

## Some New Types of $\alpha_{\beta^{-1}}$ -admissible Mappings and Fixed Point Results

Dev Raj Joshi, Piyush Kumar Tripathi, Alok Kumar Agrawal and Ashutosh Mishra

Department of Mathematics, ASAS, Amity University Uttar Pradesh Lucknow campus, Lucknow-226028, India

Email: dev.joshi@s.amity.edu, pktripathi@lko.amity.edu, akagraw@lko.amity.edu and ashutosh.mishra8@s.amity.edu

---

### Article History:

**Received:** 05-04-2024

**Revised:** 30-05-2024

**Accepted:** 14-06-2024

### Abstract:

In this paper fixed point and common fixed point theorems are proved for  $\alpha - \varphi$  contractions under the  $\alpha_{\beta^{-1}}$ -admissible and  $\mu - \alpha_{\beta^{-1}}$  admissible conditions. These results generalize the  $\alpha$ -admissible condition of self-mappings as explained by many researchers in the literature herein. In general, many research reports introduced the  $\alpha$ -admissible condition on the basis of  $\alpha(p, q) > 1$ . However, this condition is not applicable for finding the fixed points of self-mappings in many cases. We propose  $\alpha$ -admissible on the basis of  $\alpha(p, q) > \beta^{-1}$  which generalize the case we mentioned in the literature. To support our new concepts, we present two examples at the end of this paper.

**Keywords:**  $\alpha_{\beta^{-1}}$ -admissible mappings,  $\mu - \alpha_{\beta^{-1}}$ admissible mapping,  $\mu - \alpha, \varphi$  contraction.

Mathematics Subject Classification: 47H10, 54H25.

---

## 1. Introduction

In the field of fixed point theory, Banach contraction principle [1] is one of the fundamental tools to explain existence of fixed point in complete metric spaces. After this principle, many researchers have suggested different generalizations of contraction principle and their applications to solve many mathematical problems such as solution of differential equations, integral equations and non-linear analysis.

Among the generalizations of Banach contraction principle, we discuss  $\alpha - \varphi$  contraction here. It was initiated by Samet et al. [2] in which they introduced  $\alpha - \varphi$  contraction type mappings and proved fixed point theorems in complete metric spaces. This concept was further improved and modified by B. Samet [3] alone. After the paper of Samet et al. the  $\alpha - \varphi$  contraction is used by many researchers in different metric spaces. Karapinar et al. [4] used this contraction in b-metric space with  $\alpha$ -orbital admissible condition. Li. and Guan [5] introduced  $\alpha_{s^p}$  and  $\rho - \alpha_{s^p}$  admissible condition in b-metric space. Likewise, Hussain et al. [6] introduced  $\alpha$ -admissible mapping with respect to  $\eta$ . Zoto et al. [7] introduced  $\alpha_{q_s^p}$  admissible mapping and prove the fixed point theorem in b-metric space. Almost researchers in this direction have used  $\alpha$ -admissible condition. For example see [8],[9],[10], [11],[12],[13],[14],[15].

As we motivated and inspired by the papers [2],[3], [4], [5] and [7], our aim is to introduce  $\alpha_{\beta^{-1}}$  - admissible mapping to prove some fixed point theorems by using  $\alpha - \varphi$  contraction in complete

metric spaces in this paper. Additionally, we introduce  $\mu - \alpha_{\beta-1}$ admissible mappings to prove the common fixed point theorem by using  $(\mu - \alpha, \varphi)$ -contraction in complete metric space.

## 2. Preliminaries

In this section, we present some basic concepts and notations related to  $\alpha - \varphi$  contractions and  $\alpha -$  admissible conditions.

**Definition 2.1:** [16] A mapping  $\varphi: [0, \infty) \rightarrow [0, \infty)$  is called comparison function if satisfy following assumptions:

- a)  $\varphi$  is non-decreasing.
- b)  $\varphi(p) < p \quad \forall p > 0$ .
- c)  $\varphi^m(p) \rightarrow 0$  as  $m \rightarrow \infty \quad \forall p > 0$
- d)  $\varphi(0) = 0$

The set of all comparison functions is denoted by  $\emptyset$ .

**Definition 2.2:** [2] Let  $X$  be a non-empty set. The mapping  $\mu: X \rightarrow X$  is called  $\alpha$ -admissible, if there exists a function  $\alpha: X \times X \rightarrow [0, \infty)$  such that  $\alpha(a, b) \geq 1 \Rightarrow \alpha(\mu(a), \mu(b)) \geq 1$  for  $a, b \in X$ .

**Definition 2.3:** [2] Let  $\mu$  be a self-map on a metric space  $(X, d)$ . We say  $\mu$  is  $\alpha - \varphi$  contraction if there exist  $\alpha: X \times X \rightarrow [0, \infty)$  and a comparison function  $\varphi: [0, \infty) \rightarrow [0, \infty)$  such that,

$$\alpha(a, b)d(\mu(a), \mu(b)) \leq \varphi(d(a, b)) \quad \forall a, b \in X$$

**Definition 2.4:** [3] For a given  $\varphi \in \emptyset$ , Let us define a set  $R_\varphi = \{\gamma \in [0, \infty): \gamma\varphi \in \emptyset\}$

The main theorem in [2] is re stated here.

**Theorem 2.1:** Let  $(X, d)$  be complete metric space and  $\mu$  be a given continuous self- map. Suppose that there exist two functions  $\alpha: X \times X \rightarrow [0, \infty)$  and  $\varphi \in \emptyset$  such that,

- a)  $\mu$  is  $\alpha - \varphi$  contraction.
- b)  $\mu$  is  $\alpha$ -admissible.
- c) There exists  $a_0 \in X$  such that  $\alpha(a_0, \mu(a_0)) \geq 1$
- d)  $\mu$  is continuous or
- e) For every  $\{a_n\} \subset X$  such that  $a_n \rightarrow a \in X$  and  $\alpha(a_n, a_{n+1}) \geq 1$  for  $n \in N$ , we have  $\alpha(a_n, a) \geq 1$  for  $n \in N$ .

Then  $\mu$  has a fixed point. Moreover, if in addition we suppose that for every pair  $u, v \in X$  there exists  $w \in X$  such that  $\alpha(u, w) \geq 1$  and  $\alpha(v, w) \geq 1$ , we have a unique fixed point.

Following are the main results; we obtained in Samt [3]

**Theorem 2.2:** Let  $(X, d)$  be complete metric space and  $\mu$  be a given continuous self- map. Suppose that there exist two functions  $\alpha: X \times X \rightarrow [0, \infty)$  and  $\varphi \in \emptyset$  such that  $\mu$  is  $\alpha - \varphi$  contraction. Also let there exist  $\gamma \in R_\varphi$  and a finite sequence  $\{\tau_i\} \subset X, i = 1, 2 \dots p$  such that

$$\tau_0 = x_0, \tau_p = \mu(x_0), \alpha(\mu^n(\tau_i), \mu^n(\tau_{i+1})) \geq \gamma^{-1} \quad \forall n \in N, i = 1, 2 \dots p - 1 \dots \dots \dots (2. 1)$$

Then there exist a fixed point of  $\mu$ .

The uniqueness of fixed point is guaranteed by following conditions

**Theorem 2.3:** Let  $(X, d)$  be complete metric space and  $\mu$  be a given self- map. Suppose that there exist two functions  $\alpha: X \times X \rightarrow [0, \infty)$  and  $\varphi \in \emptyset$  such that  $\mu$  is  $\alpha - \varphi$  contraction. Suppose also that

- a) The fixed point set  $Fix(\mu)$  is not empty.
- b) For all  $u, v \in Fix(\mu)$  with  $u \neq v$ , if  $\alpha(u, v) < 1$  then there exist  $\gamma \in R_\varphi$  and for some  $q \in N$  There is a finite sequence  $\{\delta_i\} \subset X, i = 1, 2 \dots q$  such that  $\delta_0 = u, \delta_q = v, \alpha(\mu^n(\delta_i), \mu^n(\delta_{i+1})) \geq \gamma^{-1} \quad \forall n \in N \text{ and } i = 0, 1, 2 \dots q - 1$ .

Then  $\mu$  has a unique fixed point.

**Definition 2.5:** [17] Let  $\mu$  and  $\rho$  be two self-mappings of a complete metric space  $(X, d)$ . We say they are weakly compatible if they commute at their coincident point. That is if  $\mu(s) = \rho(s)$  for some  $s \in X$  implies  $\rho\mu(s) = \mu\rho(s)$

Here, we restate the proposition by M.Abbas and G.Jungck [17]

**Proposition 2.1:** Let  $(X, d)$  be a complete metric space and  $\mu, \rho$  are weakly compatible self-mappings on  $X$ . If there is unique coincident point  $\mu(z) = \rho(z) = u$  then  $u$  is the unique common fixed point of  $\mu$  and  $\rho$ .

The definition 2.2 was generalized by H.Aydi[18] as follows.

**Definition 2.6:** Let  $\mu$  and  $\rho$  be two self-mappings on a non-empty set  $X$ . We say the pair of mappings  $(\mu, \rho)$  is  $\alpha$ -admissible if there exist a mapping  $\alpha: X \times X \rightarrow [0, \infty)$  such that

$$\alpha(a, b) \geq 1 \Rightarrow \alpha(\mu(a), \rho(b)) \geq 1 \text{ and } \alpha(\rho(b), \mu(a)) \geq 1.$$

Here, it is noted that there is no any mathematical support to find the finite sequence satisfying (2.1). Actually we feel, it is most serious task to get such sequence. Similarly the hypothesis (b) in theorem 2.3 is not suitable for the aim of paper [3]. Because of these reasons we introduce general cases to extend and improve theorems 2.1, 2.2 and 2.3 without using condition defined in definition 2.2.

### 3. Main Results

In this section we present some new concepts, fixed point theorems and common fixed point theorems.

From definition 2.4, let us take the set  $R_{\varphi+} = \{\lambda \in R_\varphi : \varphi(p) < \lambda\varphi(p) < p \quad \forall p > 0\} \dots \dots \dots (3.1)$

It is clear that this set (3.1) non-empty because  $\varphi(p) < p \quad \forall p > 0$  and by properties of real numbers there are infinite real numbers between  $\varphi(p)$  and  $p$ . Now we can take maximum value from (3.1), say

$\beta = \max. \{\lambda \in R_{\varphi+}\}$ . Obviously,  $\beta > 1$  and it is maximum value for which  $\beta\varphi \in \emptyset$ . Thus  $\beta^{-1} < 1$  is minimum value for which  $\alpha(a, b) > \beta^{-1}$  for  $a, b \in X$ .

Now we can define  $\alpha$ -admissible mapping on the basis of  $\beta$  as follows:

**Definition 3.1:** Let  $(X, d)$  be a metric space and  $\mu$  be a given self- map. We say  $\mu$  is  $\alpha_{\beta^{-1}}$  - admissible, if there exist  $\alpha: X \times X \rightarrow [0, \infty)$  such that,  $\alpha(a, b) > \beta^{-1} \Rightarrow \alpha(\mu(a), \mu(b)) > \beta^{-1}$  for  $a, b \in X$ .

Now we prove following fixed point theorems as our main findings.

**Theorem 3.1:** Let  $(X, d)$  be a complete metric space. Let  $\beta$  is maximum of all member of  $R_{\varphi^+}$ . If a self -mapping  $\mu$  satisfies following conditions,

- a)  $\mu$  is  $\alpha - \varphi$  contraction on  $X$ .
- b)  $\mu$  is  $\alpha_{\beta^{-1}}$ - admissible
- c) There exist  $x_0 \in X$  such that  $\alpha(x_0, \mu(x_0)) > \beta^{-1}$
- d)  $\mu$  is continuous in  $X$

Then there exist a fixed point of  $\mu$  in  $X$ .

**Proof:** -Let there exist  $x_0 \in X$  such that  $\alpha(x_0, \mu(x_0)) > \beta^{-1}$ . Being  $\mu$  is  $\alpha_{\beta^{-1}}$  - admissible, so by induction we can get,

$$\alpha(\mu^n(x_0), \mu^{n+1}(x_0)) > \beta^{-1}, \forall n \in N \dots \dots \dots (3.2)$$

Now we have from condition (a),

$$\beta^{-1}d(\mu(x_0), \mu^2(x_0)) \leq \alpha(x_0, \mu(x_0))d(\mu(x_0), \mu^2(x_0)) \leq \varphi(d(x_0, \mu(x_0)))$$

$$\Rightarrow d(\mu(x_0), \mu^2(x_0)) \leq \beta\varphi(d(x_0, \mu(x_0)))$$

Let  $\beta\varphi = \sigma \in \emptyset$ , then

$$d(\mu(x_0), \mu^2(x_0)) \leq \sigma(d(x_0, \mu(x_0))) \dots \dots \dots (3.3)$$

Again

$$\beta^{-1}d(\mu^2(x_0), \mu^3(x_0)) \leq \alpha(\mu(x_0), \mu^2(x_0))d(\mu^2(x_0), \mu^3(x_0)) \leq \varphi(d(\mu(x_0), \mu^2(x_0)))$$

$$\Rightarrow d(\mu^2(x_0), \mu^3(x_0)) \leq \sigma(d(\mu(x_0), \mu^2(x_0))) , \text{ from (3.3)}$$

$$d(\mu^2(x_0), \mu^3(x_0)) \leq \sigma^2(d(x_0, \mu(x_0)))$$

Continuing this process, we have

$$d(\mu^n(x_0), \mu^{n+1}(x_0)) \leq \sigma^n(d(x_0, \mu(x_0))) \dots \dots \dots (3.4)$$

Now for  $m, n \in N$  with  $m < n$ ,

$$d(\mu^m(x_0), \mu^n(x_0))$$

$$\leq d(\mu^m(x_0), \mu^{m+1}(x_0)) + d(\mu^{m+1}(x_0), \mu^{m+2}(x_0)) + \dots + d(\mu^{n-1}(x_0), \mu^n(x_0))$$

From (3.4)

$$d(\mu^m(x_0), \mu^n(x_0)) \leq \sigma^m(d(x_0, \mu(x_0))) + \sigma^{m+1}(d(x_0, \mu(x_0))) + \dots \dots \dots + \sigma^{n-1}(d(x_0, \mu(x_0)))$$

Taking  $m \rightarrow \infty$  and by property of comparison mapping,

$$\sigma^m \left( d \left( x_0, \mu(x_0) \right) \right) = 0 .$$

Thus we have,

$$d(\mu^m(x_0), \mu^n(x_0)) = 0$$

This implies  $\{\mu^n(x_0)\}$  is Cauchy sequence in  $(X, d)$  .Since  $(X, d)$  is complete, there exist  $x_\mu \in X$  such that  $\lim_{n \rightarrow \infty} \mu^n(x_0) = x_\mu$ .

The continuity of  $\mu$  implies that,

$$\lim_{n \rightarrow \infty} \mu^{n+1}(x_0) = \mu(x_\mu) . \text{ The uniqueness of limiting value, gives } \mu(x_\mu) = x_\mu . \square$$

It is clear that our suggested hypothesis, “there exist  $x_0 \in X$  such that  $\alpha(x_0, \mu(x_0)) > \beta^{-1}$ ” is more general than the hypothesis given in theorem 2.1 in preliminaries section.

Next we show the continuity assumption of  $\mu$  does not require to exist a fixed point of  $\mu$ .

**Theorem 3.2:** Let  $(X, d)$  be a complete metric space. Let  $\beta$  is maximum of all member of  $R_{\varphi^+}$ . If the self -mapping  $\mu$  satisfies following conditions,

- a)  $\mu$  is  $\alpha - \varphi$  contraction on X.
- b)  $\mu$  is  $\alpha_{\beta^{-1}}$  -admissible.
- c) There exist  $x_0 \in X$  such that  $\alpha(x_0, \mu(x_0)) > \beta^{-1}$
- d) If  $\alpha(\mu^n(x_0), \mu^{n+1}(x_0)) > \beta^{-1}, \forall n \in N$  and  $\lim_{n \rightarrow \infty} d(\mu^n(x_0), z) = 0$  implies  $\alpha(\mu^n(x_0), z) > \beta^{-1}, \forall n \in N$

Then there exist a fixed point of  $\mu$  in X.

**Proof:** From theorem 3.1 the sequence  $\{\mu^n(x_0)\}$  is Cauchy sequence in  $(X, d)$  .Since  $(X, d)$  is complete, there exist  $x_\mu \in X$  such that  $\lim_{n \rightarrow \infty} \mu^n(x_0) = x_\mu$ . Thus  $\lim_{n \rightarrow \infty} d(\mu^n(x_0), x_\mu) = 0$ .

Now from given hypothesis (d) and from (3.2), we can get,

$$\alpha(\mu^n(x_0), x_\mu) > \beta^{-1}, \forall n \in N \dots\dots\dots (3.5)$$

Now,

$$\begin{aligned} \beta^{-1} d(\mu^{n+1}(x_0), \mu(x_\mu)) &\leq \alpha(\mu^n(x_0), x_\mu) d(\mu^{n+1}(x_0), \mu(x_\mu)) \leq \varphi(d(\mu^n(x_0), x_\mu)) \\ \Rightarrow d(\mu^{n+1}(x_0), \mu(x_\mu)) &\leq \beta \varphi(d(\mu^n(x_0), x_\mu)) \\ \Rightarrow d(\mu^{n+1}(x_0), \mu(x_\mu)) &\leq \sigma(d(\mu^n(x_0), x_\mu)) < d(\mu^n(x_0), x_\mu) \end{aligned}$$

Taking  $n \rightarrow \infty$  , we have

$$\begin{aligned} d(x_\mu, \mu(x_\mu)) &\leq d(x_\mu, x_\mu) \\ \Rightarrow d(x_\mu, f(x_\mu)) &= 0 \end{aligned}$$

$$\therefore \mu(x_\mu) = x_\mu \quad \square$$

The uniqueness of fixed point is given by following theorem.

**Theorem 3.3:** By adding the condition: “if for every pair  $u, v \in X$  there exist  $w \in X$  such that  $\alpha(u, w) > \beta^{-1}$  and  $\alpha(v, w) > \beta^{-1}$ ” in the theorem 3.1 and 3.2, then there exist unique fixed point of  $\mu$ .

**Proof:** If possible suppose there are two fixed points  $x_f$  and  $y_f$ . By our hypothesis there exist  $w \in X$  such that  $\alpha(x_\mu, w) > \beta^{-1}$  and  $\alpha(y_\mu, w) > \beta^{-1}$ .

Being  $\mu$  is  $\alpha$ -admissible,  $\alpha(x_\mu, w) > \beta^{-1}$  implies  $\alpha(x_\mu, \mu^n(w)) > \beta^{-1}$ .

Now we have,

$$\beta^{-1}d(x_\mu, \mu^n(w)) = \beta^{-1}d(\mu(x_\mu), \mu^n(w)) \leq \alpha(x_\mu, \mu^{n-1}(w))d(\mu(x_\mu), \mu^n(w)) \leq \varphi(d(x_\mu, \mu^{n-1}(w)))$$

$$\Rightarrow d(x_\mu, \mu^n(w)) \leq \sigma(d(x_\mu, \mu^{n-1}(w))).$$

Since  $\sigma$  is non-decreasing, then by induction we can have,

$$d(x_\mu, \mu^n(w)) \leq \sigma^n(d(x_\mu, w)).$$

But

$$\lim_{n \rightarrow \infty} \sigma^n(d(x_\mu, w)) = 0. \text{ Hence, } \lim_{n \rightarrow \infty} d(x_\mu, \mu^n(w)) = 0$$

Similarly we can show that  $\lim_{n \rightarrow \infty} d(y_\mu, \mu^n(w)) = 0$ .

By triangular property of metric

$$d(x_\mu, y_\mu) \leq d(x_\mu, \mu^n(w)) + d(y_\mu, \mu^n(w)). \text{ Now, taking } n \rightarrow \infty$$

$$d(x_\mu, y_\mu) = 0. \text{ Hence fixed point is unique. } \square$$

Next, we introduce some new types of  $\alpha$ -admissible mappings and contraction conditions to prove common fixed point theorem.

First of all let us denote the set of all coincident points of pair of mappings  $(\mu, \rho)$  by  $C(\mu, \rho)$ .

**Definition 3.2:** Let  $(X, d)$  be a complete metric space and  $\mu, \rho$  are self-mappings on  $X$  with  $\rho(X) \subset \mu(X)$ . We say  $\rho$  is  $\mu - \alpha, \varphi$ -contraction if there exist  $\alpha: X \times X \rightarrow [0, \infty)$  and a comparison function  $\varphi: [0, \infty) \rightarrow [0, \infty)$  such that,

$$\alpha(\mu(a), \mu(b))d(\rho(a), \rho(b)) \leq \varphi(d(\mu(a), \mu(b))) \quad \forall a, b \in X.$$

**Definition 3.3:** Let  $(X, d)$  be a metric space and  $\mu, \rho$  self-mappings on  $X$  with  $\rho(X) \subset \mu(X)$ . We say  $\rho$  is  $\mu - \alpha_{\beta^{-1}}$ -admissible if there exists a function  $\alpha: X \times X \rightarrow [0, \infty)$  such that,

$$\alpha(\mu(a), \mu(b)) > \beta^{-1} \Rightarrow \alpha(\rho(a), \rho(b)) > \beta^{-1} \text{ For } a, b \in X$$

Now we apply above results to prove the common fixed point in complete metric spaces.

**Theorem 3.4:** Let  $(X, d)$  be a complete metric space and  $\mu, \rho$  are weakly compatible self-mappings on  $X$  with  $\rho(X) \subset \mu(X)$ . Also let  $\mu(X)$  is closed subset of  $X$ . There exist  $\alpha: X \times X \rightarrow [0, \infty)$ ,  $\varphi \in \emptyset$  and  $\beta$  is maximum of all member of  $R_{\varphi+}$  such that  $\mu$  and  $\rho$  satisfy following conditions:

- a)  $\rho$  is  $\mu - \alpha, \varphi$  contraction.
  - b)  $\rho$  is  $\mu - \alpha_{\beta^{-1}}$  admissible.
  - c) There exist  $x_0 \in X$  such that  $\alpha(\mu(x_0), \rho(x_0)) > \beta^{-1}$
  - d) If for a sequence  $\{x_n\}$  in  $X$ , the sequence  $\mu(x_n)$  converges to  $\mu(x)$  and  $\alpha(\mu(x_n), \mu(x_{n+1})) > \beta^{-1}$  then there exist sub-sequence  $\mu(x_{n_k})$  of  $\mu(x_n)$  such that  $\alpha(\mu(x_{n_k}), \mu(x)) > \beta^{-1}$ .
  - e) For all  $p, q \in C(\mu, \rho)$  the condition  $\alpha(\mu(p), \mu(q)) > \beta^{-1}$  or  $\alpha(\mu(q), \mu(p)) > \beta^{-1}$  exist.
- Then  $\mu$  and  $\rho$  have a unique point of coincident. Further this point is unique common fixed point.

**Proof:** Since  $\rho(X) \subset \mu(X)$ , for  $x_0 \in X$  there exist  $x_1 \in X$  such that  $\rho(x_0) = \mu(x_1)$ . In general we can write  $s_n = \rho(x_n) = \mu(x_{n+1})$ . If  $s_n = s_{n+1}$  for some  $n \in N$ , then  $s_n = \rho(x_{n+1}) = \mu(x_{n+1})$ . This implies there is a point of coincident. Thus let  $s_n \neq s_{n+1} \forall n \in N$ . By condition (c), there exist  $x_0 \in X$  such that

$$\alpha(\mu(x_0), \rho(x_0)) > \beta^{-1}$$

$$\Rightarrow \alpha(\mu(x_0), \mu(x_1)) > \beta^{-1}$$

Since  $g$  is  $\mu - \alpha$  admissible, so

$$\alpha(\mu(x_0), \mu(x_1)) > \beta^{-1} \Rightarrow \alpha(\rho(x_0), \rho(x_1)) > \beta^{-1}$$

$$\Rightarrow \alpha(\mu(x_1), \mu(x_2)) > \beta^{-1}$$

$$\Rightarrow \alpha(\rho(x_1), \rho(x_2)) > \beta^{-1}$$

$$\Rightarrow \alpha(\mu(x_2), \mu(x_3)) > \beta^{-1}$$

Continuing this process we have,

$$\alpha(\mu(x_n), \mu(x_{n+1})) > \beta^{-1} \dots \dots (3.6)$$

From condition (a)

$$\beta^{-1}d(\rho(x_0), \rho(x_1)) \leq \alpha(\mu(x_0), \mu(x_1))d(\rho(x_0), \rho(x_1)) \leq \varphi(d(\mu(x_0), \mu(x_1)))$$

$$\Rightarrow d(\rho(x_0), \rho(x_1)) \leq \beta\varphi(d(\mu(x_0), \mu(x_1))) \quad [ \because \beta\varphi = \sigma \in \emptyset ]$$

$$\Rightarrow d(\rho(x_0), \rho(x_1)) \leq \sigma(d(\mu(x_0), \mu(x_1))) \dots \dots \dots (3.7)$$

$$\Rightarrow d(\mu(x_1), \mu(x_2)) \leq \sigma(d(\mu(x_0), \mu(x_1))) \dots \dots \dots (3.8)$$

Similarly we can show

$$d(\mu(x_2), \mu(x_3)) \leq \sigma(d(\mu(x_1), \mu(x_2)))$$

From (3.8) and non-decreasing property of  $\sigma$ , we have

$$d(\mu(x_2), \mu(x_3)) \leq \sigma^2(d(\mu(x_0), \mu(x_1)))$$

Continuing this process we have,

$$d(\mu(x_n), \mu(x_{n+1})) \leq \sigma^n(d(\mu(x_0), \mu(x_1))) \dots \dots \dots (3.9)$$

Now for any  $m, n \in N$  with  $m < n$ ,

$$d(\mu(x_m), \mu(x_n)) \leq d(\mu(x_m), \mu(x_{m+1})) + d(\mu(x_{m+1}), \mu(x_{m+2})) + \dots \dots + d(\mu(x_{n-1}), \mu(x_n))$$

From (3.9)

$$d(\mu(x_m), \mu(x_n)) \leq \sigma^m(d(\mu(x_0), \mu(x_1))) + \sigma^{m+1}(d(\mu(x_0), \mu(x_1))) + \dots \dots + \sigma^{n-1}(d(\mu(x_0), \mu(x_1)))$$

Using the property  $\sigma^m(t) = 0 \forall t > 0$  for  $m \rightarrow \infty$

$$d(\mu(x_m), \mu(x_n)) = 0 \dots \dots \dots (3.10)$$

This yield  $\{\mu(x_n)\}$  is Chauchy sequence in  $X$ . Being  $(X, d)$  is complete metric space, there exist  $u \in X$  such that  $\lim_{n \rightarrow \infty} \mu(x_n) = u$ . Here  $\rho(x_n) = \mu(x_{n+1})$ , implies  $\lim_{n \rightarrow \infty} \rho(x_n) = u$ .

As our hypothesis  $\mu(X)$  is closed sub-set of  $X$ , so  $u \in \mu(X)$  and there exist  $z \in X$  such that  $\mu(z) = u$  thus  $\lim_{n \rightarrow \infty} d(\mu(x_n) - \mu(z)) = 0 \dots \dots \dots (3.11)$

Now by hypothesis (d), from (3.6) and (3.11); there exist sub-sequence  $\mu(x_{n_k})$  of  $\mu(x_n)$  such that  $\alpha(\mu(x_{n_k}), \mu(z)) > \beta^{-1} \forall k \in N$

Now by (3.7) we have,

$$\beta^{-1}d(\rho(x_{n_k}), \rho(z)) \leq \alpha(\mu(x_{n_k}), \mu(z))d(\rho(x_{n_k}), \rho(z)) \leq \varphi(d(\mu(x_{n_k}), \mu(z)))$$

$$d(\rho(x_{n_k}), \rho(z)) \leq \sigma(d(\mu(x_{n_k}), \mu(z)))$$

Taking  $k \rightarrow \infty$

$$d(u, \rho(z)) \leq \sigma(d(\mu(z), \mu(z)))$$

$$\Rightarrow d(u, \rho(z)) = 0$$

$$\therefore \rho(z) = u = \mu(z)$$

Therefore  $z$  is a coincident of  $\mu$  and  $\rho$ .

We show  $z$  is unique coincident point. If possible suppose  $w$  is another coincident point then by hypothesis (e)

$$\alpha(\mu(z), \mu(w)) > \beta^{-1} \dots \dots \dots (3.12)$$

Applying condition (a) and from (3.7) and (3.12) we have,

$$\beta^{-1}d(\rho(z), \rho(w)) \leq \alpha(\mu(z), \mu(w))d(\rho(z), \rho(w)) \leq \varphi(d(\mu(z), \mu(w)))$$

$$\Rightarrow d(\rho(z), \rho(w)) \leq \sigma(d(\mu(z), \mu(w))) \leq d(\mu(z), \mu(w))$$

$$\Rightarrow d(\rho(z), \rho(w)) \leq d(\rho(z), \rho(w))$$

$$\Rightarrow d(\rho(z), \rho(w)) = 0$$

$$\Rightarrow \rho(z) = \rho(w)$$

$$\therefore \rho(z) = \mu(z) = \rho(w) = \mu(w)$$

Hence  $z$  is unique coincident point and  $u$  is unique point of coincident. From proposition 2.1, it is clear that  $u$  is unique common fixed point of  $\mu$  and  $\rho$   $\square$

**Example 1:** Let  $X = [0, 1]$  and metric on it is  $d(x, y) = |x - y|$ . Take the mapping  $\mu: X \rightarrow X$  defined by  $\mu(x) = \frac{x^3}{2} + \frac{7}{16}$ . It is clear that  $\mu$  is continuous but not Banach contraction at  $x = 1$  and  $y = 0.9$ . Actually it is  $\alpha - \varphi$  contraction. Where,  $\alpha: X \times X \rightarrow [0, \infty)$  defined by  $\alpha(x, y) = \frac{1}{x^2 + xy + y^2}$  and  $\varphi: X \times X \rightarrow [0, \infty)$  defined by  $\varphi(t) = \frac{t}{2}$ . Take  $\beta = 1.999 \dots$  so that  $\beta\varphi$  is again a comparison function. There exist  $x_0 = 0.1$  such that  $\alpha(x_0, \mu(x_0)) > \beta^{-1}$ . By calculation we can easily find that  $\mu$  is  $\alpha_{\beta^{-1}}$ -admissible. More over for every pair  $u, v \in X$  there exist  $w \in X$  such that  $\alpha(u, w) > \beta^{-1}$  and  $\alpha(v, w) > \beta^{-1}$ . Here  $\mu$  satisfy all condition of theorem 3.1 and 3.3. Hence there exist unique fixed point  $x = \frac{1}{2}$ .

**Example 2:** Let  $X = [0, 1]$  and metric on it is  $d(x, y) = |x - y|$ . Take two self-mappings defined by  $\mu(x) = \frac{x^3}{2} + \frac{7}{16}$  and  $\rho(x) = \frac{x^4}{3} + \frac{23}{48}$ . It is clear that  $\mu$  and  $\rho$  are continuous but neither of them are Banach contraction at  $x = 1$  and  $y = 0.9$ . Actually they satisfy  $f - \alpha, \varphi$  contraction. Where,  $\alpha: X \times X \rightarrow [0, \infty)$  defined by  $\alpha(x, y) = \frac{x^2 + xy + y^2}{x^3 + xy^2 + x^2y + y^3}$  and  $\varphi: X \times X \rightarrow [0, \infty)$  defined by  $\varphi(t) = \frac{2t}{3}$ . Take  $\beta = 1.499 \dots$  so that  $\beta\varphi$  is again a comparison function. There exist  $x_0 = 0.1$  such that  $\alpha(\mu(x_0), \rho(x_0)) > \beta^{-1}$ . By calculation we can easily find that  $\rho$  is  $\mu - \alpha_{\beta^{-1}}$ -admissible. More over for every pair  $u, v \in C(\mu, \rho)$  such that  $\alpha(\mu(u), \mu(v)) > \beta^{-1}$  or  $\alpha(\mu(v), \mu(u)) > \beta^{-1}$ . Here  $\mu$  and  $\rho$  commute at only coincident point. Hence there exist unique common fixed point  $x = \frac{1}{2}$ .

#### 4. Remarks

- a) If  $\beta = 1$  then the theorem 3.1 becomes theorem 2.1.
- b) For  $\beta = 1$  the mapping  $\mu$  becomes  $\alpha$ -admissible.
- c) If  $\mu$  is identity map and  $\beta = 1$  then the theorem 3.4 becomes theorem 2.1.

#### References

- [1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta mathematicae*, vol. 3, no. 1, pp. 133–181, 1922.
- [2] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for  $\alpha$ - $\psi$ -contractive type mappings," *Nonlinear analysis: theory, methods & applications*, vol. 75, no. 4, pp. 2154–2165, 2012.
- [3] B. Samet, "FIXED POINTS FOR  $\alpha$ - $\psi$  CONTRACTIVE MAPPINGS WITH AN APPLICATION TO QUADRATIC INTEGRAL EQUATIONS," vol. 1914, pp. 1–18, 1914.

- [4] E. Karapinar, S. Czerwik, and H. Aydi, “Research Article  $(\alpha, \psi)$ -Meir-Keeler Contraction Mappings in Generalized  $b$ -Metric Spaces,” 2018, Accessed: May 18, 2024. [Online]. Available: <https://www.academia.edu/download/72105343/3264620.pdf>
- [5] J. Li and H. Guan, “Common Fixed Point Results for Generalized  $\alpha$ -Contractive Mappings with Applications,” *Journal of Function Spaces*, vol. 2021, p. e5020027, Jun. 2021, doi: 10.1155/2021/5020027.
- [6] N. Hussain, P. Salimi, and A. Latif, “Fixed point results for single and set-valued  $\alpha$ - $\eta$ - $\psi$ -contractive mappings,” *Fixed Point Theory Appl*, vol. 2013, no. 1, p. 212, Dec. 2013, doi: 10.1186/1687-1812-2013-212.
- [7] K. Zoto, B. Rhoades, and S. Radenović, “Some generalizations for  $(\alpha$ - $\psi, \varphi)$ -contractions in  $b$ -metric-like spaces and an application,” *Fixed Point Theory and Applications*, vol. 2017, pp. 13663–2017, Nov. 2017, doi: 10.1186/s13663-017-0620-1.
- [8] B. N. Abagaro, K. K. Tola, and M. A. Mamud, “Fixed point theorems for generalized  $(\alpha, \psi)$ -contraction mappings in rectangular quasi  $b$ -metric spaces,” *Fixed Point Theory Algorithms Sci Eng*, vol. 2022, no. 1, p. 13, Dec. 2022, doi: 10.1186/s13663-022-00723-w.
- [9] J. R. Morales and E. M. Rojas, “Common fixed points for  $(\psi$ - $\varphi)$ -weak contractions type in  $b$ -metric spaces,” *Arab. J. Math.*, vol. 10, no. 3, pp. 639–658, Dec. 2021, doi: 10.1007/s40065-021-00347-9.
- [10] P. Manthena and R. Manchala, “Fixed point results for  $\mathcal{H}$ -contractions in fuzzy metric spaces via admissible,” *Malaya Journal of Matematik*, vol. 6, no. 03, pp. 588–594, 2018.
- [11] P. Salimi, N. Hussain, A. Roldan, and E. Karapinar, “On Modified  $\alpha$ - $\phi$ -Asymmetric Meir-Keeler Contractive Mappings,” *Filomat*, vol. 28, no. 9, pp. 1855–1869, 2014.
- [12] A. Latif, M. E. Gordji, E. Karapinar, and W. Sintunavarat, “Fixed point results for generalized  $(\alpha, \psi)$ -Meir-Keeler contractive mappings and applications,” *J Inequal Appl*, vol. 2014, no. 1, p. 68, Dec. 2014, doi: 10.1186/1029-242X-2014-68.
- [13] N. Hussain, E. Karapinar, P. Salimi, and P. Vetro, “Fixed point results for  $G_m$ -Meir-Keeler contractive and  $G$ - $(\alpha, \psi)$ -Meir-Keeler contractive mappings,” *Fixed Point Theory Appl*, vol. 2013, no. 1, p. 34, Dec. 2013, doi: 10.1186/1687-1812-2013-34.
- [14] N. Shahzad, M. A. Alghamdi, S. Alshehri, and I. Arandelović, “Semi-metric spaces and fixed points of  $\alpha$ - $\varphi$ -contractive maps,” *J. Nonlinear Sci. Appl.*, vol. 09, no. 05, pp. 3147–3156, May 2016, doi: 10.22436/jnsa.009.05.104.
- [15] A. FELHI, S. SAHMIM, and H. AYDI, “On common fixed points for  $(\alpha, \psi)$ -contractions and generalized cyclic contractions in  $b$ -metric-like spaces and consequences,” 2016, Accessed: May 18, 2024. [Online]. Available: [https://www.emis.de/journals/TJNSA/includes/files/articles/user\\_articles/20151029125204\\_669\\_On\\_common\\_fixed\\_points\\_for\\_\(alpha,p.pdf](https://www.emis.de/journals/TJNSA/includes/files/articles/user_articles/20151029125204_669_On_common_fixed_points_for_(alpha,p.pdf)
- [16] I. A. Rus, A. Petruşel, and G. Petruşel, *Fixed point theory*, vol. 2. Cluj University Press Cluj-Napoca, 2008.
- [17] M. Abbas and G. Jungck, “Common fixed point results for noncommuting mappings without continuity in cone metric spaces,” *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 416–420, May 2008, doi: 10.1016/j.jmaa.2007.09.070.
- [18] H. Aydi, “ $\alpha$ -implicit contractive pair of mappings on quasi  $b$ -metric spaces and an application to integral equations,” *J. Nonlinear Convex Anal*, vol. 17, no. 12, pp. 2417–2433, 2016.