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Association Schemes for Some Finite Group Rings II

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Abstract:

Association schemes have been used in coding theory and other combinatorial problems. In this paper, we construct association schemes for the abelian groups $\mathbb{Z}_2^r$, $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$, set of $n \times n$ matrices over $\mathbb{Z}_m$ and for the general linear group of order 2 over $\mathbb{Z}_2, \mathbb{Z}_4$, and $\mathbb{Z}_6$. We also obtain association schemes for symmetric groups and alternating groups of degree 4 and 5 using canonical forms.

Keywords: Group ring; Association scheme.

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1. Introduction

Association schemes (AS) introduced by Bose and Shimamoto [1], play a key role in the study of algebraic combinatorics. It has applications in graph theory, coding theory, group theory and design theory [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. Jørgensen's list of non-symmetric association schemes with classes smaller than 96 in vertices in [12], inspires us to research non-symmetric association schemes for various finite groups and group rings. In our previous work [13], we have constructed non symmetric commutative AS for symmetric groups, dihedral groups, abelian groups $\mathbb{Z}_p^r$ (where p is an odd prime), $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_r}$ ($p_i' s$ are distinct primes), finite group rings over $\mathbb{Z}_n$ and circulant matrices over $\mathbb{Z}_p$, for p prime. In the present study, we construct association schemes for the abelian groups $\mathbb{Z}_2^r = \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{r\ times}$ and $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$, the general linear group $GL(2, \mathbb{Z}_2)$, $GL(2, \mathbb{Z}_4)$, $GL(2, \mathbb{Z}_6)$, symmetric group and alternating group of order 4 and 5.

In this paper, let $\mathcal{P}$ denote the partition of $Y \times Y$, where Y is a finite set, and let $\mathbb{Z}_p^r = \underbrace{\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{r\ times}$. Some basic literature and preliminaries on association schemes are given below.

1.1. Association Scheme

Definition 1. *Let $\mathcal{P}$ be a partition of $Y \times Y$ where Y is a finite set and let $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_n$ binary relations on $\mathcal{P}$. Then $\mathcal{A} = (Y, \mathcal{P})$ forms n-class association scheme if the subsequent conditions hold:*

(1) Identity relation $\mathcal{S}_0 = \{(a, a) : a \in Y\} \in \mathcal{P}$.

(2) $\mathcal{S}^* = \{(a,b): (b,a) \in \mathcal{S}\} \in \mathcal{P}$ *for any relation* $\mathcal{S} \in \mathcal{P}$.

(3) If $(a,b) \in \mathcal{S}_k$, *the number of elements* $c \in Y$ *such that* $(a,c) \in \mathcal{S}_l$, $(c,b) \in \mathcal{S}_m$ *is a constant* p_{lm}^k *not depending on choice of* a *and* b *for all integers* $0 \le k, l, m \le n$.

The integers $\{p_{lm}^k\}_{0\le k,l,m\le n}$ are called *parameters* or *intersection numbers* of $\mathcal{A}$. If each relation $\mathcal{S}$ in $\mathcal{P}$ is a symmetric relation, that is, $\mathcal{S} = \mathcal{S}^*$, then $\mathcal{A}$ is called symmetric association scheme and if $p_{lm}^k = p_{ml}^k \; \forall \; 0 \le k, l, m \le n$, then it is called commutative association scheme. Let the set $a\mathcal{S} = \{b \in Y | \; (a,b) \in \mathcal{S} \}$ for $a \in Y$ and $\mathcal{S} \in \mathcal{P}$. The elements a and b in Y are called k^{th} *associates* if $(a,b) \in \mathcal{S}_k$ with $a \ne b$. Note that every symmetric association scheme is commutative. With regards to more basic association schemes results, refer [7, 14].

Definition 2. *A finite group* G *with the conjugacy classes* $C_0, C_1, \dots, C_d$ *produces a commutative association scheme with a class of relations on* G *defined by* $\mathcal{S}_k = \{(a,b) | \; ba^{-1} \in C_k\} \; \forall \; 0 \le k \le d$. *This scheme is called the group association scheme of* G.

Association schemes can be determined for all those groups whose conjugacy classes are known.

Lemma 1. *Let* $Y = \mathbb{Z}_n$ *and* $\mathcal{S}_k$ *defines relations on* $\mathcal{P}$ *by* $\mathcal{S}_k = (a,b) | a = k + b | a, b \in \mathbb{Z}_n \forall k \in \mathbb{Z}_n$. *Then* $(Y, \mathcal{P})$ *is a non symmetric commutative association scheme with parameters given by:*

$$p_{lm}^k = \begin{cases} 1 & if k = l + m, \\ 0 & if k \ne l + m \end{cases}$$

where $k, l, m \in Z_n$.

Proof. Since $\mathbb{Z}_n$ is an abelian group, $(\mathbb{Z}_n, \mathcal{P})$ under given relations becomes a commutative association scheme. For arbitrary relations $\mathcal{S}_l, \mathcal{S}_m, \mathcal{S}_k$ in $\mathcal{P}$, we find cardinality $p_{lm}^k = |a\mathcal{S}_l \cap b\mathcal{S}_m^*|$ whenever $(a,b) \in \mathcal{S}_k$. Let (a,b) be an arbitrary element of Y in $\mathcal{S}_k$ and let $a\mathcal{S}_l = a'$ and $b\mathcal{S}_m^* = b'$. This implies, $a = a' + l, b' = b + m$ and $a = b + k$ and we get $p_{lm}^k = 1$ if $a' = b'$ which implies $p_{lm}^k = 1$ if $k = l + m$. Further, $p_{lm}^k = 0$ if $a' \ne b'$, equivalently $p_{lm}^k = 0$, whenever $k \ne l + m$. □

In the next section of this paper, we work on non symmetric association scheme of the cyclic groups $\mathbb{Z}_2^r$, $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \dots \times \mathbb{Z}_{n_r}$ and general linear group of order 2 over $\mathbb{Z}_2$.

2. Association schemes for some finite Groups

Theorem 1. *Let* $Y = \mathbb{Z}_2^r = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$ *and* $r \ge 2$. *We define relations* $\mathcal{S}_k$ *on* $\mathcal{P}$ *by*

$$\mathcal{S}_k = \{(a,b) | \; b_s \equiv (t_s + a_s) \; mod \; 2 \; | \; t_s \in \mathbb{Z}_2 \; \forall \; 1 \le s \le r, \\ a = (a_1, a_2, \dots, a_r), b = (b_1, b_2, \dots, b_r) \in Y\}$$

where $k = 2^{r-1}t_1 + 2^{r-2}t_2 + \dots + 2t_{r-1} + t_r$.

Then $\mathcal{A} = (Y, \mathcal{P})$ *is a symmetric and commutative association scheme with parameters*

$$p_{lm}^k = \begin{cases} 1 & \text{if } t_s^{(k)} \equiv t_s^{(l)} + t_s^{(m)} mod \; 2 \; \forall \; 1 \le s \le r \\ 0 & \text{otherwise} \end{cases}$$

where $k = \sum_{s=1}^{r} 2^{r-s} t_s^{(k)}$; $l = \sum_{s=1}^{r} 2^{r-s} t_s^{(l)}$; $m = \sum_{s=1}^{r} 2^{r-s} t_s^{(m)}$ *for some* $t_s^{(k)}, t_s^{(l)}, t_s^{(m)} \in \mathbb{Z}_2$.

Proof. Observe that $|Y| = 2^r = |\mathcal{S}_k|$ for all $0 \le k \le 2^r - 1$. The relations $\mathcal{S}_k$ are disjoint and $\bigcup \mathcal{S}_k : 0 \le k \le 2^r - 1 = \mathcal{P}$. For arbitrary relations $\mathcal{S}_l, \mathcal{S}_m, \mathcal{S}_k$ in $\mathcal{P}$, we prove that the parameters $p_{lm}^k = |a\mathcal{S}_l \cap b\mathcal{S}_m^*|$ is constant for $(a,b) \in \mathcal{S}_k$. Let (a,b) be an arbitrary element of Y in $\mathcal{S}_k$ where $a = (a_1, a_2, \dots, a_r), b = (b_1, b_2, \dots, b_r)$. Let $a\mathcal{S}_l = a' = (a'_1, a'_2, \dots, a'_r)$ and $b\mathcal{S}_m^* = b' = (b'_1, b'_2, \dots, b'_r)$. Now $(a,b) \in \mathcal{S}_k$ implies $b_s \equiv \left(t_s^{(k)} + a_s\right) mod\ 2$ where $k = \sum_{s=1}^r 2^{r-s} t_s^{(k)}$ for some $t_s^{(k)} \in \mathbb{Z}_2\ \forall\ 1 \le s \le r$. Similarly, $(a,a') \in \mathcal{S}_l$ and $(b',b) \in \mathcal{S}_m$, implies $a'_s \equiv \left(t_s^{(l)} + a_s\right) mod\ 2$ where $l = \sum_{s=1}^r 2^{r-s} t_s^{(l)}$ for some $t_s^{(l)} \in \mathbb{Z}_2\ \forall\ 1 \le s \le r$, and $b_s \equiv \left(t_s^{(l)} + b'_s\right) mod\ 2$ where $m = \sum_{s=1}^r 2^{r-s} t_s^{(m)}$ for some $t_s^{(m)} \in \mathbb{Z}_2\ \forall\ 1 \le s \le r$. We observe that $p_{lm}^k = 1$ if and only if $a' = b'$ that is, if $t_s^{(k)} \equiv \left(t_s^{(l)} + t_s^{(m)}\right) mod\ 2$ for all $1 \le s \le r$.

To show $(Y, \mathcal{P})$ is a symmetric association scheme, we prove that $\mathcal{S}_k^* = \mathcal{S}_k$ for all $0 \le k < 2^r$. Let $\mathcal{S}_k^* = \mathcal{S}_K$ where $K = \sum_{s=1}^r 2^{r-s} T_s$ and $k = \sum_{s=1}^r 2^{r-s} t_s$ for some $t_s, T_s \in \mathbb{Z}_2$. If $(b,a) \in \mathcal{S}_K$, then further $a_s \equiv T_s + b_s\ mod\ 2$ and $b_s \equiv t_s + a_s\ mod\ 2$ which implies that $T_s = t_s\ \forall\ 1 \le s \le r$. Thus $K = k$, and hence, $\mathcal{S}_k^* = \mathcal{S}_k$. □

Theorem 2. *Let* $Y = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$, *where* $n_1, n_2, \dots, n_r$ *are pairwise co-prime. The relations* $\mathcal{S}_k$ *on* $\mathcal{P}$ *defined by*

$$\mathcal{S}_k = \{(a,b) |\ b_s \equiv (k + a_s)\ mod\ n_s\ \forall\ 1 \le s \le r |\ a = (a_1, a_2, \dots, a_r), b = (b_1, b_2, \dots, b_r) \in Y\}$$

where $0 \le k < n_1 n_2 \cdots n_r$,

is a non-symmetric and commutative association scheme with parameters

$$p_{lm}^k = \begin{cases} 1 & \text{if } k \equiv (l+m)\ mod\ n_s\ \forall\ 1 \le s \le r \\ 0 & \text{otherwise} \end{cases}$$

where $0 \le k, l, m \le n_1 n_2 \cdots n_r - 1$.

Proof. Observe that $|Y| = n_1 n_2 \cdots n_r = |\mathcal{S}_k|$ for all $0 \le k < n_1 n_2 \cdots n_r$. The relations $\mathcal{S}_k$ being disjoint, form partition of $\mathcal{P}$. For arbitrary relations $\mathcal{S}_l, \mathcal{S}_m, \mathcal{S}_k$ in $\mathcal{P}$, we prove that for each pair a, b with $(a,b) \in \mathcal{S}_k$, the number of elements in the set $\{c \in Y |\ (a,c) \in \mathcal{S}_l, (c,b) \in \mathcal{S}_m\}$ is invariant. Let $(a,b) \in \mathcal{S}_k$ where $a = (a_1, a_2, \dots, a_r), b = (b_1, b_2, \dots, b_r) \in Y$. Further, suppose that $a\mathcal{S}_l = a' = (a'_1, a'_2, \dots, a'_r), b\mathcal{S}_m^* = b' = (b'_1, b'_2, \dots, b'_r)$. Now $(a,b) \in \mathcal{S}_k$ implies $b_s \equiv (k + a_s)\ mod\ n_s\ \forall\ 1 \le s \le r$. Similarly, $(a,a') \in \mathcal{S}_l$ and $(b',b) \in \mathcal{S}_m$. We get $a'_s \equiv (l + a_s)\ mod\ n_s$ and $b_s \equiv (m + b'_s)\ mod\ n_s\ \forall\ 1 \le s \le r$.

The above equations give that, $p_{lm}^k = 1$ if and only if $a' = b'$. That is, $p_{lm}^k = 1$ whenever $k \equiv (l+m)\ mod\ n_s\ \forall\ 1 \le s \le r$. Hence, $(Y, \mathcal{P})$ is a non-symmetric and commutative association scheme. □

Theorem 3. *Let* $Y = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$, *where* $2 < n_1 \le n_2 \le \cdots \le n_r$ *and* $gcd(n_1, n_2, \dots, n_r) \ne 1$. *The relations* $\mathcal{S}_k$ *on* $\mathcal{P}$ *defined by*

$$\mathcal{S}_k = \{(a,b) | b_r \equiv (k + a_r) mod n_r, b_s \equiv (k + t_s + a_s) \ mod \ n_s \ \forall \ 1 \le s < r | \\ t_s \in Z_{n_s}, a = (a_1, a_2, \dots, a_r), b = (b_1, b_2, \dots, b_r) \in Y\}$$

where $k = n_2 n_3 \cdots n_r t_1 + n_3 n_4 \cdots n_r t_2 + \cdots + n_r t_{r-1} + t_r$,

is a non-symmetric and commutative association scheme with parameters

$$p_{lm}^k = \begin{cases} 1 & if k \equiv (l+m) \ mod \ n_r, \text{and} k + \ t_s^{(k)} \equiv \left(l + m + t_s^{(l)} + t_s^{(m)}\right) mod \ n_s \ \forall \ 1 \le s < r \\ 0 & \text{otherwise} \end{cases}$$

where $k = n_2 n_3 \cdots n_r t_1^{(k)} + n_3 n_4 \cdots n_r t_2^{(k)} + \cdots + n_r d_{r-1}^{(k)} + t_r^{(k)}$; $l = n_2 n_3 \cdots n_r t_1^{(l)} + n_3 n_4 \cdots n_r t_2^{(l)} + \cdots + n_r t_{r-1}^{(l)} + t_r^{(l)}$; $m = n_2 n_3 \cdots n_r t_1^{(m)} + n_3 n_4 \cdots n_r t_2^{(m)} + \cdots + n_r t_{r-1}^{(m)} + t_r^{(m)}$ *for some* $t_s^{(k)}, t_s^{(l)}, t_s^{(m)} \in \mathbb{Z}_{n_s}$ *where* $1 \le s \le r$.

For each $0 \le k < n_1 n_2 \dots n_r$, $\mathcal{S}_k^* = \mathcal{S}_K$ *can be calculated by solving the following equations:*

$$\begin{aligned} K + k &\equiv 0 \ mod \ n_r \\ K + k + T_s + t_s &\equiv 0 \ mod \ n_s \ \forall \ 1 \le s \le r-1 \end{aligned}$$

where $k = n_2 n_3 \cdots n_r t_1 + n_3 n_4 \cdots n_r t_2 + \cdots + n_r t_{r-1} + t_r$ *and* $K = n_2 n_3 \cdots n_r T_1 + n_3 n_4 \cdots n_r T_2 + \cdots + n_r T_{r-1} + T_r$ *for some* $t_s, T_s \in \mathbb{Z}_{n_s}$.

Proof. Again recall, $|Y| = n_1 n_2 \cdots n_r = |\mathcal{S}_k|$ for all $0 \le k < n_1 n_2 \cdots n_r$. Above defined relations $\mathcal{S}_k$ being disjoint, form a partition of $\mathcal{P}$. For arbitrary relations $\mathcal{S}_l, \mathcal{S}_m, \mathcal{S}_k$ in $\mathcal{P}$, we show that for each $(a,b) \in \mathcal{S}_k$, the number of elements in the set $\{c \in Y | \ (a,c) \in \mathcal{S}_l, (c,b) \in \mathcal{S}_m\}$ is invariant. Let $(a,b) \in \mathcal{S}_k$ where $a = (a_1, a_2, \dots, a_r), b = (b_1, b_2, \dots, b_r) \in Y$. Suppose $a\mathcal{S}_l = a' = (a'_1, a'_2, \dots, a'_r), b\mathcal{S}_m^* = b' = (b'_1, b'_2, \dots, b'_r)$. Then $(a,b) \in \mathcal{S}_k$ implies $b_r \equiv (k + a_r) \ mod \ n_r, b_s \equiv \left(k + t_s^{(k)} + a_s\right) mod \ n_s \ \forall \ 1 \le s \le r-1$ where $k = n_2 n_3 \cdots n_r t_1^{(k)} + n_3 n_4 \cdots n_r t_2^{(k)} + \cdots + n_r t_{r-1}^{(k)} + t_r^{(k)}$ for some $t_s \in Z_{n_s} \ \forall \ 1 \le s \le r$.

Since $(a, a') \in \mathcal{S}_l$ and $(b', b) \in S_m$,

we have $a'_r \equiv (l + a_r) \ mod \ n_r$, $a'_s \equiv \left(l + t_s^{(l)} + a_s\right) mod \ n_s \ \forall \ 1 \le s \le r-1$ where $l = n_2 n_3 \cdots n_r t_1^{(l)} + n_3 n_4 \cdots n_r t_2^{(l)} + \cdots + n_r t_{r-1}^{(l)} + t_r^{(l)}$ for some $t_s^{(l)} \in \mathbb{Z}_{n_s} \forall 1 \le s \le r$, and

$b_r \equiv (m + b'_r) \ mod \ n_r, b_s \equiv \left(m + t_s^{(m)} + b'_s\right) mod \ n_s \ \forall \ 1 \le s \le r-1$ where $m = n_2 n_3 \cdots n_r t_1^{(m)} + n_3 n_4 \cdots n_r t_2^{(m)} + \cdots + n_r t_{r-1}^{(m)} + t_r^{(m)}$ for some $t_s^{(m)} \in \mathbb{Z}_{n_s} \ \forall \ 1 \le s \le r$.

We observe that $p_{lm}^k = 1$ if and only if $a' = b'$. That is, $p_{lm}^k = 1$ if $k \equiv (l+m) \ mod \ n_r$ and $k + t_s^{(k)} \equiv \left(l + m + t_s^{(l)} + t_s^{(m)}\right) mod \ n_s \ \forall \ 1 \le s \le r-1$ Hence, $(Y, \mathcal{P})$ is a non-symmetric and commutative association scheme.

Next, we find $\mathcal{S}_k^*$. Let $\mathcal{S}_k^* = \mathcal{S}_K$ where $K = n_2 n_3 \cdots n_r T_1 + n_3 n_4 \cdots n_r T_2 + \cdots + n_r T_{r-1} + T_r$ for some $T_s \in \mathbb{Z}_{n_s}$. If $(b,a) \in \mathcal{S}_K$, then $a_r \equiv K + b_r \ mod \ n_r$; $a_s \equiv K + T_s + b_s \ mod \ n_s \ \forall \ 1 \le s \le r-1$. Also, since $(a,b) \in \mathcal{S}_k$, we get $b_r \equiv k + a_r \ mod \ n_r$; $b_s \equiv k + t_s + a_s \ mod \ n_s \ \forall \ 1 \le s \le r-1$.

This implies that $K+k \equiv 0 \, mod \, n_r$ and $K+k+T_s+t_s \equiv 0 \, mod \, n_s \, \forall \, 1 \le s \le r-1$. Solving these equations, we get the value of K as claimed. □

Corollary 1. *Let* $Y = \mathcal{M}_n(\mathbb{Z}_m)$ *be the set of all* $n \times n$ *matrices over* $\mathbb{Z}_m$ *where* $m \ge 2$ *and* $n \ge 2$. *Any element of* $\mathcal{M}_n(\mathbb{Z}_m)$ *can be written as* n^2*-tuple in* $\mathbb{Z}_m^{n^2}$. *Therefore, relations defined on* $\mathbb{Z}_m^{n^2}$ *will form an association scheme over* $\mathcal{M}_n(\mathbb{Z}_m)$.

Proof. Follows from Theorem *1* when $m = 2$, and from Theorem *3* when $m > 2$. □

Presentations of some general linear groups $GL(2, \mathbb{Z}_n)$ are given in [15] for $n = 4, 6, 8, 10$. With the help of these presentations, we can compute the canonical form for these groups (provided in Table 1 Canonical forms of some general linear groups

The the canonical form of the presentation of $GL(2, \mathbb{Z}_2)$ is A^aB^b: $0 \le a \le 1, 0 \le b \le 2$. The group is isomorphic onto S_3, symmetric group on 3-symbols. Hence, another familiar presentation can be given as $GL(2, \mathbb{Z}_2) = \langle A, B | A^2 = B^3 = I, AB = B^{-1}A \rangle$. Using this presentation, we get a non symmetric association scheme.

Table 1 Canonical forms of some general linear groups

Group	Generators	Presentation	Canonical form
$GL(2,\mathbb{Z}_2)$	A,B	$\tau^2 = \sigma^3 = (\tau\sigma)^3 = 1$	A^aB^b: $0 \le a \le 1, 0 \le b \le 2$
$GL(2,\mathbb{Z}_4)$	A,B,C	$A^2 = B^2 = C^4 = X^3 = Y^4 = (CBA)^4 = I$, $C^2A = AC^2, C^2B = BC^2, BC = C^{-1}B$ where $X = AB$ and $Y = XCX$	$A^aC^bY^cX^d$: $0 \le a \le 1, 0 \le b, c \le 3$, $0 \le d \le 2$
$GL(2,\mathbb{Z}_6)$	A,B,C	$A^2 = B^2 = C^4 = X^{12} = Y^6 = (AB)^3 = (BC)^2 = I$, $C^A = AC^2, C^2B = BC^2$, $(AC)^{12} = C^2$, $(CA)^2B(CA)^2 = (AC)^2B(AC)^2$ where $X = B(CA)^2B$ and $Y = CAB$	$C^aX^bY^cB^d$: $0 \le a, d \le 1$, $0 \le b \le 11$, $0 \le c \le 5$

Theorem 4. *Let* $Y = GL(2, \mathbb{Z}_2)$. *Define relations* $\mathcal{S}_k$ *on* $\mathcal{P}$ *by*

$$\mathcal{S}_k = \{(A^aB^b, A^{k+a}B^{k+b}) |\ 0 \le a \le 1, 0 \le b \le 2\} \textit{ for all } 0 \le k \le 5.$$

Then $(Y, \mathcal{P})$ *is a non symmetric association scheme with parameters as follows:*

$$p_{lm}^k = \begin{cases} 1 & \text{if } k = l+m, \\ 0 & \text{if } k \ne l+m \end{cases}$$

where $0 \le k, l, m \le 5$.

Proof. Observe that $\mathcal{S}_0 = \{(M, M) |\ M \in GL(2, \mathbb{Z}_2)\}$ is an identity relation. For arbitrary relations $\mathcal{S}_l, \mathcal{S}_m, \mathcal{S}_k \in \mathcal{P}$, we find the cardinality p_{lm}^k such that $|M\mathcal{S}_l \cap N\mathcal{S}_m^*| = p_{lm}^k$ for all $(M, N) \in \mathcal{S}_k$. Let $(M, N) \in \mathcal{S}_k$ and let $M\mathcal{S}_l = M'$ and $N\mathcal{S}_m^* = N'$. That is, $M = A^aB^b, N = A^{k+a}B^{k+b}; M = A^{a_1}B^{b_1}, M' = A^{l+a_1}B^{l+b_1}$; $N' = A^{a_2}B^{b_2}$, $N = A^{m+a_2}B^{m+b_2}$ where $0 \le a, a_1, a_2 \le 1$ and $0 \le$

$b, b_1, b_2 \leq 2$. Since every pair (M, N) in Y are k^{th} associates for exactly one k, we find that p_{lm}^k can be either 0 or 1. Hence, with the help of the above equations, we obtain $p_{lm}^k = 1$ if $M' = N'$ equivalently, if $k = l + m$. Moreover, $p_{lm}^k = 0$ if $M' \neq N'$ that is, whenever $k \neq l + m$. Also it can be easily proved that the relations are not symmetric and hence $(Y, \mathcal{P})$ is a non symmetric association scheme. □

Theorem 5. *Let* $Y = GL(2, \mathbb{Z}_4)$ *and* $\mathcal{P}$ *be a partition of* $Y \times Y$. *For all* k *written in form of* $48s + 16n + 4t + r$ $(s \in \mathbb{Z}_2; n \in \mathbb{Z}_3; r, t \in \mathbb{Z}_4)$, *the relations* $\mathcal{S}_k$ *in* $\mathcal{P}$ *defined by*

$$\mathcal{S}_k = \{(A^a C^b Y^c X^d, A^{a+s} C^{b+t} Y^{c+r} X^{d+n}) \mid a \in \mathbb{Z}_2; b, c \in \mathbb{Z}_4; d \in \mathbb{Z}_3\}$$

is a non-symmetric association scheme with parameters

$$p_{lm}^k = \begin{cases} 1 & if\ s^{(k)} \equiv \left(s^{(l)} + s^{(m)}\right) mod\ 2, \\ & n^{(k)} \equiv \left(n^{(l)} + n^{(m)}\right) mod\ 2, \\ & t^{(k)} \equiv \left(t^{(l)} + t^{(m)}\right) mod\ 4, and \\ & r^{(k)} \equiv \left(r^{(l)} + r^{(m)}\right) mod\ 4 \\ 0 & otherwise \end{cases}$$

Proof. Observe that $|\mathcal{S}_k| = |Y|$ for all $0 \leq k < 96$. The relations $\mathcal{S}_k$ are disjoint and $\cup\, \mathcal{S}_k{:}0 \leq k < 96 = \mathcal{P}$.

$\mathcal{S}_0 = \{(M, M){:}\, M \in GL(2, \mathbb{Z}_4)\}$ is an identity relation. Let $\mathcal{S}_l, \mathcal{S}_m, \mathcal{S}_k$ be arbitrary relations in $\mathcal{P}$ and (M, N) be any element of $\mathcal{S}_k$. Since $M \in Y$, M is of the form $A^a C^b Y^c X^d$ for some a, b, c, d (from Table *[table:canonical forms An,Sn]*). Suppose $M\mathcal{S}_l = M'$ and $N\mathcal{S}_m^* = N'$ where $M', N' \in Y$.

Now $(M, N) \in \mathcal{S}_k$, $(M, M') \in \mathcal{S}_l$ and $(N', N) \in \mathcal{S}_m$, implies

$$N = A^{\left(a+s^{(k)}\right)_{mod\,2}} C^{\left(b+t^{(k)}\right)_{mod\,4}} Y^{\left(c+r^{(k)}\right)_{mod\,4}} X^{\left(d+n^{(k)}\right)_{mod\,3}},$$

$$M' = A^{\left(a+s^{(l)}\right)_{mod\,2}} C^{\left(b+t^{(l)}\right)_{mod\,4}} Y^{\left(c+r^{(l)}\right)_{mod\,4}} X^{\left(d+n^{(l)}\right)_{mod\,3}}, \text{ and}$$

$$N' = A^{\left(a+s^{(k)}-s^{(m)}\right)_{mod\,2}} C^{\left(b+t^{(k)}-t^{(m)}\right)_{mod\,4}} Y^{\left(c+r^{(k)}-r^{(m)}\right)_{mod\,4}} X^{\left(d+n^{(k)}-n^{(m)}\right)_{mod\,3}}$$

respectively, where $k = 48s^{(k)} + 16n^{(k)} + 4t^{(k)} + r^{(k)}$, $l = 48s^{(l)} + 16n^{(l)} + 4t^{(l)} + r^{(l)}$ and $m = 48s^{(m)} + 16n^{(m)} + 4t^{(m)} + r^{(m)}$, for some $s^{(k)}, s^{(l)}, s^{(m)} \in \mathbb{Z}_2; n^{(k)}, n^{(l)}, n^{(m)} \in \mathbb{Z}_3$ and $t^{(k)}, t^{(l)}, t^{(m)}, r^{(k)}, r^{(l)}, r^{(m)} \in \mathbb{Z}_4$.

Since every pair (M, N) in $\mathcal{P}$ are k^{th} associates for exactly one k, we find that p_{lm}^k can be either 0 or 1. Hence we obtain that $p_{lm}^k = 1$ if and only if $M' = N'$ that is, if $s^{(k)} \equiv \left(s^{(l)} + s^{(m)}\right) mod\ 2$, $t^{(k)} \equiv \left(t^{(l)} + t^{(m)}\right) mod\ 4$, $r^{(k)} \equiv \left(r^{(l)} + r^{(m)}\right) mod\ 4$ and $n^{(k)} \equiv \left(n^{(l)} + n^{(m)}\right) mod\ 3$. □

Theorem 6. *Let* $Y = GL(2, \mathbb{Z}_6)$ *and* $\mathcal{P}$ *be a partition of* $Y \times Y$. *For all* k *written in form of* $144s + 72n + 12t + r$ $(s, n \in \mathbb{Z}_2, t \in \mathbb{Z}_6, r \in \mathbb{Z}_{12})$, *the relations* $\mathcal{S}_k$ *in* $\mathcal{P}$ *defined by*

$$\mathcal{S}_k = \{(C^a X^b Y^c B^d, C^{a+s} X^{b+r} Y^{c+t} B^{d+n}) \mid a, d \in \mathbb{Z}_2; b \in \mathbb{Z}_{12}; c \in \mathbb{Z}_6\}$$

is a non-symmetric association scheme with parameters

$$p_{lm}^{k} = \begin{cases} 1 & if\ s^{(k)} \equiv \left(s^{(l)} + s^{(m)}\right) mod\ 2, \\ & n^{(k)} \equiv \left(n^{(l)} + n^{(m)}\right) mod\ 2, \\ & t^{(k)} \equiv \left(t^{(l)} + t^{(m)}\right) mod\ 6, and \\ & r^{(k)} \equiv \left(r^{(l)} + r^{(m)}\right) mod\ 12 \\ 0 & otherwise \end{cases}$$

Proof. Here, $|c_k| = |Y|$ for all $0 \leq k < 288$. The relations $\mathcal{S}_k$ are disjoint and $\cup\ \mathcal{S}_k{:}0 \leq k < 288 = \mathcal{P}$. Proceeding as in proof of theorem *5*, we can find cardinality p_{lm}^{k} such that for all $(x, y) \in \mathcal{S}_k$, $|x\mathcal{S}_l \cap y\mathcal{S}_m^*| = p_{lm}^{k}$ is a constant. □

Note: Let $SL(2, \mathbb{Z}_p)$ be the special linear group over $\mathbb{Z}_p$ and $GL(2, \mathbb{Z}_p)$ be general linear group over $\mathbb{Z}_p$. The structure of these conjugacy classes are worked out in detail in [16]. Using these classes one can compute group association scheme for $SL(2, \mathbb{Z}_p)$ and $GL(2, \mathbb{Z}_p)$ using Definition *2*.

Let us represent S_n as the symmetric group of degree n. Group association scheme for symmetric groups have been described by Tomiyama and Yamazaki in [17]. In [13], Sabharwal et al. identified the association schemes for the symmetric groups S_3 and S_4 without using the conjugacy classes. In next theorem, we have determined non symmetric commutative association scheme for the symmetric groups S_4 and S_5 and alternating groups A_3, A_4 and A_5 without using conjugacy classes.

Lemma 2. *Let $X = A_3 = \sigma^i{:}0 \leq i \leq 2$ where $\sigma^3 = 1$. Then the relations $\mathcal{S}_k$ on $\mathcal{P}$ defined by*

$$\mathcal{S}_k = \{(\sigma^i, \sigma^{k+i}) \mid\ 0 \leq i \leq 2\} \text{ for all } 0 \leq k \leq 2$$

is a non-symmetric commutative association scheme and intersection numbers of this association scheme are as follows:

$$p_{lm}^{k} = \begin{cases} 1 & \text{if } k = l + m, \\ 0 & \text{if } k \neq l + m \end{cases}$$

Proof. Since A_3 is isomorphic to $\mathbb{Z}_3$ by mapping $\sigma^i \mapsto i$, and the set of relations $\{(i, k+i) \mid i = 0,1,2\}$ forms AS for $\mathbb{Z}_3$, we can conclude the result. □

Presentations of alternating and symmetric groups of degree less than 8 are given in [18]. With the help of these presentations, we can compute the canonical form for these groups (provided in Table 2 Canonical forms of alternating and symmetric groups of degree 4 and 5). In [13], canonical form of S_4 is discussed.

Table 2 Canonical forms of alternating and symmetric groups of degree 4 and 5

Group	Generators	Presentation	Canonical form
A_4	τ, σ	$A^2 = B^3 = I, AB = B^{-1}A$	$\tau^a\sigma\tau^b\sigma^c$: $0 \leq a, b \leq 1$, $0 \leq c \leq 2$
S_4	τ, σ	$\tau^2 = \sigma^3 = (\tau\sigma)^3 = 1$	$A^2\tau^a\sigma^b\tau\sigma^c$: $0 \leq a \leq 1, 0 \leq b \leq 2$, $0 \leq c \leq 3$

A_5	τ,σ	$\tau^2=\sigma^3=\gamma^5=1$ where $\gamma=\tau\sigma$	$\tau^a\sigma\tau\sigma^2\gamma^b\tau^c\sigma\tau\sigma^d$: $0\le a,c\le 1$, $0\le b\le 4, 0\le d\le 2$
S_5	τ,σ	$\tau^5=\sigma^6=(\gamma)^2=(\delta)^2=1$ where $\gamma=\tau\sigma$ and $\delta=\tau^2\sigma^2$	$\tau^a\sigma\gamma^b\tau^4\sigma^5\delta^c\tau\sigma^d$: $0\le a\le 4, 0\le b,c\le 1$, $0\le d\le 5$

Theorem 7. *Let $Y=A_4$ and $\mathcal{P}$ be a partition of $Y\times Y$. For all k written in form of $3n+t$ $(n\in\mathbb{Z}_4, t\in\mathbb{Z}_3)$, the relations $\mathcal{S}_k$ on $\mathcal{P}$ defined by*

$$\mathcal{S}_k=\left\{\left(\alpha_i^{(j)},\alpha_{(i+t)_{mod\,3}}^{(j+n)_{mod\,4}}\right)\mid 0\le i\le 2, 0\le j\le 3\right\}$$

where $\alpha_i^{(0)}=\sigma^{i+1}$, $\alpha_i^{(1)}=\sigma\tau\sigma^i$, $\alpha_i^{(2)}=\tau\sigma^{i+1}$, $\alpha_i^{(3)}=\tau\sigma\tau\sigma^i$

is a non-symmetric association scheme with parameters

$$p_{lm}^k=\begin{cases}1 & if\ \ n^{(k)}\equiv\left(n^{(l)}+n^{(m)}\right) mod\ 4, and\\ & t^{(k)}\equiv\left(t^{(l)}+t^{(m)}\right) mod\ 3\\ 0 & otherwise\end{cases}$$

where $k=3n^{(k)}+t^{(k)}; l=3n^{(l)}+t^{(l)}; m=3n^{(m)}+t^{(m)}$; for some $t^{(k)},t^{(l)},t^{(m)}\in\mathbb{Z}_3$ and $n^{(k)},n^{(l)},n^{(m)}\in\mathbb{Z}_4$.

Proof. Observe that $|\mathcal{S}_k|=|Y|$ for all $0\le k<12$. The relations $\mathcal{S}_k$ are disjoint and $\cup\,\mathcal{S}_k: 0\le k<12=\mathcal{P}$.

$\mathcal{S}_0=\left\{\left(\alpha_i^{(j)},\alpha_i^{(j)}\right): 0\le i\le 2, 0\le j\le 3\right\}$ is an identity relation. For arbitrary relations $\mathcal{S}_l,\mathcal{S}_m,\mathcal{S}_k$ in $\mathcal{P}$, we prove that for all $(x,y)\in\mathcal{S}_k$, $|x\mathcal{S}_l\cap y\mathcal{S}_m^*|$ is constant. Now let (x,y) be an arbitrary element of $\mathcal{S}_k$. Since $x\in Y$, x is of the form $\tau^a\sigma\tau^b\sigma^c$ for some a,b,c (from Table *[table:canonical forms An,Sn]*) and further $x=\alpha_i^{(j)}$ for some i,j. Suppose $x\mathcal{S}_l=x'$ and $y\mathcal{S}_m^*=y'$ where $x',y'\in Y$.

Now $(x,y)\in\mathcal{S}_k$ implies $y=\alpha_{\left(i+t^{(k)}\right)_{mod\,3}}^{\left(j+n^{(k)}\right)_{mod\,4}}$ where $k=3n^{(k)}+t^{(k)}$ for some $t^{(k)}\in\mathbb{Z}_3, n^{(k)}\in\mathbb{Z}_4$. Similarly, $(x,x')\in\mathcal{S}_l$ and $(y',y)\in\mathcal{S}_m$, implies $x'=\alpha_{\left(i+t^{(l)}\right)_{mod\,3}}^{\left(j+n^{(l)}\right)_{mod\,4}}$ where $l=3n^{(l)}+t^{(l)}$ for some $t^{(l)}\in\mathbb{Z}_3, n^{(l)}\in\mathbb{Z}_4$, and $y'=\alpha_{\left(i+t^{(k)}-t^{(m)}\right)_{mod\,3}}^{\left(j+n^{(k)}-n^{(m)}\right)_{mod\,4}}$ where $m=3n^{(m)}+t^{(m)}$ for some $t^{(m)}\in\mathbb{Z}_3, n^{(m)}\in\mathbb{Z}_4$. Since every pair (x,y) in $\mathcal{P}$ are k^{th} associates for exactly one *k*, we find that p_{lm}^k can be either 0 or 1. Hence we obtain that $p_{lm}^k=1$ if and only if $x'=y'$ that is, if $t^{(k)}\equiv\left(t^{(l)}+t^{(m)}\right) mod\ 3$ and $n^{(k)}\equiv\left(n^{(l)}+n^{(m)}\right) mod\ 4$. □

Theorem 8. *Let $Y=A_5$ and $\mathcal{P}$ be a partition of $Y\times Y$. For all k written in form of $12n+4t+r$ $(n\in\mathbb{Z}_5, t\in\mathbb{Z}_3, r\in\mathbb{Z}_4)$, the relations $\mathcal{S}_k$ on $\mathcal{P}$ defined by*

$$\mathcal{S}_k=\left\{\left(\alpha_{ij}^{(s)},\alpha_{(i+n)_{mod\,5}(j+t)_{mod\,3}}^{(s+r)_{mod\,4}}\right)\mid i\in\mathbb{Z}_5, j\in\mathbb{Z}_3, s\in\mathbb{Z}_4\right\}$$

where $\alpha_{ij}^{(0)} = \sigma\tau\sigma^2\gamma^i\sigma\tau\sigma^j, \alpha_{ij}^{(1)} = \sigma\tau\sigma^2\gamma^i\tau\sigma\tau\sigma^j, \alpha_{ij}^{(2)} = \tau\sigma\tau\sigma^2\gamma^i\sigma\tau\sigma^j, \alpha_{ij}^{(3)} = \tau\sigma\tau\sigma^2\gamma^i\tau\sigma\tau\sigma^j$

is a non-symmetric association scheme with parameters

$$p_{lm}^k = \begin{cases} 1 & \text{if } n^{(k)} \equiv \left(n^{(l)} + n^{(m)}\right) mod\ 5, \\ & t^{(k)} \equiv \left(t^{(l)} + t^{(m)}\right) mod\ 3, and \\ & r^{(k)} \equiv \left(r^{(l)} + r^{(m)}\right) mod\ 4 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Observe that $|\mathcal{S}_k| = |Y|$ for all $0 \le k < 60$. The relations $\mathcal{S}_k$ are disjoint and $\cup \{\mathcal{S}_k : 0 \le k < 60\} = \mathcal{P}$.

$\mathcal{S}_0 = \left\{\left(\alpha_{ij}^{(s)}, \alpha_{ij}^{(s)}\right) : 0 \le i \le 4, 0 \le j \le 2, 0 \le s \le 3\right\}$ is an identity relation. Let $\mathcal{S}_l, \mathcal{S}_m, \mathcal{S}_k$ be arbitrary relations in $\mathcal{P}$ and (x, y) be any element of $\mathcal{S}_k$. Since $x \in Y$, x is of the form $\tau^a\sigma\tau\sigma^2\gamma^b\tau^c\sigma\tau\sigma^d$ for some a, b, c, d (from Table *[table:canonical forms An,Sn]*) and further $x = \alpha_{ij}^{(s)}$ for some i, j, s. Suppose $x\mathcal{S}_l = x'$ and $y\mathcal{S}_m^* = y'$ where $x', y' \in Y$.

Now $(x, y) \in \mathcal{S}_k$ implies $y = \alpha_{(i+n^{(k)})_{mod\ 5}(j+t^{(k)})_{mod\ 3}}^{(s+r^{(k)})_{mod\ 4}}$ where $k = 12n^{(k)} + 4t^{(k)} + r^{(k)}$ for some $t^{(k)} \in \mathbb{Z}_3$, $n^{(k)} \in \mathbb{Z}_5$, $r^{(k)} \in \mathbb{Z}_4$. Similarly, $(x, x') \in \mathcal{S}_l$ and $(y', y) \in \mathcal{S}_m$, implies $x' = \alpha_{(i+n^{(l)})_{mod\ 5}(j+t^{(l)})_{mod\ 3}}^{(s+r^{(l)})_{mod\ 4}}$ where $l = 12n^{(l)} + 4t^{(l)} + r^{(l)}$ for some $t^{(l)} \in \mathbb{Z}_3, n^{(l)} \in \mathbb{Z}_5, r^{(l)} \in \mathbb{Z}_4$; and $y' = \alpha_{(i+n^{(k)}-n^{(m)})_{mod\ 5}(j+t^{(k)}-t^{(m)})_{mod\ 3}}^{(s+r^{(k)}-r^{(m)})_{mod\ 4}}$ where $m = 12n^{(m)} + 4t^{(m)} + r^{(m)}$ for some $t^{(m)} \in \mathbb{Z}_3, n^{(m)} \in \mathbb{Z}_5, r^{(m)} \in \mathbb{Z}_4$. Since every pair (x, y) in $\mathcal{P}$ are k^{th} associates for exactly one *k*, we find that p_{lm}^k can be either 0 or 1. Hence we obtain that $p_{lm}^k = 1$ if and only if $x' = y'$ that is, if $t^{(k)} \equiv \left(t^{(l)} + t^{(m)}\right) mod\ 3$, $r^{(k)} \equiv \left(r^{(l)} + r^{(m)}\right) mod\ 4$ and $n^{(k)} \equiv \left(n^{(l)} + n^{(m)}\right) mod\ 5$. □

Theorem 9. *Let* $Y = S_4$ *and* $\mathcal{P}$ *be a partition of* $Y \times Y$. *For all* k *written in form of* $4n + t$ $(n \in \mathbb{Z}_6, t \in \mathbb{Z}_4)$, *the relations* $\mathcal{S}_k$ *on* $\mathcal{P}$ *defined by*

$$\mathcal{S}_k = \left\{\left(\alpha_i^{(j)}, \alpha_{(i+t)_{mod\ 4}}^{(j+n)_{mod\ 6}}\right) \mid i \in \mathbb{Z}_4, j \in \mathbb{Z}_6\right\}$$

where $\alpha_i^{(0)} = \tau\sigma^i$, $\alpha_i^{(1)} = \sigma\tau\sigma^i$, $\alpha_i^{(2)} = \sigma^2\tau\sigma^i$, $\alpha_i^{(3)} = \sigma^i$, $\alpha_i^{(4)} = \tau\sigma\tau\sigma^i$, $\alpha_i^{(5)} = \tau\sigma^2\tau\sigma^i$

is a non-symmetric association scheme with parameters

$$p_{lm}^k = \begin{cases} 1 & if\ \ n^{(k)} \equiv \left(n^{(l)} + n^{(m)}\right) mod\ 6, and \\ & t^{(k)} \equiv \left(t^{(l)} + t^{(m)}\right) mod\ 4 \\ 0 & otherwise \end{cases}$$

Proof. $\mathcal{S}_0 = \left\{\left(\alpha_i^{(j)}, \alpha_i^{(j)}\right) : 0 \le i \le 2, 0 \le j \le 3\right\}$ is an identity relation. It can be verified that $(Y, \mathcal{P})$ is an association scheme and the relations are non-symmetric.

Let $\mathcal{S}_l, \mathcal{S}_m, \mathcal{S}_k \in \mathcal{P}$. To find cardinality p_{lm}^k such that for all $(x, y) \in \mathcal{S}_k$, $|x\mathcal{S}_l \cap y\mathcal{S}_m^*| = p_{lm}^k$.

Let $(x, y) \in \mathcal{S}_k$, $x\mathcal{S}_l = x'$ and $y\mathcal{S}_m^* = y'$ where $x', y' \in Y$. Since $x \in Y$, x is of the form $\tau^a\sigma^b\tau\sigma^c$ for some a, b, c (from Table *[table:canonical forms An,Sn]*) and further $x = \alpha_i^{(j)}$ for some i, j. Proceeding as in proof of previous theorem, we get $y = \alpha_{(i+t^{(k)})_{mod\,4}}^{(j+n^{(k)})_{mod\,6}}$ where $k = 4n^{(k)} + t^{(k)}$ for some $t^{(k)} \in \mathbb{Z}_4, n^{(k)} \in \mathbb{Z}_6$; $x' = \alpha_{(i+t^{(l)})_{mod\,4}}^{(j+n^{(l)})_{mod\,6}}$ where $l = 4n^{(l)} + t^{(l)}$ for some $t^{(l)} \in \mathbb{Z}_4, n^{(l)} \in \mathbb{Z}_6$, and $y' = \alpha_{(i+t^{(k)}-t^{(m)})_{mod\,4}}^{(j+n^{(k)}-n^{(m)})_{mod\,6}}$ where $m = 4n^{(m)} + t^{(m)}$ for some $t^{(m)} \in \mathbb{Z}_4, n^{(m)} \in \mathbb{Z}_6$. Using these equations, we have $p_{lm}^k = 1$ if and only if $x' = y'$ that is, if $t^{(k)} \equiv \left(t^{(l)} + t^{(m)}\right) mod\ 4$ and $n^{(k)} \equiv \left(n^{(l)} + n^{(m)}\right) mod\ 6$. □

Theorem 10. *Let $Y = S_5$ and $\mathcal{P}$ be a partition of $Y \times Y$. For all k written in form of $24t + 4n + r$ $(t \in \mathbb{Z}_5, n \in \mathbb{Z}_6, r \in \mathbb{Z}_4)$, the relations $\mathcal{S}_k$ on $\mathcal{P}$ defined by*

$$\mathcal{S}_k = \left\{\left(\alpha_{ij}^{(s)}, \alpha_{(i+t)_{mod\,5}(j+n)_{mod\,6}}^{(s+r)_{mod\,4}}\right) \mid i \in \mathbb{Z}_5, j \in \mathbb{Z}_6, s \in \mathbb{Z}_4\right\}$$

where $\alpha_{ij}^{(0)} = \tau^i\sigma\tau^4\sigma^5\tau\sigma^j$, $\alpha_{ij}^{(1)} = \tau^i\sigma\gamma\tau^4\sigma^5\tau\sigma^j$, $\alpha_{ij}^{(2)} = \tau^i\sigma\tau^4\sigma^5\delta\tau\sigma^j$, $\alpha_{ij}^{(3)} = \tau^i\sigma\gamma\tau^4\sigma^5\delta\tau\sigma^j$

is a non-symmetric association scheme with parameters

$$p_{lm}^k = \begin{cases} 1 & \text{if } n^{(k)} \equiv \left(n^{(l)} + n^{(m)}\right) mod\ 6, \\ & t^{(k)} \equiv \left(t^{(l)} + t^{(m)}\right) mod\ 5, and \\ & r^{(k)} \equiv \left(r^{(l)} + r^{(m)}\right) mod\ 4 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Here, $|\mathcal{S}_k| = |Y|$ for all $0 \le k < 120$. The relations $\mathcal{S}_k$ are disjoint and $\cup\, \mathcal{S}_k : 0 \le k < 120 = \mathcal{P}$. Proceeding as in proof of theorem *8*, we can find cardinality p_{lm}^k such that for all $(x, y) \in \mathcal{S}_k$, $|x\mathcal{S}_l \cap y\mathcal{S}_m^*| = p_{lm}^k$ is a constant. □

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