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On the Oscillation of a Class of Conformable Schrodinger Equations

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Abstract

In this article, we have derived a new oscillation criteria for a class of conformable Schrodinger equations. Based on the generalized Riccati technique, the results were obtained here. Also we have extended the Hartman-Winter oscillation criteria to conformable Schrodinger equation.

Keywords:. Oscillation, Conformable Schrodinger equation, Elliptic partial differential equation.

AMS Subject Classification: 34A08, 34A34, 34K11, 35B05, 35R20.

1. INTRODUCTION

The area of research that has grown the fastest in recent years is differential equations in fractional calculus. Although there are various fractional derivative notions, including Riemann-Liouville and Caputo fractional derivatives, which are based on singular integrals and non-locality, they are commonly utilized. In 2014, Khalil et al. [13] developed the conformable fractional derivative, which was based on a limit concept similar to that of integer order derivatives.

The Conformable derivative of Khalil was quickly made general by Katugampola fractional derivative or alpha fractional derivative [11, 12]. It has wide application in biophysics, quantum mechanics, wave theory and polymers and it is a crucial tool for simulating a variety of physical phenomena, including electromagnetic waves and viscoelastic systems [9, 14]. Numerous studies have been done in the literature on the oscillation of conformable fractional differential equations [2, 4, 8, 17]. In mathematically oriented sciences like physics and engineering, conformable partial differential equations are widely encountered [19, 20]. For instance, they form the basis of current scientific understanding of diffusion, electrostatics, materials, dynamical theory, hydrodynamics, electrodynamics, viscoelasticity and quantum mechanics. Moreover, fractional partial differential equations have gained popularity in recent years as a tool for mathematical modelling. The oscillation of conformable partial differential equations has been researched by numerous authors, see [5, 6].

The concept of elliptic equation has undergone an important growth over the last two centuries. Together with electro statistics heat and mass diffusion, hydrodynamics and many other applications it has become one of the most richly enhanced field of mathematics. Numerous authors have been motivated by the oscillation theory of elliptic equations in recent years [1, 7]. Over the past few years, there has been a lot of attention paid to the issue of oscillation and non-oscillation of differential equation solutions [10, 18].

A linear partial differential equation called the Schrodinger equation controls how a quantum mechanical system behaves in terms of its wave function. The Schrodinger equation is the cornerstone of quantum mechanics, the study of microscopic events. The Schrodinger equation, created in 1926 by the Austrian scientist Erwin Schrodinger, is as essential to understanding quantum mechanics as Newton's laws of motion are to understanding large-scale classical mechanics occurrences. Later, Noussair [21] used the n-dimensional Emden-Fowler method to generate solutions to the nonlinear Schrodinger equation in the outer domain Bn. E.Muller- Pfeiffer [22] also obtained oscillation criteria for the Schrodinger equation in sobolev space and generalized the derivative of order 2 summable on every compact subdomain of G. By using the Emden-Fowler equation, Hirashi Onose [23] has explored some conclusions on the sublinear Schrodinger equation. Swanson [24] used a modified sublinear hypothesis to build the article in an emden-Fowler type sublinear equation. Zhang [25] has introduced unique standards for the absence of positive results by using the Perturbed Schrodinger equation.

In this paper, we are concerned with conformable elliptic equations is of the type

$$\Delta^{\alpha}_{\underline{x}} u + p(x)u = 0, \quad \Delta^{\alpha}_{\underline{x}} u = \sum_{i=1}^{n} \frac{\partial^{2\alpha} u}{\partial x_i^{2\alpha}} \tag{1.1}$$

where $\alpha \in (0,1), x = (x_1, x_2, \dots x_n), \Delta^{\alpha}_{\underline{x}}$ is the conformable nabla operator and $p(x) : \mathbb{R}^n \to \mathbb{R}$ is potential function and each compact subset of Ω.

Define the set

$$\Omega(a) = \{x \in \mathbb{R}^n : a \le r\},$$
$$\Omega(a,b) = \{x \in \mathbb{R}^n : a \le r \le b\},$$
$$S(a) = \{x \in \mathbb{R}^n : r = a\},$$

$u(x) : \Omega \to \mathbb{R}$ is almost always absolutely continuous in α-fractional derivative of compact subsets that fulfills equation (1.1) on Ω is almost everywhere.

A constrained area $G \subset \Omega$ is a nodal domain for (1.1) if there exists a nontrivial function $u \in C^2(G; \mathbb{R}) \cap C(\bar{G}; \mathbb{R})$ such that 1.1 is equal to zero and $u = 0$ on ∂G. If for each $r > 0$ equation (1.1) has a nodal domain contained and enclosed in $\Omega_r = \Omega \cap \{x \in \mathbb{R}^n : |x| > r\}$, then equation (1.1) is called nodally oscillatory. If the function f(x) has zero outside of arbitrary ball in $\mathbb{R}^n$ that is centered in the origin, is said to be oscillatory; if not, it is said non-oscillatory. We get the function P(t) form the Hartman-Winter Theorem

$$P(r) = \frac{1}{r^{\alpha}} \int_1^r \int_{\Omega(1,r)} r^{1-n+\lambda} p(x) d_{\alpha}x d_{\alpha}r \tag{1.2}$$

We distinguish two cases,

(i) There is a finite limit

$$\lim_{r \to \infty} P(r) = P_0 \tag{1.3}$$

(ii) The (1.3) condition is fails to hold and $\liminf_{r\to\infty} P(r) > -\infty$.

To my knowledge, aware, there is no literature exists on the oscillation of the conformable elliptic equation. Inspired by Robert Marik [16] and Lomtatidze [3, 15] we investigating the following conformable elliptic equation of the form

$$M(r) = r\left(\alpha P_o - \int_{\Omega(1,r)} r^{1-n+\lambda} p(x) d_\alpha x\right) \qquad (1.4),$$

$$N(r) = \frac{1}{r}\int_{\Omega(1,r)} r^{3-n+\lambda} p(x) d_\alpha x. \qquad (1.5)$$

2. PRELIMINARIES

In order to make our method clear, We provide certain fundamental definitions, properties and lemmas

Definition 2.1. Given $u: [0,\infty) \to \mathbb{R}$. Conformable fractional derivative of u of order α is given by

$$T_\alpha(u)(x) = \lim_{\epsilon\to 0} \frac{u(x+\epsilon x^{1-\alpha}) - u(x)}{\epsilon} \quad \forall x > 0, \quad \alpha \in (0,1).$$

If u can be α-differentiable in some $(0,a), a > 0$ and $\lim_{x\to 0^+} u^\alpha(x)$ exists, then we define

$$u^\alpha(0) = \lim_{x\to 0^+} u^\alpha(x).$$

Definition 2.2. $I_\alpha^a(u)(x) = I_1^\alpha(x^{\alpha-1})(u) = \int_a^x \frac{u(x)}{x^{1-\alpha}} dx$, where the integral is the standard Riemann improper integral and $\alpha \in (0,1)$.

Properties 2.1. Let $\alpha \in (0,1]$ and at some point $x > 0$. $u$and v will eventually be α differentiable. Then,

(1) $T_\alpha(a_1 u + a_2 v) = a_1 T_\alpha(u) + a_2 T_\alpha(v), \quad \forall a_1, a_2 \in \mathbb{R}$

(2) $T_\alpha(uv) = uT_\alpha(v) + vT_\alpha(u)$

(3) $T_\alpha(x^p) = px^{p-\alpha}, \quad \forall p \in \mathbb{R}$

(4) $T_\alpha(a) = 0, \quad u(x) = a$ for every constant functions.

(5) $T_\alpha\left(\frac{u}{v}\right) = \frac{vT_\alpha(u) - uT_\alpha(v)}{v^2}$

(6) If u is differential, then $T_\alpha(u(x)) = x^{1-\alpha}\frac{du(x)}{dx}$.

Proof . Refer [13]

Definition 2.3. Let u be a function with m variable $x_1, \dots\dots, x_m$, and the conformable partial derivative of u of order $0 < \alpha \le 1$ in x_i is defined as follows

$$\frac{\partial^\alpha}{\partial x_i^\alpha} u(x_1, \dots\dots, x_m) = \lim_{\epsilon\to 0} \frac{u(x_1, \dots, x_{i-1}, x_i + \epsilon x_i^{1-\alpha} \dots, x_m) - u(x_1, \dots\dots, x_m)}{\epsilon}$$

Lemma 2.1. If $\vec{r} = \overrightarrow{x_i} + \overrightarrow{y_j} + \overrightarrow{z_k}$ and $r = |\vec{r}|$ then $\mathrm{grad}_\alpha\, f(r) = r^{1-\alpha}\mathrm{grad}\, f(r)$.

Proof. If f is differentiable then by using the properties of 2.1 we get

$$\text{grad}_\alpha\ f(r) = r^{1-\alpha}\text{grad}\ f(r).$$

Hence the proof is complete.

First we introduce the Riccati method. There exists a $\Omega_r = \{x \in \mathbb{R}^n \colon \| x \| \geq r\}$ and a solution u of (1.1) that is non negative on Ω_r. Let $\overrightarrow{W} = \frac{\text{grad}_\alpha \text{u}}{\text{u}}$ be the vector function representing the solution to the Riccati equation defined on the set Ω_r.

$$\text{div}_\alpha\ \overrightarrow{W} + p(x) + \| W \|^2 = 0 \tag{2.1}$$

The operator div_α is typical divergent operator, i.e. for $\overrightarrow{W} = (W_1, \dots, W_n)$ where the common Euclidean norm in $\mathbb{R}^n$ is represented by $\|.\|$.

Lemma 2.2. Let equation (1.1) be non-oscillatory, i.e, (1.1) has a positive solution on Ω_a for some $a \leq 1$. The below statements are equivalent:

i) Its

$$\int_{\Omega(a,\infty)} r^{1-n+\lambda} \| W \|^2\ d_\alpha x < \infty \tag{2.2}$$

ii) There's a finite limit

$$\lim_{r\to\infty} P(r) = P_0 \tag{2.3}$$

iii) Its holds

$$\lim\inf_{r\to\infty} P(r) > -\infty \tag{2.4}$$

Proof. Let equation (1.1) be non-oscillatory. There is a number a $\in \mathbb{R}^+$and a solution u of (1.1) that is non negative on Ω_a. Let $\overrightarrow{\text{W}} = \frac{\text{grad}_\alpha\ u}{u}$ be the vector function representing the solution of Riccati equation defined on Ω_a and using the Gauss divergence theorem and the identity

$$\begin{aligned}&\int_{S(r)} r^{\alpha-n+\lambda}\langle W, e_i\rangle dS - \int_{S(a)} r^{\alpha-n+\lambda}\langle W, e_i\rangle dS\\ &+\int_{\Omega(a,r)} r^{\alpha-n+\lambda} p(x)dx + \int_{\Omega(a,r)} r^{\alpha-n+\lambda} \| W \|^2\ dx\\ &-\int_{\Omega(a,r)} (\alpha-n+\lambda) r^{-n+\lambda}\langle W, e_i\rangle dx. \qquad (2.5)\end{aligned}$$

i) ⇒ ii) If (2.2) holds. then the Cauchy inequality gives

$$\begin{aligned}\int_{\Omega(a,r)} r^{-n+\lambda} \| W \|\ dx &\leq \left(\int_{\Omega(a,r)} r^{\alpha-n+\lambda} \| W \|^2\ dx\right)^{\frac{1}{2}} \left(\int_{\Omega(a,r)} r^{-\alpha-n+\lambda} dx\right)^{\frac{1}{2}}\\ &= \left(\int_{\Omega(a,r)} r^{\alpha-n+\lambda} \| W \|^2\ dx\right)^{\frac{1}{2}} \left(\omega_n \int_a^r r^{\alpha+\lambda-3} dr\right)^{\frac{1}{2}}.\end{aligned}$$

here, ω_n is represented as the measure of the sphere in $\mathbb{R}^n$ and $\omega_n = \frac{2\Pi^{\frac{n}{2}}}{\Gamma\frac{n}{2}}$,

$$\int_{\Omega(a,\infty)} r^{-n+\lambda}\langle W, e_i\rangle dx < \infty \tag{2.6}$$

not diverges. Evaluate of (2.5) and (2.6) gives

$$\begin{aligned}\hat{P} - \int_{\Omega(1,r)} r^{\alpha-n+\lambda} p(x)dx &= \int_{S(r)} r^{\alpha-n+\lambda}\langle W, e_i\rangle dS \\ &+ \int_{\Omega(r,\infty)} (\alpha - n + \lambda) r^{-n+\lambda}\langle W, e_i\rangle dx - \int_{\Omega(r,\infty)} r^{\alpha-n+\lambda} \parallel W \parallel^2 dx,\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}\hat{P} = \int_{S(a)} r^{\alpha-n+\lambda}\langle W, e_i\rangle dS + \int_{\Omega(a,\infty)} (\alpha - n + \lambda) r^{-n+\lambda}\langle W, e_i\rangle dx \\ + \int_{\Omega(1,a)} r^{\alpha-n+\lambda} p(x)dx - \int_{\Omega(a,\infty)} r^{\alpha-n+\lambda} \parallel W \parallel^2 dx,\end{aligned}$$

is a finite number. We will show that

$$\hat{P} = \alpha P_0. \tag{2.8}$$

Then it follows that $\hat{P}$ actually does not always depend on the choice of the number for a. Using (2.7) and the inequality

$$|b + c + d|^2 \leq 4|b|^2 + 4|c|^2 + 4|d|^2.$$

Taking integration from $a \to R$ and multiply by $\frac{1}{R^\alpha}$ on both side

$$\frac{1}{R^\alpha}\int_a^R \left|\hat{P} - \int_{\Omega(1,r)} r^{\alpha-n+\lambda} p(x)dx\right|^2 d_\alpha r = \frac{4}{R^\alpha}\int_a^R \left|\int_{s(r)} r^{\alpha-n+\lambda}\langle W, e_i\rangle dS\right|^2 d_\alpha r \quad (2.9)$$

$$+\frac{4(\alpha - n + \lambda)^2}{R^\alpha}\int_a^R \left|\int_{\Omega(r,\infty)} r^{-n+\lambda}\langle W, e_i\rangle dx\right|^2 d_\alpha r \quad (2.10)$$

$$+\frac{4}{R^\alpha}\int_a^R \left|\int_{\Omega(r,\infty)} r^{\alpha-n+\lambda} \parallel W \parallel^2 dx\right|^2 d_\alpha r = 0. \quad (2.11)$$

The terms (2.10) and (2.11) tends to zero for $r \to \infty$, according to the L'Hospital rule, (2.2) and (2.6). If follows from the Cauchy inequality

$$\begin{aligned}&\frac{1}{R^\alpha}\int_a^R \left|\int_{S(r)} r^{\alpha-n+\lambda}\langle W, e_i\rangle dS\right|^2 d_\alpha r \\ &\leq \frac{1}{R^\alpha}\int_a^R \left(\int_{S(r)} r^{\alpha-n+\lambda} \parallel W \parallel^2 dS\right)\left(\omega_n \int_{S(r)} r^{\alpha+\lambda-1} dS\right) d_\alpha r.\end{aligned}$$

and the term (2.9) tends to zero by using (2.2). Hence

$$\frac{1}{R^\alpha}\int_a^R \left|\hat{P} - \int_{\Omega(1,r)} r^{\alpha-n+\lambda} p(x)dx\right|^2 d_\alpha r \to 0 \text{ for } R \to \infty \tag{2.12}$$

Therefore

$\left|\frac{1}{R^{\alpha}}\int_a^R\left(\hat{P}-\int_{\Omega(1,r)} r^{\alpha-n+\lambda}p(x)dx\right)d_{\alpha}r\right| \le \left(\frac{1}{R^{\alpha}}\int_a^R\left|\hat{P}-\int_{\Omega(1,r)} r^{\alpha-n+\lambda}p(x)dx\right|^2 d_{\alpha}r\right)^{\frac{1}{2}}$ and from (2.12) it follows that (2.3),

$$\hat{P}=\alpha P_0 \text{ holds.}$$

ii) ⇒ iii) is trivial. iii) ⇒ i) If the (2.4) holds and equation (2.2) does not hold.

Denoting

$$\chi(r):=\int_a^r\int_{\Omega(a,r)} r^{\alpha-n+\lambda}\parallel W\parallel^2\,dx d_{\alpha}r$$

this function satisfies

$\lim_{r\to\infty}\frac{\chi(r)}{r}\to\infty$ for $r\to\infty$ (2.13)

$$\int_{\Omega(a,\infty)} r^{\alpha-n+\lambda}\parallel W\parallel^2\,dx=\infty.$$

From (2.5) we integrate from $a\to$ R and multiply by $\frac{1}{R^{\alpha}}$ we get

$$\begin{aligned}&|\frac{1}{R^{\alpha}}\chi(R)-\frac{1}{R^{\alpha}}\int_a^R\int_{\Omega(a,r)}(\alpha-n+\lambda)r^{-n+\lambda}\langle W,e_i\rangle dx d_{\alpha}r\\&+\frac{1}{R^{\alpha}}\int_a^R\int_{s(r)} r^{\alpha-n+\lambda}\langle W,e_i\rangle dS d_{\alpha}r|=|-P(R)+\frac{1}{R^{\alpha}}\int_a^R\int_{s(a)} r^{\alpha-n+\lambda}\langle W,e_i\rangle dS d_{\alpha}r\,|\end{aligned}$$

If iii) holds, $\lim_{r\to\infty}\inf P(r)>-\infty$ hold.

Less than $\frac{1}{4R^{\alpha}}\chi(R)$ and the right-hand side is bounded from above if (2.4) is valid. Hence

$$\begin{aligned}\frac{3\chi(R)}{4R^{\alpha}}\le\left|\frac{1}{R^{\alpha}}\int_a^R\int_{\Omega(a,r)}(\alpha-n+\lambda)r^{-n+\lambda}\langle W,e_i\rangle dx d_{\alpha}r\right|+\\ \left|\frac{1}{R^{\alpha}}\int_a^R\int_{s(r)} r^{\alpha-n+\lambda}\langle W,e_i\rangle dS d_{\alpha}r\right|. \qquad (2.14)\end{aligned}$$

By the Cauchy inequality,

$$\frac{1}{R^{\alpha}}\int_a^R\int_{s(r)} r^{\alpha-n+\lambda}\langle W,e_i\rangle dS d_{\alpha}r\le(T_{\alpha}\chi(R))^{\frac{1}{2}}\left(\frac{\omega_n R^{\lambda+2\alpha}}{(\lambda+\alpha)(\lambda+2\alpha)}\right)^{\frac{1}{2}}$$

$$\frac{1}{R^{\alpha}}\int_a^R\int_{\Omega(a,r)}(\alpha-n+\lambda)r^{-n+\lambda}\langle W,e_i\rangle dx d_{\alpha}r$$

$$\le(\alpha-n+\lambda)(\chi(R))^{\frac{1}{2}}\left(\frac{\omega_n R^{\lambda}}{\lambda(\lambda-\alpha)}\right)^{\frac{1}{2}}.$$

From (2.13) it follows that

$$\frac{\chi(R)}{4^2(\alpha-n+\lambda)^2R^\lambda} \leq \frac{\omega_n}{\lambda(\lambda-\alpha)} \qquad (2.15)$$

for R large enough, therefore

$$\frac{1}{R^\alpha}\int_a^R \int_{\Omega(a,r)} (\alpha - n + \lambda)r^{-n+\lambda}\langle W, e_i\rangle dxd_\alpha r = \frac{\chi(R)}{4} \qquad (2.16)$$

Combining the above computations above for R large enough, we obtain

$$\frac{\chi^2(R)}{4} \leq \frac{\omega_n}{(\lambda+\alpha)(\lambda+2\alpha)}T_\alpha\chi(R)R^{\lambda+2\alpha}$$

and from here we get

$$4\omega_n\frac{T_\alpha\chi(R)}{\chi^2(R)} \geq \frac{(\lambda+\alpha)(\lambda+2\alpha)}{R^{\lambda+2\alpha}} \text{ for } R \text{ large enough.}$$

Integration from $r_1 \to \infty$ gives a divergent integral on the right-hand side and convergent integral on the left-hand side. This contradicts our proof.

Introducing this function $\rho(r)$ is defined

$$\rho(r) = \int_{S(r)} r^{\alpha-n+\lambda}\langle W, e_i\rangle dS \qquad (2.17)$$

Lemma: 2.3. Let (2.3) holds. Let the equation (1.1) have a non-oscillatory solution. Then,

$$M(r) - \left(\frac{(\alpha-n+\lambda)}{\alpha}+1\right)g + \frac{r^{1-(\alpha+\lambda)}g^2}{\omega_n(\lambda+\alpha)} \leq 0$$

and

$$r^{-1}\big(rN(r) - \tau_\epsilon N(\tau_\epsilon) - \tau_\epsilon^2\rho(\tau_\epsilon)\big) - G\left(\frac{(\alpha-n+\lambda)r^{(1-\alpha)}}{2-\alpha} + \frac{(2r^{\alpha-1})}{\alpha} - 1\right) + \frac{r^{1-(\alpha+\lambda)}G^2}{\omega_n(2-\alpha-\lambda)} \leq 0.$$

are solvable.

Proof. Let $\overrightarrow{W}$ represent the solution of (2.1), which is based on W_a for each $a \in \mathbb{R}$.

Cauchy inequality gives us

$$\rho^2(r) = \omega_n r^{\lambda+\alpha-1}\int_{S(r)} r^{\alpha-n+\lambda} \parallel W \parallel^2 dS \qquad (2.18)$$

Introducing the Notation

$$g = \text{liminf} r\rho(r) G = \text{limsup} r\rho(r)$$

Obviously, to any $0 < \epsilon < min\{g, 1-G\}$ there exists $\tau_\epsilon > r_0$ and $r_\epsilon > \tau_\epsilon$ such that

$$g - \epsilon < r\rho(r) < G + \epsilon \qquad (2.19)$$

From (2.7), (2.8) we easily find that

$$r\rho(r) = M(r) - (\alpha - n + \lambda) r \int_r^\infty \rho(s) s^{-\alpha} ds + \frac{r}{\omega_n} \int_r^\infty \rho^2(s) s^{1-(\lambda+\alpha)} ds \quad (2.20)$$

Taking the new account and the above argument, we get

$$g - \epsilon \geq M(r) - \frac{(\alpha - n + \lambda)}{\alpha}(g - \epsilon) + \frac{r^{1-(\alpha+\lambda)}(g-\epsilon)^2}{\omega_n(\lambda+\alpha)}$$

Taking differentiation, Multiply by r^2 and integrating from $\tau \rightarrow$ R we get

$$R\rho(R) = R^{-1}(\tau^2\rho(\tau) - RN(R) + \tau^2 N(\tau)) + R^{-1}\int_\tau^R 2s^\alpha \rho(s) ds +$$
$$R^{-1}\int_\tau^R (\alpha - n + \lambda) s^{2-\alpha}\rho(s) ds - R^{-1}\int_\tau^R \frac{s^{(3-\lambda-\alpha)}\rho^2(s)}{\omega_n} ds$$

here substituting $R = r$ and $\tau = \tau_\epsilon$

$$r\rho(r) = r^{-1}\left(\tau_\epsilon^2 \rho(\tau_\epsilon) - rN(r) + \tau_\epsilon^2 N(\tau_\epsilon)\right) + r^{-1}\int_\tau^r 2s^\alpha \rho(s) ds +$$
$$r^{-1}\int_\tau^r (\alpha - n + \lambda) s^{2-\alpha}\rho(s) ds - r^{-1}\int_\tau^r \frac{s^{(3-\lambda-\alpha)}\rho^2(s)}{\omega_n} ds \quad (2.21)$$
$$G + \epsilon \leq r^{-1}\left(\tau_\epsilon^2 \rho(\tau_\epsilon) - rN(r) + \tau_\epsilon N(\tau_\epsilon)\right) +$$
$$(G+\epsilon)\left(\frac{2r^{(\alpha-1)}}{\alpha} + \frac{(\alpha - n + \lambda) r^{1-\alpha}}{2-\alpha} - \frac{(G+\epsilon) r t^{1-(\lambda+\alpha)}}{\omega_n(2-\alpha-\lambda)}\right)$$

Hence

$$M(r) - \left(\frac{(\alpha - n + \lambda)}{\alpha} + 1\right) g + \frac{r^{1-(\alpha+\lambda)} g^2}{\omega_n(\lambda+\alpha)} \leq 0$$

and

$$r^{-1}\left(rN(r) - \tau_\epsilon N(\tau_\epsilon) - \tau_\epsilon^2 \rho(\tau_\epsilon)\right) - G\left(\frac{(\alpha - n + \lambda) r^{(1-\alpha)}}{2-\alpha} + \frac{(2r^{\alpha-1})}{\alpha} - 1\right)$$
$$+ \frac{r^{1-(\alpha+\lambda)} G^2}{\omega_n(2-\alpha-\lambda)} \leq 0$$

Hence it is proved.

The Hartman Winter theorem and newly discovered oscillation requirements for conformable elliptic equations are covered in the following session.

3. MAIN RESULTS

In this section, the following results has been established. The below theorem is an oscillation criterion of the Hartman Winter type.

Theorem 3.1. If $-\infty < \liminf_{r\to\infty} P(r) < \limsup_{r\to\infty} P(r) \leq \infty$ (3.1) or if $\lim_{r\to\infty} P(r) = \infty$ (3.2)

Then (1.1) is oscillatory

Proof. Assume 3.1 is true by contradiction and $\exists r$ there is a non negative solution of 1.1 on Ω_r exists. It follows that the Riccati equations corresponding solution is defined on R. Lemma 2.1' s (iii) ⇒ (ii) portion and the first inequality in 3.1 indicates the existence of a finite limit $\lim_{r\to\infty} P(r)$ which is in opposition to 3.1. The same proof applies to 3.2

Theorem 3.2. Let equation (1.1) have oscillatory solution u. Then

$$M(r) > B + \epsilon + \frac{(\alpha - n + \lambda)^2 \omega_n}{4(\alpha - \lambda)} r^{\lambda - \alpha + 1}$$

and

$$N(r) > r^{-1}\left(r_\epsilon^2 \rho(r_\epsilon) + r_\epsilon^2 N(r_\epsilon)\right) - \frac{\omega_n r^{-1}}{4}\left(\frac{(\alpha - n + \lambda)^2 r^{\lambda + 2 - \alpha}}{\lambda - \alpha + 2} + \frac{4r^{\lambda + 3\alpha - 2}}{\lambda + 3\alpha - 2} + \frac{4(\alpha - n + \lambda) r^{\alpha + \lambda}}{\alpha + \lambda}\right) - A + \epsilon$$

are oscillatory.

Moreover,

$$\lim\inf r\rho(r) \geq A, \square \mathrm{liminf} r\rho(r) \leq B$$

where A is the least non-negative root of equation and B is the largest root of equation.

Proof. Assume the contradiction. Let equation (1.1) have the non-oscillatory solution. From lemma 2.1, (2.5) and (2.7) there exists $r_\epsilon > r_0$ such that

$$A - \epsilon < r\rho(r) < B + \epsilon \text{ for } r > r_\epsilon$$

Integrating from $r \to \infty$ and taking into account of (2.5) and (2.7) we get that

$$r\rho(r) = M(r) - (\alpha - n + \lambda) r \int_r^\infty \rho(s) s^{-\alpha} ds + \frac{r}{\omega_n}\int_r^\infty \rho^2(s) s^{1-(\lambda+\alpha)} ds$$

$$M(r) = r\rho(r) - r\left(\frac{r}{\omega_n}\int_r^\infty \rho^2(s) s^{1-(\lambda+\alpha)} ds - (\alpha - n + \lambda)\int_r^\infty \rho(s) s^{-\alpha} ds\right)$$

$$M(r) < B + \epsilon + \frac{(\alpha - n + \lambda)^2 \omega_n}{4(\alpha - \lambda)} r^{\lambda - \alpha + 1} \qquad (3.3)$$

Similarly, from (2.5)

$$rN(r) = \tau_\epsilon^2 \rho(\tau_\epsilon) + \tau_\epsilon N(\tau_\epsilon) - \int_r^{\tau_\epsilon}\left(\frac{s^{3-\alpha-\lambda}\rho^2(s)}{\omega_n} - \rho(s)((\alpha - n + \lambda)s^{2-\alpha} + 2s^\alpha)\right) ds - r^2 \rho(r)$$

$$N(r) < r^{-1}\left(r_\epsilon^2 \rho(r_\epsilon) + r_\epsilon^2 N(r_\epsilon)\right) - \frac{\omega_n r^{-1}}{4}\left(\frac{(\alpha - n + \lambda)^2 r^{\lambda + 2 - \alpha}}{\lambda - \alpha + 2} + \frac{4r^{\lambda + 3\alpha - 2}}{\lambda + 3\alpha - 2} + \frac{4(\alpha - n + \lambda) r^{\alpha + \lambda}}{\alpha + \lambda}\right) - A + \epsilon$$

which contradicts the theorem and it is proved.

Theorem 3.3. Let equation (1.1) have oscillatory solution u. Then

$$M(r) > B + \epsilon + (A + \epsilon)\left(\frac{(\alpha - n + \lambda)r^{(1-\alpha)}}{\alpha} - \frac{(A + \epsilon)r^{1-(\lambda+\alpha)}}{\omega_n(\lambda + \alpha)}\right)$$

and

$N(r) > \epsilon - A + r^{-1}\left(r_\epsilon^2\rho(r_\epsilon) + r_\epsilon N(r_\epsilon)\right) + (B + \epsilon)r^{-1}\left((\alpha - n + \lambda)\frac{r^{2-\alpha}}{(2-\alpha)} + \frac{2r^\alpha}{\alpha} - \frac{(B+\epsilon)r^{2-\alpha-\lambda}}{\omega_n(2-\alpha-\lambda)}\right)$ are oscillatory.

Proof. Assume the contradiction. Let equation (1.1) have the non-oscillatory solution. By lemma 2.1, (2.5) and (2.7)

$$M(r) = r\rho(r)) + (\alpha - n + \lambda)r\int_r^\infty \rho(s)s^{-\alpha}ds - \frac{r}{\omega_n}\int_r^\infty \rho(s)^2 s^{1-(\lambda+\alpha)}ds$$

$$M(r) < B + \epsilon + (A + \epsilon)\left(\frac{(\alpha - n + \lambda)r^{(1-\alpha)}}{\alpha} - \frac{(A + \epsilon)r^{1-(\lambda+\alpha)}}{\omega_n(\lambda + \alpha)}\right) \quad (3.4)$$

Similarly, we obtain from (2.5)

$$-N(r) = t\rho(r) - r^{-1}(\tau^2\rho(\tau) + \tau N(\tau)) +$$
$$r^{-1}\int_\tau^r \left(\frac{s^{(3-\lambda-\alpha)}\rho(s)^2}{\omega_n} - (\alpha - n + \lambda)s^{2-\alpha}\rho(s) - 2s^\alpha\rho(s)\right)ds$$
$$N(r) < \epsilon - A + r^{-1}\left(t_\epsilon^2\rho(r_\epsilon) + r_\epsilon N(r_\epsilon)\right)$$
$$\square + (B + \epsilon)r^{-1}\left((\alpha - n + \lambda)\frac{r^{2-\alpha}}{(2-\alpha)} + \frac{2r^\alpha}{\alpha} - \frac{(B + \epsilon)r^{2-\alpha-\lambda}}{\omega_n(2 - \alpha - \lambda)}\right) \quad (3.5)$$

which contradicts the theorem and it is proved.

4. CONCLUSION

Using the traditional Riccati Substitution, this work presents some oscillation results for the class of conformable Schrodinger equations. The result demonstrates that Hartman-Winter criteria may be effectively applied in the theorem to derive oscillation criterion. Our newly obtained results in this study have improved, extending and adopting a broad perspective of certain known results that are already there in the literature.

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