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Subclass of Analytic Functions Associated with Differential Operator

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Abstract:

Introduction: **:** In this work, we introduce and investigate a new class $\tilde{\Theta}S_s^m(\varsigma,\hbar,\wp,\varrho,t)$ of analytic functions in the open unit disc U with negative coefficients. The object of the present paper is to determine coefficient estimates, neighborhoods and partial sums for functions f belonging to this class.

Keywords: **:** analytic function, uniformly starlike function, coefficient estimate, neighborhood, partial sums.

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1. Introduction

Let A denote the class of analytic functions f defined on the unit disk $U = \{z: |z| < 1\}$ with normalization $f(0) = 0$ and $f'(0) = 1$. Such a function has the Taylor series expansion about the origin in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1),$$

denoted by S, the subclass of A consisting of functions that are univalent in U.

For $f \in A$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \qquad (1.2)$$

their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \ \ (z \in U). \qquad (1.3)$$

Note that $f * g \in A$.

A function $f \in A$ is said to be in $US(\varrho)$, the class of uniformly starlike functions of order $\varrho, 0 \le \varrho < 1$, if satisfies the condition

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right| + \varrho, \tag{1.4}$$

and a function $f \in A$ is said to be in $UC(\varrho)$, the class of uniformly convex functions of order $\varrho, 0 \leq \varrho < 1$, if satisfies the condition

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right| + \varrho. \tag{1.5}$$

Uniformly starlike and uniformly convex functions were first introduced by Goodman [8] and then studied by various authors.

In , Sakaguchi [11] defined the class S_s of starlike functions with respect to symmetric points as follows:

Let $f \in A$. Then f is said to be starlike with respect to symmetric points in U if and only if

$$\Re\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > 0,\ (z \in U).$$

Recently, Owa et al. [10] defined the class $S_s(\varsigma, t)$ as follows:

$$\Re\left\{\frac{(1-t)zf'(z)}{f(z) - f(tz)}\right\} > \varsigma,\ (z \in U),$$

where $0 \leq \varsigma < 1, |t| \leq 1, t \neq 1$. Note that $S_s(0,-1) = S_s$ and $S_s(\varsigma,-1) = S_s(\varsigma)$ is called Sakaguchi function of order ς.

In , Darus and Faisal [5] introduced the following differential operator.

For a function $f \in A$,

$$\begin{aligned}
&\mathcal{D}_{\wp}^{0}(\varsigma,\hbar)f(z) = f(z)\\
&\mathcal{D}_{\wp}^{1}(\varsigma,\hbar)f(z) = \left(\frac{\varsigma - \hbar - \wp}{\varsigma}\right)f(z) + \left(\frac{\hbar + \wp}{\varsigma}\right)zf'(z)\\
&\mathcal{D}_{\wp}^{2}(\varsigma,\hbar)f(z) = \mathcal{D}\left(\mathcal{D}_{\wp}^{1}(\varsigma,\hbar)f(z)\right)\\
&\qquad\vdots\\
&\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z) = \mathcal{D}_{\wp}\left(\mathcal{D}_{\wp}^{m-1}(\varsigma,\hbar)f(z)\right)
\end{aligned}$$

where $\varsigma, \hbar, \wp \geq 0, \varsigma \neq 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

If f is given by (1.1) then from the definition of the operator $\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f$ it is easy to see that

$$\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z) = z + \sum_{n=2}^{\infty} \phi_n\,(\varsigma,\hbar,\wp,m)a_n z^n \tag{1.6}$$

where

$$\phi_n(\varsigma,\hbar,\wp,m) = \left(\frac{\varsigma + (\hbar + \wp)(n-1)}{\varsigma}\right)^m \tag{1.7}$$

By specializing the parameters of $\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)$, we get the following differential operators. If we substitute

(i) $\varsigma = 1$and $\hbar = 0$, we get $\mathcal{D}^{m}f(z) = z + \sum_{n=2}^{\infty}\left(1+\wp(n-1)\right)^{m} a_n z^n$ oof differential operator given by Al-Oboudi [1].

(ii) $\varsigma = 1, \hbar = 0$ and $\wp = 1$, we get $\mathcal{D}^{m}f(z) = z + \sum_{n=2}^{\infty}(n)^{m} a_n z^n$ of Salagean differential operator[12].

Now, by making use of the differential operator $\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f$, we define a new subclass of functions belonging to the class A.

Definition 1. A function $f \in A$ is said to be in the class $\Theta S_s^m(\varsigma,\hbar,\wp,\varrho,t)$ if for all $z \in U$

$$\Re\left\{\frac{(1-t)z\left(\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)\right)'}{\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)-\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(tz)}\right\} \geq \left|\frac{(1-t)z\left(\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)\right)'}{\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)-\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(tz)}-1\right|+\varrho,$$

for $\wp \geq 0,\ m, |t| \leq 1, t \neq 1, 0 \leq \varrho < 1$.

Furthermore, we say that a function $f \in US_s^m(\varsigma,\hbar,\wp,\varrho,t)$ is in the subclass $\tilde{\Theta} S_s^m(\varsigma,\hbar,\wp,\varrho,t)$ if $f(z)$ is of the following form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0, n \in \mathbb{N},\ z \in U. \tag{1.8}$$

The aim of the present paper is to study the coefficient bounds, partial sums and certain neighborhood results of the class $\tilde{\Theta} S_s^m(\varsigma,\hbar,\wp,\varrho,t)$.

Firstly, we shall need the following lemmas.

Lemma 2. *Let* $w = u + iv$. *Then* $\Re\,(w) \geq \beta$ if and only if $|w-(1+\beta)| \leq |w+(1-\varsigma)|$.

Lemma 3. *Let* $w = u + iv$ *and* ς, ϱ *be real numbers. Then* $\Re\,(w) > \beta|w-1| + \varrho$ if and only if $\Re\{w(1+\beta e^{i\theta})-\beta e^{i\theta}\} > \varrho$

2 Coefficient bounds

Theorem 4. *The function* f *defined by* (1.8) *is in the class* $\tilde{\Theta} S_s^m(\varsigma,\hbar,\wp,\varrho,t)$ if and only if

$$\sum_{n=2}^{\infty} \phi_n\,(\varsigma,\hbar,\wp,m)|2n-u_n(1+\varrho)|a_n \leq 1-\varrho, \tag{2.1}$$

where $\wp \geq 0,\ m,k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \varrho < 1$ *and* $u_n = 1 + t + \cdots + t^{n-1}$.
The result is sharp for the function $f(z)$ *given by* $f(z) = z - \frac{1-\varrho}{\phi_n(\varsigma,\hbar,\wp,m)|2n-u_n(1+\varrho)|}z^n$.

Proof. By Definition 1, we get

$$\Re\left\{\frac{(1-t)z\left(\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)\right)'}{\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)-\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(tz)}\right\} \geq \left|\frac{(1-t)z\left(\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)\right)'}{\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)-\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(tz)}-1\right|+\varrho.$$

Then by Lemma 3, we have

$$\Re\left\{\frac{(1-t)z\left(\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)\right)'}{\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)-\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(tz)}\left(1+e^{i\theta}\right)-e^{i\theta}\right\}\geq \varrho,\ -\pi<\theta\leq\pi$$

or equivalently

$$\Re\left\{\frac{(1-t)z\left(\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)\right)'\left(1+e^{i\theta}\right)}{\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)-\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(tz)}-\frac{e^{i\theta}\left[\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)-\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(tz)\right]}{\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)-\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(tz)}\right\}\geq \varrho. \qquad (2.2)$$

Let $F(z)=(1-t)z\left(\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)\right)'\left(1+e^{i\theta}\right)-e^{i\theta}\left[\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)-\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(tz)\right]$
and $E(z)=\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)-\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(tz)$.
By Lemma 2 , (2.2) is equivalent to

$$|F(z)+(1-\varrho)E(z)|\geq|F(z)-(1+\varrho)E(z)|,\ \text{ for } 0\leq\varrho<1.$$

But

$$\begin{aligned}|F(z)+(1-\varrho)E(z)|= \quad & |(1-t)\{(2-\varrho)z-\sum_{n=2}^{\infty}\phi_n\,(\varsigma,\hbar,\wp,m)\left(n+u_n(1-\varrho)\right)a_nz^n\\ & -e^{i\theta}\sum_{n=2}^{\infty}\phi_n\,(\varsigma,\hbar,\wp,m)(n-u_n)a_nz^n\}|\\ \geq \quad & |1-t|\{(2-\varrho)|z|-\sum_{n=2}^{\infty}\phi_n\,(\varsigma,\hbar,\wp,m)|n+u_n(1-\varrho)|a_n|z^n|\\ & -\sum_{n=2}^{\infty}\phi_n\,(\varsigma,\hbar,\wp,m)|n-u_n|a_n|z^n|\}.\end{aligned}$$

Also

$$\begin{aligned}|F(z)-(1+\varrho)E(z)|= \quad & |(1-t)\{-\varrho z-\sum_{n=2}^{\infty}\phi_n\,(\varsigma,\hbar,\wp,m)\left(n-u_n(1+\varrho)\right)a_nz^n\\ & -e^{i\theta}\sum_{n=2}^{\infty}\phi_n\,(\varsigma,\hbar,\wp,m)(n-u_n)a_nz^n\}|\\ \leq \quad & |1-t|\{\varrho|z|+\sum_{n=2}^{\infty}\phi_n\,(\varsigma,\hbar,\wp,m)|n-u_n(1+\varrho)|a_n|z^n|\\ & +\sum_{n=2}^{\infty}\phi_n\,(\varsigma,\hbar,\wp,m)|n-u_n|a_n|z^n|\}.\end{aligned}$$

So

$$\begin{aligned} & |F(z)+(1-\varrho)E(z)|-|F(z)-(1+\varrho)E(z)| \\ \geq \quad & |1-t|\{2(1-\varrho)|z|-\sum_{n=2}^{\infty}\phi_n\,(\varsigma,\hbar,\wp,m)[|n+u_n(1-\varrho)|+|n-u_n(1+\varrho)|+2|n-u_n|\,]a_n|z^n|\} \\ \geq \quad & 2(1-\varrho)|z|-\sum_{n=2}^{\infty}2\,\phi_n(\varsigma,\hbar,\wp,m)|2n-u_n(1+\varrho)|a_n|z^n|\geq 0 \end{aligned}$$

or

$$\sum_{n=2}^{\infty}\phi_n\,(\varsigma,\hbar,\wp,m)|2n-u_n(1+\varrho)|a_n\leq 1-\varrho.$$

Conversely, suppose that (2.1) holds. Then we must show

$$\Re\left\{\frac{(1-t)z\left(\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)\right)'\left(1+e^{i\theta}\right)-e^{i\theta}\left[\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)-\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(tz)\right]}{\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(z)-\mathcal{D}_{\wp}^{m}(\varsigma,\hbar)f(tz)}\right\}\geq\varrho.$$

Upon choosing the values of z on the positive real axis where $0\leq|z|=r<1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\varrho)-\sum_{n=2}^{\infty}\phi_n\,(\varsigma,\hbar,\wp,m)\left[n\left(1+e^{i\theta}\right)-u_n\left(\varrho+e^{i\theta}\right)\right]a_nz^{n-1}}{1-\sum_{n=2}^{\infty}\phi_n\,(\varsigma,\hbar,\wp,m)u_na_nz^{n-1}}\right\}\geq 0.$$

Since $\Re\left(-e^{i\theta}\right)\geq-\left|e^{i\theta}\right|=-1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\varrho)-\sum_{n=2}^{\infty}\phi_n\,(\varsigma,\hbar,\wp,m)[2n-u_n(1+\varrho]a_nr^{n-1}}{1-\sum_{n=2}^{\infty}\phi_n\,(\varsigma,\hbar,\wp,m)u_na_nr^{n-1}}\right\}\geq 0.$$

Letting $r\to 1^-$, we have desired conclusion.

Corollary 5. *If* $f(z)\in\tilde{\Theta}S_s^m(\varsigma,\hbar,\wp,\varrho,t)$ *then* $a_n\leq\frac{1-\varrho}{\phi_n(\varsigma,\hbar,\wp,m)|2n-u_n(1+\varrho)|}$ *where* $\wp\geq 0,\ m,|t|\leq 1,t\neq 1,0\leq\varrho<1$ *and* $u_n=1+t+\cdots+t^{n-1}$.

3 Neighborhood property

Following the earlier investigations (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [7], Srinivas et al [16] , Altintas et al [2, 3]. and others including Srivastava et al.[15] , Orhan [9], Deniz et al. [6], Catas [4].

Definition 6. Let $\wp\geq 0,\ m,|t|\leq 1,t\neq 1,0\leq\varrho<1,\varsigma\geq 0$ and $u_n=1+t+\cdots+t^{n-1}$. We define the ς −neighborhood of a function $f\in A$ and denote by $N_{\varsigma}(f)$ consisting of all functions $g(z)=z-\sum_{n=2}^{\infty}b_n\,z^n\in S(b_n\geq 0,n\in\mathbb{N})$ satisfying

$$\sum_{n=2}^{\infty}\frac{\phi_n(\varsigma,\hbar,\wp,m)|2n-u_n(1+\varrho)|}{1-\varrho}|a_n-b_n|\leq 1-\varsigma.$$

Theorem 7. *Let* $f(z) \in \tilde{\Theta}S_s^m(\varsigma, \hbar, \wp, \varrho, t)$ *and for all real* θ *we have* $\varrho\left(e^{i\theta}-1\right)-2e^{i\theta} \neq 0$. *For any complex number* ϵ *with* $|\epsilon| < \varsigma(\varsigma \geq 0)$, *if* f *satisfies the following condition:* $\frac{f(z)+\epsilon z}{1+\epsilon} \in \tilde{\Theta}S_s^m(\varsigma, \hbar, \wp, \varrho, t)$ *then* $N_\varsigma(f) \subset \tilde{\Theta}S_s^m(\varsigma, \hbar, \wp, \varrho, t)$.

Proof. It is obvious that $f \in \tilde{\Theta}S_s^m(\varsigma, \hbar, \wp, \varrho, t)$ if and only if

$$\left|\frac{(1-t)z\left(\mathcal{D}_{\wp}^m(\varsigma,\hbar)f(z)\right)'\left(1+e^{i\theta}\right)-\left(e^{i\theta}+1+\varrho\right)\left(\mathcal{D}_{\wp}^m(\varsigma,\hbar)f(z)-\mathcal{D}_{\wp}^m(\varsigma,\hbar)f(tz)\right)}{(1-t)z\left(\mathcal{D}_{\wp}^m(\varsigma,\hbar)f(z)\right)'\left(1+e^{i\theta}\right)+\left(1-e^{i\theta}-\varrho\right)\left(\mathcal{D}_{\wp}^m(\varsigma,\hbar)f(z)-\mathcal{D}_{\wp}^m(\varsigma,\hbar)f(tz)\right)}\right|<1,$$

$$(-\pi < \theta \leq \pi),$$

for any complex number s with $|s| = 1$, we have

$$\frac{(1-t)z\left(\mathcal{D}_{\wp}^m(\varsigma,\hbar)f(z)\right)'\left(1+e^{i\theta}\right)-\left(e^{i\theta}+1+\varrho\right)\left(\mathcal{D}_{\wp}^m(\varsigma,\hbar)f(z)-\mathcal{D}_{\wp}^m(\varsigma,\hbar)f(tz)\right)}{(1-t)z\left(\mathcal{D}_{\wp}^m(\varsigma,\hbar)f(z)\right)'\left(1+e^{i\theta}\right)+\left(1-e^{i\theta}-\varrho\right)\left(\mathcal{D}_{\wp}^m(\varsigma,\hbar)f(z)-\mathcal{D}_{\wp}^m(\varsigma,\hbar)f(tz)\right)} \neq s.$$

In other words, we must have

$$(1-s)(1-t)z\left(\mathcal{D}_{\wp}^m(\varsigma,\hbar)f(z)\right)'\left(1+e^{i\theta}\right)-\left(e^{i\theta}+1+\varrho+s\left(-1+e^{i\theta}+\varrho\right)\right)$$

$$\times\left(\mathcal{D}_{\wp}^m(\varsigma,\hbar)f(z)-\mathcal{D}_{\wp}^m(\varsigma,\hbar)f(tz)\right) \neq 0.$$

which is equivalent to

$$z-\sum_{n=2}^{\infty}\frac{\phi_n(\varsigma,\hbar,\wp,m)\left((n-u_n)\left(1+e^{i\theta}-ske^{i\theta}\right)-s(n+u_n)-u_n\varrho(1-s)\right)}{\varrho(s-1)-2s}z^n \neq 0.$$

However, $f \in \tilde{\Theta}S_s^m(\varsigma, \hbar, \wp, \varrho, t)$ if and only $\frac{(f*h)}{z} \neq 0, z \in U-\{0\}$, where $h(z) = z - \sum_{n=2}^{\infty} c_n z^n$ and

$$c_n = \frac{\phi_n(\varsigma,\hbar,\wp,m)\left((n-u_n)\left(1+e^{i\theta}-se^{i\theta}\right)-s(n+u_n)-u_n\varrho(1-s)\right)}{\varrho(s-1)-2s}$$

we note that

$$|c_n| \leq \frac{\phi_n(\varsigma,\hbar,\wp,m)|2n-u_n(1+\varrho)|}{1-\varrho}$$

since $\frac{f(z)+\epsilon z}{1+\epsilon} \in \tilde{\Theta}S_s^m(\varsigma, \hbar, \wp, \varrho, t)$, therefore $z^{-1}\left(\frac{f(z)+\epsilon z}{1+\epsilon} * h(z)\right) \neq 0$, which is equivalent to

$$\frac{(f*h)(z)}{(1+\epsilon)z}+\frac{\epsilon}{1+\epsilon} \neq 0. \qquad (3.1)$$

Now suppose that $\left|\frac{(f*h)(z)}{z}\right| < \varsigma$. Then by (3.1), we must have

$$\begin{aligned}\left|\frac{(f*h)(z)}{(1+\epsilon)z}+\frac{\epsilon}{1+\epsilon}\right| &\geq \frac{|\epsilon|}{|1+\epsilon|}-\frac{1}{|1+\epsilon|}\left|\frac{(f*h)(z)}{z}\right| \\ &> \frac{|\epsilon|-\varsigma}{|1+\epsilon|}\geq 0,\end{aligned}$$

this is a contradiction by $|\epsilon| < \varsigma$ and however, we have $\left|\frac{(f*h)(z)}{z}\right| \geq \varsigma$. If $g(z) = z - \sum_{n=2}^{\infty} b_n z^n \in N_{\varsigma}(f)$, then

$$\begin{aligned}\varsigma - \left|\frac{(g*h)(z)}{z}\right| &\leq \left|\frac{\big((f-g)*h\big)(z)}{z}\right| \leq \sum_{n=2}^{\infty} |a_n - b_n|\,|c_n||z^n| \\ &< \sum_{n=2}^{\infty} \frac{\phi_n(\varsigma,\hbar,\wp,m)|2n-u_n(1+\varrho)|}{1-\varrho}|a_n-b_n| \leq \varsigma.\end{aligned}$$

4 Partial sums

In this section, applying methods used by Silverman [13] and Silvia [14] , we investigate the ratio of a function of the form (1.8) to its sequence of partial sums $f_m(z) = z + \sum_{n=2}^{m} a_n z^n$.

Theorem 8. *If f of the form* (1.1) *satisfies the condition* (2.1) *then*

$$\Re\left\{\frac{f(z)}{f_m(z)}\right\} \geq 1 - \frac{1}{\delta_{m+1}} \tag{4.1}$$

and

$$\delta_n = \begin{cases} 1, & if\ n = 2,3 \cdots m \\ \delta_{m+1}, & if\ n = m+1, m*2, \cdots \end{cases} \tag{4.2}$$

Where $\delta_n = \frac{\phi_n(\varsigma,\hbar,\wp,m)|2n-u_n(1+\varrho)|}{1-\varrho}$. (4.3)

The result in (4.1) *is sharp for every* m, *with the extremal function*

$$f(z) = z + \frac{z^{m+1}}{\delta_{m+1}}. \tag{4.4}$$

Proof. Define the function w, we may write

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} &= \delta_{m+1}\left\{\frac{f(z)}{f_m(z)} - \left(1-\frac{1}{\delta_{m+1}}\right)\right\} \qquad (4.5)\\ &= \left\{\frac{1+\sum_{n=2}^{m} a_n z^{n-1} + \delta_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}}{1+\sum_{n=2}^{m} a_n z^{n-1}}\right\}.\end{aligned}$$

Then, from (4.5), we can obtain

$$w(z) = \frac{\delta_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}}{2+2\sum_{n=2}^{m} a_n z^{n-1} + \delta_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}}$$

and

$$|w(z)| \leq \frac{\delta_{m+1}\sum_{n=m+1}^{\infty} a_n}{2-2\sum_{n=2}^{m} a_n - \delta_{m+1}\sum_{n=m+1}^{\infty} a_n}.$$

Now $|w(z)| \leq 1$ if

$$2\delta_{m+1}\sum_{n=m+1}^{\infty} a_n \leq 2-2\sum_{n=2}^{m} a_n,$$

which is equivalent to

$$\sum_{n=2}^{m} a_n + \delta_{m+1}\sum_{n=m+1}^{\infty} a_n \leq 1. \qquad (4.6)$$

It is suffices to show that the left hand side of (4.6) is bounded above by $\sum_{n=2}^{\infty} \delta_n\, a_n$, which is equivalent to

$$\sum_{n=2}^{m}(\delta_n - 1)\, a_n + \sum_{n=m+1}^{\infty}(\delta_n - \delta_{m+1})\, a_n \geq 0.$$

To see that the function given by (4.4) gives the sharp result, we observe that for $z = re^{i\pi/n}$,

$$\frac{f(z)}{f_m(z)} = 1 + \frac{z^m}{\delta_{m+1}} \qquad (4.7).$$

Taking $z \to 1^-$, we have

$$\frac{f(z)}{f_m(z)} = 1 - \frac{1}{\delta_{m+1}}.$$

This completes the proof of Theorem 8.

We next determine bounds for $\frac{f_m(z)}{f(z)}$.

Theorem 9. *If* f *of the form* (1.1) *satisfies the condition* (2.1) *then*

$$\Re\left\{\frac{f_m(z)}{f(z)}\right\} \geq \frac{\delta_{m+1}}{1+\delta_{m+1}}. \qquad (4.8)$$

The result is sharp with the function given by (4.4).

Proof. We may write

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} &= (1+\delta_{m+1})\left\{\frac{f_m(z)}{f(z)} - \frac{\delta_{m+1}}{1+\delta_{m+1}}\right\}\\ &= \left\{\frac{1+\sum_{n=2}^{m} a_n\, z^{n-1} - \delta_{m+1}\sum_{n=m+1}^{\infty} a_n\, z^{n-1}}{1+\sum_{n=2}^{\infty} a_n\, z^{n-1}}\right\},\end{aligned}$$

where

$$w(z)=\frac{(1+\delta_{m+1})\sum_{n=m+1}^{\infty}a_n\,z^{n-1}}{-(2+2\sum_{n=2}^{m}a_n\,z^{n-1}-(1-\delta_{m+1})\sum_{n=m+1}^{\infty}a_n\,z^{n-1})}$$

and

$$|w(z)|\le\frac{(1+\delta_{m+1})\sum_{n=m+1}^{\infty}a_n}{2-2\sum_{n=2}^{m}a_n+(1-\delta_{m+1})\sum_{n=m+1}^{\infty}a_n}\le 1.$$

This last inequality is equivalent to

$$\sum_{n=2}^{m}a_n+\delta_{m+1}\sum_{n=m+1}^{\infty}a_n\le 1. \qquad (4.9)$$

It is suffices to show that the left hand side of (4.9) is bounded above by $\sum_{n=2}^{\infty}\delta_n\,a_n$, which is equivalent to

$$\sum_{n=2}^{m}(\delta_n-1)\,a_n+\sum_{n=m+1}^{\infty}(\delta_n-\delta_{m+1})\,a_n\ge 0.$$

This completes the proof of Theorem .

We next turn to ratios involving derivatives.

Theorem 10. *If f of the form* (1.1) *satisfies the condition* (2.1) *then*

$$\Re\left\{\frac{f'(z)}{f_m{}'(z)}\right\}\ge 1-\frac{m+1}{\delta_{m+1}} \qquad (4.10)$$

$$\Re\left\{\frac{f_m{}'(z)}{f'(z)}\right\}\ge\frac{\delta_{m+1}}{1+m+\delta_{m+1}} \qquad (4.11)$$

where

$$\delta_n\ge\begin{cases}1, & if\ n=2,3\cdots m\\ n\,\dfrac{\delta_{m+1}}{m+1}, & if n=m+1,m*2,\cdots\end{cases}$$

and δ_n is defined by [4.3]. *The estimates in* (4.10) *and* (4.11) *are sharp with the extremal function given by*(4.4).

Proof. Firstly, we will give proof of (4.10). We write

$$\frac{1+w(z)}{1-w(z)}=\delta_{m+1}\left\{\frac{f'(z)}{f_m{}'(z)}-\left(1-\frac{m+1}{\delta_{m+1}}\right)\right\}$$

$$=\left\{\frac{1+\sum_{n=2}^{m}n\,a_nz^{n-1}+\dfrac{\delta_{m+1}}{m+1}\sum_{n=m+1}^{\infty}n\,a_nz^{n-1}}{1+\sum_{n=2}^{m}a_n\,z^{n-1}}\right\},$$

where

$$w(z)=\frac{\frac{\delta_{m+1}}{m+1}\sum_{n=m+1}^{\infty} n\,a_n z^{n-1}}{2+2\sum_{n=2}^{m} n\,a_n z^{n-1}+\frac{\delta_{m+1}}{m+1}\sum_{n=m+1}^{\infty} n\,a_n z^{n-1}}$$

and

$$|w(z)|\le\frac{\frac{\delta_{m+1}}{m+1}\sum_{n=m+1}^{\infty} n\,a_n}{2-2\sum_{n=2}^{m} n\,a_n+\frac{\delta_{m+1}}{m+1}\sum_{n=m+1}^{\infty} n\,a_n}.$$

Now $|w(z)|\le 1$ if and only if

$$\sum_{n=2}^{m} n\,a_n+\frac{\delta_{m+1}}{m+1}\sum_{n=m+1}^{\infty} n\,a_n\le 1, \qquad (4.12)$$

since the left hand side of (4.12)is bounded above by $\sum_{n=2}^{\infty}\delta_n\,a_n$.

The proof of (4.11) follows the pattern of that in Theorem 9.

This completes the proof of Theorem .

Refrences

[1] Al-oboudi, F.M., On univalent functions defined by a generalized Salagean operator, Int. J. Math. Math. Sci., (2004) Article ID 172525, 1429–436, 2004.

[2] Altinta, S. O. and Owa, S., Neighborhoods of certain analytic functions with negative coefficients, Int. J. Math. and Math. Sci., 19, 797–800, 1996.

[3] Altinta, S. O., Ozkan, E. and Srivastava, H. M., Neighborhoods of a class of analytic functions with negative coefficients, Appl. Math. Let., 13, 63–67, 2000.

[4] Cata, S. A., Neighborhoods of a certain class of analytic functions with negative coefficients, Banach J. Math. Anal., 3 (1), 111–121, 2009.

[5] Darus, M. , and Faisal, I., Characrerization properties for a class of analytic aunctions defined by generalized Cho and Srivastava operator, In Proc. 2nd Inter. Conf. Math. Sci., Kuala Lumpur, Malaysia, (2010), 1106–1113.

[6] Deniz, E. and Orhan. H., Some properties of certain subclasses of analytic functions with negative coefficients by using generalized Ruscheweyh derivative operator, Czechoslovak Math. J., 60 (135), 699 –713, 2010.

[7] Goodman, A. W., Univalent functions and nonanalytic curves, Proc. Amer. Math. Soc., 8, 598–601, 1957

[8] Goodman, A. W., On uniformly starlike functions, J. Math. Anal. Appl., 155, 364–370, 1991.

[9] Orhan, H., On neighborhoods of analytic functions defined by using hadamard product, Novi Sad J. Math., 37(1), 17–25, 2007.

[10] Owa, S., Sekine, T. and Yamakawa. R., On Sakaguchi type functions, Appl. Math. Comput., 187, 356–361, 2007.

[11] Sakaguchi, K., On a certain univalent mapping, J. Math. Soc. Japan, 11, 72 –75, 1959.

[12] Salagean, G.S., Subclasses of univalent functions, Lecture Notes in Math., 1013, Springer-verlag (1983), 362–372.

[13] Silverman, H., Partial sums of starlike and convex functions, J. Anal. Appl., 209, 221–227, 1997.

[14] Silvia, E. M., Partial sums of convex functions of order α, Houston J. Math., 11 (3), 397–404, 1985.

[15] Srivastava, H. M. and Aouf, M. K., A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients- I , J. Math. Anal. Appl., 171, 11-13, 1992.

[16] Srinivas,T., Reddy,P.T.and Madhavi.B.,Neighbourhoods of a certain subclass of uniformly starlike functions, Tamkang Journal of Mathematics, 43(4), 587-594, 2012.