (3.3) , we obtain

$D_1(u,v)\sigma(r)\,\delta_2(v,w) + \tau(u)D_1(r,v)\delta_2(v,w) = 0.$

Using (3.3) and σ is an automorphism of R, we get

$D_1(u,v)r\,\delta_2(v,w) = 0, \forall\, u,v,r,w \in R\,.$

In view of Lemma 1, we can have

$D_1(u,v)r\delta_2(v,w) = 0 = \delta_2(v,w)rD_1(u,v)$ (3.4)

and $D_1(u,v)\delta_2(v,w) = 0 = \delta_2(v,w)D_1(u,v).$ (3.5)

Again by the definition of orthogonality of δ_1 and δ_2, we have

$\delta_1(u,v)r\delta_2(v,w) = 0 = \delta_2(u,v)r\delta_1(v,w)$, for all $u,v,w \in R.$ (3.6)

By Lemma (1), we can have $\delta_1(u,v)\delta_2(v,w) = \delta_2(v,w)\delta_1(u,v) = 0$

and also $\delta_2(u,v)\delta_1(v,w) = 0.$

Consider $\delta_2(u,v)\delta_1(v,w)=0, \forall\, u,v,w \in R$. (3.7)

Replacing u by ur, $r \in R$ in the equation (3.7) and using (3.6), we obtain

$\tau(r)D_2(u,v)\delta_1(v,w) = 0, \forall\, u,v,w,r \in R.$

Left Multiplying the above equation by $D_2(u,v)\delta_1(v,w)$ and using the semiprimeness of R, we get $D_2(u,v)\delta_1(v,w) = 0.$ (3.8)

Replacing u by ru,$r \in R$ in (3.8) and using (3.8), we get

$D_2(u,v)\sigma(r)\delta_1(v,w) = 0$, for all $u,v,w,r \in R.$

$D_2(u,v)\sigma(r)\delta_1(v,w) = 0.$

Since σ is an automorphism, we obtain

$D_2(u,v)\,r\delta_1(v,w) = 0 = \delta_1(v,w)\,rD_2(u,v), \forall\, u,v,w,r \in R.$ (3.9)

By lemma 1, $D_2(u,v)\delta_1(v,w) = 0 = \delta_1(v,w)D_2(u,v)$, for all $u,v,w \in R.$ (3.10)

From (3.5) and (3.10), we can have $D_1(u,v)\delta_2(v,w) + D_2(u,v)\delta_1(v,w) = 0.$

Hence condition (ii) is Proved.

Conversely,

Suppose the conditions (i) $\delta_1(u,v)\delta_2(v,w) + \delta_2(u,v)\,\delta_1(v,w) = 0$ and (3.11)

(ii) $D_1(u,v)\delta_2(v,w) + D_2(u,v)\,\delta_1(v,w) = 0$ holds good. (3.12)

We prove that (δ_1, D_1) and (δ_2, D_2) are orthogonal generalized symmetric reverse bi-(σ,τ)-derivations of R.

Replacing u by ru, $r \in R$ in (3.11) and using (3.12), we get

$\delta_1(u,v)\sigma(r)\delta_2(v,w) + \delta_2(u,v)\sigma(r)\delta_1(v,w) = 0$, for all $u,v,w,r \in R$.

Since σ is an automorphism, we get

$\delta_1(u,v)r\delta_2(v,w) + \delta_2(u,v)r\delta_1(v,w) = 0$, for all $u,v,w,r \in R$.

By Lemma 1, we can conclude that δ_1 and δ_2 are orthogonal.

THEOREM 2:

If (δ_1, D_1) and (δ_2, D_2) are two orthogonal generalized symmetric reverse bi-(σ,τ)-derivations of R, then (δ_1, D_1) and (δ_2, D_2) are orthogonal iff $\delta_1(u,v)\delta_2(v,w) = 0 = D_1(u,v)\delta_2(v,w)$.

Proof: Suppose that (δ_1, D_1) and (δ_2, D_2) are orthogonal generalized symmetric reverse bi-(σ,τ)-derivations of R.

By the definition of orthogonality, we have $\delta_1(u,v)R\delta_2(v,w) = \{0\}$.

$\delta_1(u,v)r\delta_2(v,w) = 0$ and so $\delta_1(u,v)\delta_2(v,w) = 0$, for all $u,v,w \in R$. (By Lemma1) (3.13)

From the equation (3.5) of Theorem 1, we have $D_1(u,v)\delta_2(v,w) = 0$.

Hence, we conclude that $\delta_1(u,v)\delta_2(v,w) = 0 = D_1(u,v)\delta_2(v,w)$, for all $u,v,w \in R$.

Conversely,

Suppose that $\delta_1(u,v)\delta_2(v,w) = D_1(u,v)\delta_2(v,w) = 0$, for all $u,v,w \in R$. (3.14)

We have to prove that δ_1 and δ_2 are orthogonal

Consider $\delta_1(u,v)\delta_2(v,w) = 0$. (From (3.14)) (3.15)

Replacing u by ru, $r \in R$ in (3.15) and using (3.14), we get

$\delta_1(u,v)\sigma(r)\delta_2(v,w) = 0$.

Since σ is an automorphism, we get

$\delta_1(u,v)r\delta_2(v,w) = 0$, for all $u,v,w,r \in R$.

Therefore, δ_1 and δ_2 are orthogonal.

THEOREM 3:

If (δ_1, D_1) and (δ_2, D_2) are two orthogonal generalized symmetric reverse bi-(σ,τ)-derivations of R, then (δ_1, D_1) and (δ_2, D_2) are orthogonal if and only if $\delta_1(u,v)\delta_2(v,w) = 0$ and $D_1\delta_2 = 0 = D_1D_2$.

Proof: Suppose that (δ_1, D_1) and (δ_2, D_2) are orthogonal generalized symmetric reverse bi-(σ,τ)-derivations of R.

By the definition of orthogonality, it is evident that

$\delta_1(u,v)r\delta_2(v,w) = 0$ and so $\delta_1(u,v)\delta_2(v,w) = 0$. (By using Lemma 1)

To Prove that $D_1\delta_2 = 0$:

Consider $\delta_2(v,w)rD_1(u,v) = 0$ (By the equation (3.4) of Theorem1)

$\delta_1(\delta_2(v,w)rD_1(u,v), m) = 0$

$\delta_1(rD_1(u,v), m)\,\sigma(\delta_2(v,w)) + \tau\big(rD_1(u,v)\big)D_1(\delta_2(v,w), m) = 0$

$(\delta_1(D_1(u,v), m)\sigma(r) + \tau\big(D_1(u,v)\big)D_1(r,m))\,\sigma(\delta_2(v,w) + \tau\big(rD_1(u,v)\big)D_1(\delta_2(v,w), m) = 0.$

Using $D_1\tau = \tau D_1$, $\sigma\delta_2 = \delta_2\sigma$ and σ and τ are automorphisms, we can have

$\delta_1(D_1(u,v), m)r\delta_2(v,w) + D_1(u,v)D_1(r,m)\delta_2(v,w) + rD_1(u,v)D_1(\delta_2(v,w), m) = 0.$ (3.16)

Using the condition of orthogonality of δ_1 and δ_2 , we can have $\delta_1(D_1(u,v), m)r\delta_2(v,w) = 0$ and by Theorem 2, we can have $D_1(r,m)\,\delta_2(v,w) = 0$, for all $v, w, r, m \in R$.

By applying the above conditions in (3.16), we get

$rD_1(u,v)D_1(\delta_2(v,w), m) = 0$, for all $r, u, v, w \in R$.

Left Multiplying the above equation by $D_1(u,v)D_1(\delta_2(v,w), m)$ and using the semiprimeness of R, we get $D_1(u,v)D_1(\delta_2(v,w), m) = 0$

$D_1(u,v)D_1\delta_2(v,w) = 0.$ (3.17)

Replacing u by $u\delta_2(v,w)$ in (3.17), we get

$(D_1(\delta_2(v,w), v)\sigma(u) + \tau\big(\delta_2(v,w)\big)\,D_1(u,v))D_1\delta_2(v,w) = 0.$

Using (3.17) and σ is an automorphism of R, we obtain

$D_1(\delta_2(v,w), v)uD_1\delta_2(v,w) = 0$, for all $u, v, w \in R$

$D_1\delta_2(v,w)\,u\,D_1\delta_2(v,w) = 0$

$D_1\delta_2(v,w)RD_1\delta_2(v,w) = 0$

$D_1\,\delta_2 = 0.$ (By the semiprime ness of R)

To Prove that $D_1D_2 = 0$:

Let (δ_1, D_1) and (δ_1, D_2) are two orthogonal generalized symmetric reverse bi-(σ,τ)-derivations of a semi prime ring R.

First we prove D_1 and D_2 are orthogonal.

By the definition of orthogonality of δ_1 and δ_2 , we have

$\delta_1(u,v)r\delta_2(v,w) = 0$, for all $u, v, w \in R$.

By Lemma 1, we have $\delta_1(u,v)\delta_2(v,w) = 0.$ (3.18)

Replacing u by ru, $r \in R$ in the equation (3.18), we get

$\delta_1(u,v)\sigma(r)\delta_2(v,w) + \tau(u)D_1(r,v)\delta_2(v,w) = 0$, for all $u, v, w, r \in R$.

Since σ and τ are automorphisms of R, we can have

$\delta_1(u,v)r\,\delta_2(v,w) + uD_1(r,v)\delta_2(v,w) = 0$, for all $u, v, w, r \in R$. (3.19)

Replacing w by rw, $r \in R$ in equation (3.19) and using the fact that σ and τ are automorphisms , we get

$\delta_1(u,v)r\,\delta_2(v,w)r + \delta_1(u,v)rwD_2(v,r) + uD_1(r,v)\delta_2(v,w)r + uD_1(r,v)\,wD_2(v,r) = 0.$

Using the condition of orthogonality of δ_1 and δ_2 , equation (3.9) of Theorem 2 and also by Theorem 2, the first three terms of the above equation are zero.

The above equation reduces to $uD_1(r,v)\,wD_2(v,r) = 0$, for all $u,v,r \in R$.

Left Multiplying the above equation by $D_1(r,v)\,wD_2(v,r)$ and using the semipriness of R, we obtain $D_1(r,v)wD_2(v,r) = 0$.

In particular, $D_1(u,v)\,wD_2(v,u) = 0$

$D_1(u,v)\,wD_2(u,v) = 0$

$D_1(u,v)\,wD_2(v,w) = 0$. (By lemma 2)

Therefore, D_1 and D_2 are orthogonal.

By Lemma 4, we can have $D_1D_2 = 0$.

Hence, the two conditions are proved.

Conversely

Suppose that $\delta_1(u,v)\delta_2(v,w) = 0$ and $D_1\delta_2 = 0 = D_1D_2$. (3.20)

Let $D_1\delta_2 = 0$.

$D_1\delta_2(uv,w) = D_1(\delta_2(uv,w),m)$, for all $u,v,w,m \in R$

$= D_1(\delta_2(v,w)\sigma(u) + \tau(v)D_2(u,w),m)$.

Since σ and τ are automorphisms, we get

$=D_1(\delta_2(v,w)u + vD_2(u,w),m)$

$=D_1(u,m)\sigma\big(\delta_2(v,w)\big) + \tau(u)D_1(\delta_2(v,w),m) + D_1(D_2(u,w),m)\sigma(v) +$

$\tau\big(D_2(u,w)\big)D_1(v,m)$, for all $u,v,w,m \in R$.

Again using the fact that σ and τ are automorphisms, $\delta_2\sigma = \sigma\,\delta_2$, $\tau D_2 = D_2\tau$ we get

$=D_1(u,m)\delta_2(v,w) + uD_1(\delta_2(v,w),m) + D_1(D_2(u,w),m)v + D_2(u,w)D_1(v,m)$

$=D_1(u,m)\delta_2(v,w) + uD_1\delta_2(v,w) + D_1D_2(u,w)v + D_2(u,w)D_1(v,m), \forall\, u,v,w,m \in R.$ (3.21)

By hypothesis, we have $D_1\delta_2 = 0 = D_1D_2$.

By Lemma 4, we have $D_1D_2 = 0$ implies D_1, D_2 are orthogonal and hence

$D_2(u,w)D_1(v,m) = 0, \forall\, u,v,w,m \in R$.

Therefore, equation (3.21) becomes

$D_1\delta_2(uv,w) = D_1(u,m)\delta_2(v,w)$, for all $u,v,w,m \in R$

But $D_1\delta_2 = 0$ and hence we have $D_1(u,m)\delta_2(v,w)=0$. (3.22)

Replacing u by ru, r ∈ R in the equation (3.22) and using the fact that σ and τ are automorphisms of R, we obtain,

$D_1(u,m)\, r\delta_2(v,w) + u_1 D_1(r,m)\delta_2(v,w) = 0$

$D_1(u,m)\, r\, \delta_2(v,w) = 0$. (By the equation (3.22), $\forall\, u,m,r \in R$

By Lemma (1), we have $D_1(u,m)\, \delta_2(v_1,w) = 0 = \delta_2(v,w) D_1(u,m)$.

In Particular, $D_1(u,v)\, \delta_2(v,w) = 0 = \delta_2(v,w)\, D_1(u,v)$. (3.23)

From (3.20) and (3.23), we have $\delta_1(u,v)\delta_2(v,w) = 0 = D_1(u,v)\, \delta_2(v,w)$.

By Theorem 2, we have δ_1 and δ_2 are orthogonal.

THEOREM 4: If (δ_1, D_1) and (δ_2, D_2) are two orthogonal generalized symmetric reverse bi-(σ,τ)-derivations of R, then (i) D_1 and D_2 are orthogonal reverse bi-(σ,τ)-derivations (ii) $\delta_2 D_1 = 0$ (iii) $\delta_1 D_2 = 0$ (iv) $D_2\delta_1 = 0$ (v) $\delta_1\delta_2 = 0$ (vi) $\delta_2\delta_1 = 0$.

Proof: (i) To Prove $\delta_2 D_1 = 0$:

Suppose that δ_1, δ_2 are two orthogonal generalized symmetric reverse bi-(σ,τ)-derivations of R.

Then, by the definition of orthogonality, we have $\delta_1(u,v)r\delta_2(v,w) = 0$, $\forall\, u,v,w,r \in R$.

By Lemma 1, we can write $\delta_1(u,v)\delta_2(v,w) = 0$.

Replacing u by ru, r ∈ R in the above equation, we get

$\delta_1(ru,v)\delta_2(v,w) = 0$

$\delta_1(u,v)\sigma(r)\delta_2(v,w) + \tau(u)D_1(r,v)\delta_2(v,w) = 0$. (3.24)

Replacing w by rw, r ∈ R in equation (3.24),

$\delta_1(u,v)\sigma(r)\delta_2(v,rw) + \tau(u)D_1(r,v)\delta_2(v,rw) = 0$

$\delta_1(u,v)\sigma(r)\delta_2(v,w)\sigma(r) + \delta_1(u,v)\sigma(r)\tau(w)D_2(v,r) + \tau(u)D_1(r,v)\delta_2(v,w)\sigma(r)$

$+ \tau(u)D_1(r,v)\tau(w)D_2(v,r) = 0, \forall\, u,v,w,r \in R$. (3.25)

Since σ, τ are automorphisms of R,

Using equations (3.1), (3.9) and (3.5), the first three terms are zero, then equation (3.25) reduces to $uD_1(r,v)wD_2(v,r) = 0, \forall\, u,v,w,r \in R$

Then, $D_1(r,v)wD_2(v,r)uD_1(r,v)wD_2(v,r) = 0, \forall\, u,v,w,r \in R$.

By the semiprimeness of R, we get

$D_1(r,v)wD_2(v,r) = 0$ which is same as $D_1(r,v)wD_2(r,v) = 0$.

Using Lemma 2, we can write $D_1(r,v)wD_2(v,u) = 0, \forall\, u,v,w,r \in R$.

By Lemma1, $D_1(r,v)D_2(v,u) = 0 = D_2(v,u)D_1(r,v) = 0, \forall\, u,v,r \in R$. (3.26)

which shows that D_1 , D_2 are orthogonal.

(ii)To Prove $\delta_2 D_1 = 0$:

Since, δ_1, δ_2 are two orthogonal generalized symmetric reverse bi-(σ, τ)-derivation of R.

By equation (3.4) of Theorem 1, we have $D_1(u, v)r\, \delta_2(v, w) = 0 = \delta_2(v, w)rD_1(u, v)$.

Consider, $\delta_2(v, w)rD_1(u, v) = 0,\ \forall\, u, v, w, r \in R$

Then, $\delta_2(\delta_2(v, w)rD_1\,(u, v), m) = 0, \forall\, u, v, w, r, m \in R$

$(\delta_2(rD_1(u, v), m)\sigma(\delta_2(v, w)) + \tau(rD_1(u, v)D_2(\delta_2(v, w), m) = 0$

$\delta_2(D_1(u, v), m)\sigma(r)\sigma(\delta_2(v, w)) + \tau(D_1(u, v))D_2(r, m)\sigma(\delta_2(v, w)) +$
$\tau(rD_1\,(u, v)D_2(\,\delta_2(v, w), m) = 0.$

Using $\sigma\delta_2 = \delta_2\sigma$; $\tau D_1 = D_1\tau$; σ, τ are automorphisms of R and using the fact that D_1,D_2 are orthogonal, we have

$\delta_2(D_1(u, v), m)r\, \delta_2(v, w) = 0, \forall\, u, v, w, r \in R$

$\delta_2 D_1(u, v)r\, \delta_2(v, w) = 0.$

Replacing w by $w\delta_2(v, w)$ in the above equation, we obtain

$\delta_2 D_1(u, v)r\, \delta_2(D_1(u, v), v)w + \delta_2 D_1(u, v)r\, D_1(u, v)D_2(v, w) = 0.$

Since D_1 , D_2 are orthogonal are orthogonal, we get

$\delta_2 D_1(u, v)r\, \delta_2 D_1(u, v)w = 0, \forall\, u, v, w, r \in R$

$\delta_2 D_1(u, v)r\, \delta_2 D_1(u, v)w\delta_2 D_1(u, v)r\, \delta_2 D_1(u, v) = 0.$

Using the semiprimeness of R, we get

$\delta_2 D_1\,(u, v)r\, \delta_2 D_1\,(u, v) = 0$ (Since R is semiprime)

$\delta_2 D_1\,(u, v) = 0$

$\delta_2 D_1 = 0.$

(iii)To Prove $\delta_1 D_2 = 0$:

By (3.9) of Theorem 1, we have $\delta_1(v, w)\, rD_2(u, v) = 0,\ \forall\, u, v, w, r \in R$

$\delta_1(\delta_1(v, w)rD_2(u, v), m) = 0,\ \forall\, u, v, w, m, r \in R$

$\delta_1(rD_2(u, v), m)\sigma(\delta_1(v, w)) + \tau(rD_2(u, v))D_1(\delta_1(v, w), m) = 0$

$\delta_1(D_2(u, v), m)\sigma(r)\sigma(\delta_1(v, w)) + \tau(D_2(u, v))D_1(r, m)\sigma(\delta_1(v, w)) + \tau(rD_2(u, v))D_1(\delta_1(v, w), m) = 0,\ \forall\, u, v, w, r \in R.$

Using $\sigma\delta_1 = \delta_1\sigma$; $\tau D_2 = D_2\tau$; and σ and τ are automorphisms of R, we get

$\delta_1(D_2(u, v), m)r\delta_1(v, w) + D_2(u, v)D_1(r, m)\delta_1(v, w) + rD_2(u, v)D_1(\delta_1(v, w), m) = 0,$

$\forall\, u, v, w, r \in R.$

Using (3.26), the above equation reduces to $\delta_1(D_2(u, v), m)r\delta_1(v, w) = 0$

$\delta_1 D_2(u,v) r \delta_1(v,w) = 0, \forall\, u,v,w,r \in R.$

Replacing w by $wD_2(u,v)$ in the above equation and using the fact that σ and τ are automorphisms of R, we obtain

$\delta_1 D_2(u,v) r \delta_1\big(v, D_2(u,v)\big) w + \delta_1 D_2(u,v) r D_2(u,v) D_1(v,w) = 0, \ \forall\, u,v,w,r \in R.$

Using (3.26), the above equation reduces to

$\delta_1 D_2(u,v) r \delta_1\big(v, D_2(u,v)\big) w = 0$

$\delta_1 D_2(u,v) r \delta_1\big(v, D_2(u,v)\big) w \delta_1 D_2(u,v) r \delta_1\big(v, D_2(u,v)\big) = 0.$

Using the Semiprimeness of R, we get

$\delta_1 D_2(u,v) r \delta_1\big(v, D_2(u,v)\big) = 0$

$\delta_1 D_2(u,v) r \delta_1(D_2(u,v), v) = 0$

$\delta_1 D_2(u,v) r \delta_1 D_2(u,v) = 0$

Again using the semiprimeness of R, we get $\delta_1 D_2 = 0$.

(iv)To Prove $D_2\delta_1 = 0$:

Since, δ_1, δ_2 are two orthogonal generalized symmetric reverse(σ,τ) biderivations of R.

By (3.9) of Theorem 1, we can have $\delta_1(v,w)\, rD_2(u,v) = 0$

$\delta_2(\delta_1(v,w)\, rD_2(u,v), m) = 0.$

By expanding the above equation and using the fact that σ and τ are automorphisms of R, we obtain $\delta_2(D_2(u,v), m) r \delta_1(v,w) + D_2(u,v) D_2(r,m) \delta_1(v,w) + rD_2(u,v) D_2\delta_1(v,w) = 0.$

Using (3.1) and (3.10), the first two terms are zero and hence we get

$rD_2(u,v) D_2\delta_1(v,w) = 0$

$D_2(u,v) D_2\delta_1(v,w) r D_2(u,v) D_2\delta_1(v,w) = 0, \ \forall\, u,v,w,r \in R.$

By the semiprimeness of R, we get

$D_2(u,v) D_2\delta_1(v,w) = 0, \forall\, u,v,w \in R.$ (3.27)

Replacing $v = v\delta_1(v,w)$ in the above equation , we get

$D_2\big(u, v\delta_1(v,w)\big) D_2\delta_1(v,w) = 0$

$D_2(\delta_1(v,w), u) v D_2\delta_1(v,w) + \delta_1(v,w) D_2(u,v) D_2\delta_1(v,w) = 0.$

Using (3.27), we get

$D_2(\delta_1(v,w), u) v D_2\delta_1(v,w) = 0$

$D_2\delta_1(v,w) v D_2\delta_1(v,w) = 0, \ \forall\, v,w \in R$. Hence, $D_2\delta_1 = 0$.

(v)To Prove $\delta_1\delta_2$=0 :

Since δ_1, δ_2 are orthogonal, we can have $\delta_1(u,v)\, r\delta_2(v,w) = 0, \forall\, u, v, w, r \in R$

$\delta_1(\delta_1(u,v)\, r\delta_2(v,w), m) = 0, \forall\, u, v, w, r, m \in R$

$\delta_1(\delta_2(v,w), m)\sigma(r)\sigma(\delta_1(u,v) + \tau(\delta_2(v,w))D_1(r,m)\sigma(\delta_1(u,v)) + \tau(r\delta_2(v,w)\, D_1(\delta_1(u,v), m) = 0.$

Using $\sigma\delta_1 = \delta_1\sigma$; $\tau\delta_2 = \delta_2\tau$; and σ and τ are automorphisms of R.

$\delta_1(\delta_2(v,w), m)r\delta_1(u,v) + \delta_2(v,w)D_1(r,m)\,\delta_1(u,v) + r\delta_2(v,w)\, D_1(\delta_1(u,v), m) = 0.$

Using (3.5) of Theorem 1, the last two terms of the above equation becomes zero, then we get

$\delta_1\delta_2(v,w)r\,\delta_1(u,v) = 0.$

In Particular if we put $u = \delta_2(v,w)$ in the above equation, we get

$\delta_1\delta_2(v,w)r\delta_1(\delta_2(v,w), v) = 0$

$\delta_1\delta_2(v,w)r\delta_1\delta_2(v,w) = 0$

$\delta_1\delta_2(v,w) = 0$ (By Semiprimeness of R.) and so $\delta_1\delta_2 = 0.$

(vi)To Prove $\delta_2\delta_1 = 0$:

Since δ_1, δ_2 are orthogonal , we can have $\delta_2(u,v)\, r\delta_1(v,w) = 0$, $\forall\, u, v, w, r \in R$

$\delta_2(\delta_2(u,v)\, r\delta_1(v,w), m) = 0, \forall\, u, v, w, r, m \in R.$

By following the similar procedure as we adopted in the previous case, we can easily obtain the result.

4. DISCUSSION:

The study of derivations and their generalizations plays a significant role in ring theory. In this manuscript, we focus on the concept of generalized symmetric reverse bi-(σ,τ)-derivations and explore the conditions for their orthogonality within the framework of semi prime rings. Let R be a semi prime ring. A symmetric bi-additive mapping δ_1:R×R→R is termed a generalized symmetric reverse bi-(σ,τ)-derivation if there exists a symmetric reverse bi-(σ,τ)-derivation D_1 on R such that for all $u, v, w \in R$, $\delta_1(uv,w) = \delta_1(v,w)\sigma(u) + \tau(v)\, D_1(u,w)$. This definition extends the classical notion of generalized symmetric reverse bi-derivations by incorporating the actions of two automorphisms, σ and τ, thus providing a richer and more flexible structure for analysis.

We consider two generalized symmetric reverse bi-(σ,τ)-derivations [δ_1,D_1] and [δ_2,D_2] of R with associated reverse bi-(σ,τ)-derivations D_1 an D_2 The primary goal of this paper is to establish equivalent conditions for the orthogonality between these two mappings. Orthogonality in this context refers to the condition where the products of the mappings and their associated derivations satisfy specific nullity conditions.

Through our analysis, we derive several equivalent conditions that characterize the orthogonality of generalized symmetric reverse bi-(σ,τ)-derivations in semi prime rings. These conditions provide insights into the underlying algebraic structures and their inter relationships. Specifically, we show that orthogonality can be characterized in terms of commutativity and specific interaction properties between the mappings and their associated derivations.

The exploration of generalized symmetric reverse bi-(σ,τ)-derivations opens several avenues for future research. One potential direction is the investigation of these derivations in the context of non-associative algebras or rings with additional structures or other ring-theoretic properties, such as ideals and radicals etc. While this study focuses on semiprime rings, the concepts can be extended to other classes of rings offering a broader applicability of the results.

5. CONCLUSION:

In conclusion, this manuscript contributes to the field of ring theory by providing a detailed analysis of the orthogonality conditions for generalized symmetric reverse bi-(σ,τ)-derivations in semi prime rings. The equivalence conditions established here deepen our understanding of these mappings and pave the way for further explorations into the rich landscape of derivations and their generalizations in other algebraic structures.

REFERENCES:

[1] A. Ali, D. Filippis, and F. Shujat, "Results concerning symmetric generalized biderivations of prime and semiprime rings," Mathematical Bechak, 66(4) (2014), 410–417.
[2] C. Jaya Subba Reddy and B.Ramoorthy Reddy,"Commutativity of prime ring witorthogonal symmetric biderivations," Mathematical Journal of Interdisciplinary Sciences, 7(2) (2019),117–120.
[3] C. Jaya Subba Reddy and B. Ramoorthy Reddy,"Orthogonal symmetric bi- (σ,τ)-derivations in semiprime rings," International Journal of Algebra, 10(9) (2016),423–428.
[4] C. Jaya Subba Reddy and B. Ramoorthy Reddy, "Orthogonal generalized symmetric bi-derivations of semiprime rings," Contemporary Mathematics, 4(1)(2017), 21–27.
[5] C.Jaya Subba Reddy, B.Ramoorthy Reddy,"Orthogonal symmetric biderivations in semiprime rings", International Journal of Mathematics and Satistics Studies,4(1) (2016), 22-29.
[6] C.Jaya Subba Reddy and B.Ramoorthy Reddy, K.Chennakesavulu, "Orthogonality of generalized (σ,τ) symmetric biderivations in semiprime rings", International Journal of Educational Science and Research, 8(6) (2018), 45-52.
[7] C.Jaya Subba Reddy and V.S.V.Krishna Murty, "Orthogonal symmetric reverse bi-(σ,τ)-derivations in semi prime rings", Tuijin Jishu /Journal of Propulsion Technology, 45(1) (2024), 5133-5138.
[8] C.Jaya Subba Reddy and V.S.V. Krishna Murty, "Orthogonality of generalized reverse (σ, τ)-derivations in semiprime Γ-rings", Journal of Nonlinear Analysis and Optimization:Theory and Applications, 15 (4)1,(2024),20-28.
[9] E.C.Posner, "Derivations in prime rings", Proc.Amer.Mth.Soc., 8(1957), 1093-1100.
[10] E. Koc, "Notes on ideals and orthogonal generalized (σ,τ) derivations", East Asian mathematical journal, 24(4) (2008), 389–398.
[11] J.C.Chang, "On the identity h(x) = af(x) + g(x)b", Taiwanese J. Math.7(1)(2003),103-113.
[12] K. Kaya, E. Guven, and M. Soyturk, "On (σ,τ) derivations of prime rings", The Pure and Applied Mathematics,13(3) (2006) 189–195.
[13] M. Ashraf, " On (σ,τ)-derivations in prime rings," Archivum Mathematicum, 38(4) (2002), 259–264.
[14] M. Bresar and J.Vukman,"Orthogonal derivations and an extension of a theorem of Posner", Radovi Mathematicki," 5 (1991), 237-246.
[15] M. N. Daif, M. T. El-Sayiad, and C. Haetinger, "Orthogonal Derivations and Biderivations", JMI International Journal of Mathematical Sciences, 1(1) (2010), 23–34.
[16] M. N. Daif, M. S. T. El-Sayiad, and C. Haetinger, "Reverse, jordan and left biderivations", Oriental Journal Of Mathematics, 2(2) (2010) 65–81.
[17] N. Argaç, A. Nakajima, and E. ALBAŞ, "On orthogonal generalized derivations of semiprime rings", Turkish Journal of Mathematics, 28(2) (2004) 185–194.
[18] O. Golbaşi and N. Aydin, "Orthogonal generalized (σ,τ) -derivations of semiprime rings", Siberian Mathematical Journal, 48(6) (2007), 979-983.