

# Modelling the Formation and Propagation of Nonlinear Waves in Dynamical Systems with Sources

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The problem of describing the dynamics of wave processes in hydrodynamic systems with sources remains generally open. The known results in constructing the corresponding models do not have sufficient generality and are applicable to the analysis of real systems only under special restrictions on the explicit form of a function for the source intensity. The absence of solution-constants for unperturbed flows in the presence of a mass source creates challenges for the effective use of asymptotic analysis methods. Another challenge is associated with the possibility of a delayed response to flow disturbances introduced by the source.

The main objective of this paper is to develop a fairly general mathematical model allowing for identifying a set of parameters for controlling the propagation of nonlinear wave processes in dynamic systems with sources without specifying an explicit form of the source function.

To solve the problem, the paper develops approaches based on the adaptation of methods for deriving the perturbed Korteweg-de-Vries equation taking into account the nonlocality of the response based on the Whitham integro-differential equation. The contribution and novelty of this work are that a general concept of a mathematical model for describing the formation and propagation of nonlinear waves in systems with sources has been submitted, and the asymptotic analysis to establish the main control parameters has been performed too. The results of the work may be useful in the theory and practice of controlling heat and mass transfer processes in through-flowing reactors.

## 1. Introduction

The development of the nonlinear physics has been going on for several decades. And the famous Korteweg-de Vries equation (KdV) is almost 130 years old (Kappeler and Pöschel, 2013). One of its possible expression applying to thin film flows reads as follows (Brener et al., 2020):

$$U_X - \frac{3}{2} \frac{\theta_X \theta_T}{H} U U_\theta + \frac{H \theta_X}{4} \left( \theta_T^2 - \frac{1}{3} H \theta_X^2 \right) U_{\theta\theta\theta} = 0. \quad (1)$$

Nevertheless, the interest of researchers in this problem does not decrease, which can be explained by the importance of the problems being solved for science and for practical applications (Chattopadhyay, 2021). The field of research here is simply limitless (Chatzigiannakis, 2021).

It is obvious that in their full form, almost all mathematical models of hydrodynamics are nonlinear. Despite such close attention of researchers, a lot of problems that can be attributed both to fundamental and to applied problems remain insufficiently studied (Pototsky and Maksymov, 2023). For such problems, it is advisable to make efforts for solving them with the help of approximate models, reflecting the fundamentally important features of the processes being described.

Among such problems, the solution of which is possible within the framework of approximate models and is important for the creation of engineering calculation methods (Wang et al., 2020), are the problems of the propagation of nonlinear waves with evolving wave characteristics (Demekhin, 2021) in the presence of heat and mass sources (Czernek and Witczak, 2020).

These problems become especially important in mathematical modelling wave propagation in flowing systems for processes accompanied by the intensive transfer phenomena (Xie H. et al., 2021). For the high-intensive processes, the choice of the transient stage and the stage of stable control parameters becomes often uncertain (Cheng-Chuan Lin & Fu-Ling Yanga, 2020). However, in the heat and mass transfer apparatuses, the hydrodynamic picture can be quite complex. At the same time, the wave characteristics of the films flow significantly affect the interaction of various liquid structures (Li D. et al., 2023).

Account of the influence of external disturbances, in particular the influence of the mass source in the system, on the evolution of nonlinear waves turns out to be a difficult problem (Zaitsev et al., 2022). This is due to the fact that the presence of a source leads to an absence of solutions-constants (Lerisson et al., 2020). This aspect, in turn, complicates the use of asymptotic analysis methods (Choudhury et al., 2020). From the point of view of control problems, the optimal choice of control parameters loses the uniqueness (Chinnov and Kabov, 2021). Another aspect of the problem that complicates the theoretical description is that the influence of sources can be accompanied by delay effects and can lead to non-local phenomena. Under modeling the transfer processes in nanosystems and thin films, the need in allowing for the nonlocality becomes very important for not only high-intensive (Wei et al., 2020) but for mean-intensive processes also (Gopalakrishnan and Narendra, 2013). In these situations, the interesting and promising tool for building the models may be the Whitham equation (Whitham, 1999).

The Whitham integro-differential equation is a model that effectively describes nonlinear waves in highly dispersed media. This equation contains a characteristic nonlinearity of the convective type in combination with dispersion of an arbitrary type, and its general form reads as follows:

$$U_t + U_{XX} + \int_{-\infty}^{+\infty} K(X - X_1) U_{X_1}(X_1, t) dX_1 = 0 \quad (2)$$

However, G. B. Whitham proposed his equation without derivation and specific interpretation. It was shown early that an equation of this type can be derived when modelling heat and mass transfer with the propagation of nonlinear waves in media with spatial nonlocality based on the relaxation transfer kernels method (Brener, 2006). In this paper, an attempt is made to adapt approaches on the basis of which both the Korteweg-de Vries equation and Whitham equation can be derived for describing thin film flow (Hu et al., 2024) in the presence of mass sources (Brener, 2006).

The main objective of the work is to construct a mathematical model of the nonlinear waves propagation in thin liquid layers in the presence of mass sources distributed over the supporting surface, taking into account the possibility of the nonlocal response of the system to a flow rate disturbance in the liquid flow. The goal is to derive also, on the basis of the model, a general form of the equation for the propagation of nonlinear waves.

## 2. Theoretical details and discussion

In this section, two different cases for the mathematical model building are considered. In the first case, the theoretical description remains within the framework of the local approach to modeling transport phenomena (Liu C. et al., 2022). In the second case, an attempt is made to change the structure of the model in order to describe a possible non-local response to the non-uniformity of spatial distribution in the source intensity (Xie Q. et al., 2021).

### 2.1 Local case description

Let us consider the potential flow of a perfect liquid thin layer over a support surface with variable profile. The approach used follows in the main points the scheme proposed in (Brener et al., 2020).

The govern equations describing such a flow with a free surface will look as follows.

Continuity equation

$$\varphi_{xx} + \varphi_{yy} = 0. \quad (3)$$

Kinematic boundary condition on a free surface

$$\eta_t + \varphi_x \eta_x - \varphi_y = 0. \quad (4)$$

Dynamic boundary condition on a free surface

$$\varphi_t + q\eta + \frac{1}{2}(\varphi_x^2 + \varphi_y^2) = 0. \quad (5)$$

Subscripts  $x$ ,  $y$  and  $t$  in equations (3 - 5) and in the further expressions denote partial derivatives with respect to the corresponding variables.

Earlier, it was shown that the condition of proportionality between the washout rate (i.e. the normal component of the liquid velocity directly at the bottom) and the tangential component of the liquid velocity nearby the bottom region can be proposed as a boundary condition for the flow over the supporting surface (Brener et al., 2020). Such a condition has a more general physical meaning, since the usual adherence or slippage conditions on a solid wall in the presence of mass source at the bottom are not quite correct (Rivera et al., 2022).

So, the appropriate condition at  $y = 0$  reads

$$\varphi_x h_x + \varphi_y = k(\varphi_x + \varphi_y h_x). \quad (6)$$

Further, only the case of long-wave disturbances of the free surface will be considered. This means that the disturbance wavelength  $l$  is much greater than the average thickness of the liquid layer  $h_0$ . Such an assumption is quite correct for thin layers of liquid, when the characteristic longitudinal scale of the flow significantly exceeds the thickness of the layer.

In order for remaining within the framework of weak nonlinearity, an assumption of a small disturbance value  $a$  of the liquid film thickness should be also introduced.

These two assumptions can be described by own small parameter each

$$\mu = h_0^2 / l^2 \ll 1, \quad (7)$$

$$\varepsilon = a / h_0 \ll 1. \quad (8)$$

Further, for stricter control over orders of magnitude when deriving equations and for correct asymptotic analysis, it is advisable to switch to dimensionless variables (Brener et al., 2013):

$$x \rightarrow xl, \varphi \rightarrow \varphi \frac{a}{h_0} \sqrt{gh_0}, t \rightarrow t \sqrt{\frac{l}{gh_0}}, y \rightarrow yh_0, \eta \rightarrow \eta a, h \rightarrow hh_0. \quad (9)$$

Then, it is advisable to use further the slow variables  $X = \varepsilon x$ ,  $T = \varepsilon t$  and the special fast variable  $\theta(X, T)$ .

Such an approach allows in turn for applying the secular perturbation theory methods (Brener et al., 2013). And, in order to ensure weak nonlinearity of the perturbed equation, a condition for a slow possible change in the profile of the flow support surface should be also set.

The solution for the current function can be looked for in the form of a Taylor series expansion

$$\varphi = F(x, t) + \varphi_y (y + h) + \frac{1}{2} \varphi_{yy} (y + h)^2 + \frac{1}{6} \varphi_{yyy} (y + h)^3 + \dots \quad (10)$$

If both small parameters  $\mu$  and  $\varepsilon$  have a comparable orders, then, in order for saving the general structure of the modified KdV equation, the coefficient describing the mass source intensity  $k$  should be of higher order of smallness. Particularly, the case when  $k = k_1 \varepsilon \mu$  has been considered in more details (Yegenova et al, 2022). After substituting expansion (10) into system (3 - 6) and dividing into powers of the small parameters, the general structure of the control equation for the function  $F$  can be presented in the form:

$$F_{\theta\theta}(\theta_T^2 - H\theta_X^2) = \varepsilon \Phi(F_x, F_{xx}, F_{xxx}, F_t, F_{xt}, F_{xt}, F_{xxt}), \quad (11)$$

Based on this assumption, the following dispersion relation (Nayfeh, 2024) can be derived:

$$\theta_T^2 - H\theta_X^2 = 0. \quad (12)$$

Dispersion relation (12) is necessary for the solution of equation (11) to be satisfied in the zeroth order. By the way of eliminating the secular terms (Jazar, 2021) in the next order of expansion, the following control equation for the propagation of wave amplitudes disturbances has been obtained

$$U_X - \frac{3}{2} \frac{\theta_X \theta_T}{H} U U_\theta + \frac{H \theta_X}{4} \left( \theta_T^2 - \frac{1}{3} H \theta_X^2 \right) U_{\theta\theta\theta} = \frac{\theta_T - H \theta_{XX} + \theta_X (k_1 - H_X)}{2H\theta_X} U. \quad (13)$$

Here  $U = F_X$ ,  $H = 1 + h$ .

It follows from (13) that for the accepted order of the source intensity smallness, its influence on the propagation of nonlinear surface waves will be described within the framework of the general structure of the perturbed Korteweg-de Vries equation (Brener et al., 2020) when choosing  $\theta_T < 0$ ,  $\theta_X > 0$ .

## 2.2 Nonlocal case description

The next novel point in this work consists in further generalization of the nonlinear waves propagation model in the presence of mass sources. Namely, it is proposed to take into account the possible spatial nonlocality of the response of integral flow characteristics to the non-uniformity of spatial distribution in the source intensity (Jou et al., 2010). Such nonlocality may be induced by the distributed mass sources which cause to different delay times of the system's response to disturbances.

In order to perform the appropriate asymptotic analysis, it is advisable to rewrite equation (13) in a more compact form

$$U_X + A(X, T) U U_\theta + B(X, T) U_{\theta\theta\theta} + C(X, T) U = 0. \quad (14)$$

The representation of the non-local response can be written in the form of an integral operator proposed in the work (Brener, 2006) on the base of the Whitham model approach (2)

$$U_X + A(X, T) U U_\theta + B(X, T) U_{\theta\theta\theta} + \int_{\Omega} K(|\theta - s|, U) U_s(s, X) ds = 0, \quad (15)$$

where  $\Omega$  is the range of integration in the considered flow area.

The asymptotic analysis of an equation of this type for the case when the desired function can be presented in the gradient form, has been earlier carried out (Brener, 2006). However, in a more general case, such an assumption may be too strong.

Unlike the mentioned above work the further suppositions and appropriate transformations in this paper only in the approximation of fast decay of spatial nonlocality has been carried out:

$$\lim_{z \rightarrow \infty} K(z, U) = 0, \quad (16)$$

where  $z = |\theta - s|$ .

Next, the following assumption of fairness of the assessment is used

$$K(z, U) = O(\exp(-z)). \quad (17)$$

At the same time, this assumption is consistent with the conclusion obtained in the work (Brener, 2006) based on the gradient law.

Then, after transforming the integral operator in relation (15) with integration by parts, the following relation can be obtained

$$U_X + A(X, T) U U_\theta + B(X, T) U_{\theta\theta\theta} + \int_{\Omega} U \frac{\partial K(z, U)}{\partial z} dz = 0. \quad (18)$$

A typical way to further developing the model is to expand the kernel of the integrand in (18) into a Taylor series by the variable  $U$  (Brener, 2006). Such an approach can be justified in the vicinity of a resting point (Nayfeh, 2024).

Then, in the approximation of a weak-intensity source, under restricting to the first order of smallness, the appropriate expression for  $K(z, U)$  takes the form

$$K(z, U) = (U - U_0) \exp(-z). \quad (19)$$

Finally, the expression (15) can be transformed to the form of the perturbed Korteweg-de-Vries equation with the explicit right-hand side.

$$U_X + A(X, T)UU_\theta + B(X, T)U_{\theta\theta\theta} = \int_{\Omega} U(U - U_0) \exp(-z) dz. \quad (20)$$

The resulting equation (20) should be considered as a general form. It can be further specified for various processes and situations with mass sources in flows of different physical nature. In this case, the model can be supplemented with the corresponding conservation laws (Yegenova et al., 2022).

In some cases, the corresponding "hints" regarding the model structure can be obtained based on the characteristics of the technological process under study (Lavalle et al., 2020). The correct approximation can be also obtained based on the rheology features of the flowing medium (Jeong et al., 2021). In particular, this includes problems of describing suspension flows (Jones, 2022). In these cases, additional information can be obtained based on experimental data on the peculiarities of the liquid and flow (Tashimov et al., 2018).

### 3. Conclusions

The paper shows that approaches to constructing models for describing the propagation of nonlinear waves in thin liquid films in the presence of mass sources should consider both the case of a local source influence and the case of a model adapted to take into account the possible nonlocality due to the effect of a response delay on the flow structure. It is shown that for a weak-intensity source, the structure of the models in both cases corresponds to the perturbed Korteweg-de Vries equation. However, for a nonlocal model, the control equation leads to a kind of synthesis of the Korteweg-de Vries equation and the Whitham equation.

The preliminary asymptotic analysis in order to identify the main control parameters has been carried out. The results of the work can be applied to the theoretical and practical methods for the control of heat and mass transfer processes in flow reactors. Further development of this work involves conducting computer simulation and numerical experiments based on the new model. This will make it possible to develop a mathematical description of nonlinear wave processes in specific technological processes, conduct a correct comparison with known experimental data and propose methods for calculating optimal modes.

### Nomenclature

$a$ – wave amplitude, m	
$h$ – film thickness, m	$x, y$ – cartesian coordinates
$l$ – characteristic flow length, m	$\eta$ – dimensionless film thickness
$q$ – dimensionless source intensity	$\varphi$ – flow potential
$t$ – time, s	

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