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Ontogenesis of the Being of Mathematics: The *Model*

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Abstract:

This article presents the result of an ontogenetic investigation on the Being of Mathematics from a Marxist-Lukácsian perspective. Understanding the categories as determinations of being, modes of existence, the course of the investigation of the social complex of Mathematics-identified categories such as abstraction of abstraction, genericity, materialization and formal. To this end, this complex's nexuses and structure were made explicit as the basis of an ontogenetic investigation. The understanding of mathematics as a social complex enables an understanding of the mutual impregnation between mathematics and mathematics education.

Keywords: Lukácsian method; Critical ontology; Philosophy of Mathematics Education.

Constituting the problem

From the historical concreteness that we start, seeing in it a key to the understanding of previous mathematics, it is possible to glimpse many specialized fields, from arithmetic to the theory of abelian groups, from geometry to topology, from calculus to analysis. All are fields of Mathematics, some very distinct, some similar and others a synthesis of their previous ones. Its substance, its historical character, is determined by its Being, by the Being of Mathematics, which we define as a *model*. We will seek to expose its genesis and its predominant nexuses in this article.

To understand the genesis of the *model,* it is necessary to pay attention to the genesis of science as a whole. Lukács (1978) explains that religious knowledge, due to the low productivity of work and, consequently, insufficient free moments for the suspension of daily life, is conceived as the ideal personification of concrete social relations. The creator of the universe follows the dictates of the creative work of the human being, notwithstanding that God Yahweh worked six days and rested on the seventh. Religion is thus the result of the human inability to reach the generic being from science, even though "the critique of religion is the presupposition of all criticism" (Marx, 2005, p. 145). However, religious knowledge configures an advance of consciousness over magical knowledge, as it is the first movement of the ideology of the Social Being to ontologically discern the Social Being from the Natural Being. While in magic the human being figured as a total part of the dialectic of nature, in religion the human being figures as a social dialectic detached from the dialectic of nature.

It is with the continuous development of the productive forces and, consequently, the liberation of expenditure from labor to leisure that the character of science can be established with more emphasis. The sciences emerge as an important part of the creative work process, namely, investigations of the means to bring about the end of the social objectification of work (Lukács, 2013). The development of these productive forces, and the increase of free time, allows human beings to practice another suspension of daily life, the investigation of the means as an immediate end in itself, and mediate in relation to the metabolism with nature. It is in these moments that homogenization becomes more present.

Homogenization in science for Lukács (1978, 2012, 2013) is characterized as a deanthropomorphization of everyday life. In mathematics it is no different; the concepts of number and figure have existed since primitive peoples, as a process of *abstraction of abstraction*. Number, for example, is at first a collection of singular objects (sticks, for example) that are then abstracted from their singular properties to become universal. Thus we have a collection of general bodies that can be applied to different particularities. On the other hand, the concept of number is configured as a new abstraction, however, a *quantitative abstraction* of a previous *qualitative abstraction* of material entities. Number is conceived as an abstraction isolating the quantitative character of *bodies in general.* Engels (1976) understands this continuous movement as the dialectic of the transformation of the qualitative into the quantitative and vice versa. It is from a process of abstraction of abstraction that number begins to be configured as *the genericity* of the quantitative qualities of material entities.

Number is not a quality that is found in material entities; it is above all a mirroring/refraction of quantitative qualities of material entities that is configured as a possibility of higher and later achievements. The abstraction of abstraction in this movement is a negation of the negation in a process of production of the new. When the first abstraction of material entities for the concept of bodies in general occurs, the syncretic and everyday character of these entities is denied. The second abstraction, by taking the latter as social objectivity, denies its material character, configuring a new type of abstraction, of a higher order, in a form in which the content is configured as "*formless*" (Lefebvre, 1991) and where it is present *in nuce,* as a character to be informed. Einstein already said[1] that politics serves a moment in the present, but an equation is eternal. However, mathematics as investigations of the means is directed by a *must-be* inculcated from an *objectively* social value, external to the subjects who make it. Thus, in the process of abstraction of abstraction, the value of mathematical practice as a regulator of the whole process is already present. This is the point that needs to be better explored now.

In the ancient Orient, the movements of construction of the concepts of number measure/figure; consequently, the fields of algebra-arithmetic and geometry are mediated consequences of the consolidation of a division of labor in its proper modes of production. These are based on systems of communal land ownership with the defense of a centralizing state that kept part of the surpluses while the cultivation of the land took place on a subsistence basis.

Mathematics developed as practical sciences of the bureaucrats (Struik, 1992) responsible for calculating taxes, organizing public works, constructing the calendar, etc., in which "practical arithmetic and measurement were naturally emphasized" (p. 47). In this

[1] This statement has already become recurrent, despite the lack of its source. Here only its existence for didactic purposes is appropriate.

aspect, Struik (1992) explains that the consolidation of a special stratum allowed the development of more abstract tendencies, gradually cultivating a study beyond the immediate questions, now linked to its own interest as a field of knowledge.

The Rhind and Golonishev papyri, during the consolidation of Egyptian scribes and bureaucrats as a social class, represent the movements in search of a more comprehensive *genericity*. However, the first problems that served as a model for solving others were still strongly linked, mediately, with the social division of labor and with the latter's ultimate realization. It is in this aspect that the process of promoting the *genericity* of mathematical *praxis* brings with it its opposite extreme: *materialization.* Both categories are already present in the process of abstraction of abstraction, since at a given moment it must "return" to everyday life so that it ceases to be pseudo-concreteness and becomes concrete thought, a synthesis of multiple determinations (Kosik, 1976). Materialization in this process inculcates in the subject who exercises the moment of abstraction the *ought-to-be* for the production of a *genericity*, that is, the ascent to *genericity* is the alternative movement that aims to actualize the *ought-to-be* in its objective form of value, namely the moment of materialization. It is the materialization, as the objective of the setting of the end of *mathematical praxis*, that defines the *value* of the *effective genericity* and its correction. In this aspect, we strongly distance ourselves from the Wittgensteinian (Gottschalk, 2004, and Skovsmose, 1999) understandings, in which there is an ontological understanding that Mathematics materializes in a symbolically violent way, normalizing the historical-social dynamics.[2]

In these papyri, *genericity* and materialization are confused as categories *to be developed*, since the papyri themselves are *genericities* of materially continuous problems in their societies. The abstractions of abstractions in the Ancient East, due to their connection to the practicality of the daily life of their societies, stagnate. It is not appropriate here to say in an anachronistic and evaluative way that the peoples of the Ancient East were incapable of developing Mathematics, but it is important to understand it as part of a complex of the Social Being of these civilizations. The character of these categories as categories *for development* is not a moral or ethnic issue; it is first and foremost linked to the reproduction of these societies. The mathematics of these civilizations was highly developed for the social function that they should fulfill within the reproduction of their Social Being.

Marx (2011), when analyzing the Asian mode of production, configured it as stagnant, not because it is incapable of surviving, but because its character of reproducibility is seemingly endless, different from the mode of production of capital that expands and tends to destroy everything. In short, the productive forces and the dynamics of the social division of labor, communal property and centralized state as defense were duly linked to a subsistence society, where certain classes appropriated surpluses from communities. However, the lack of private property and the villages as an extension of the countryside made these societies self-sufficient and with historical elasticity. For Marx (2011), the non-existence of private property configured an even more peculiar division of labor in which manufacturing and agriculture were not segregated into countryside and city, but were always tasks of the countryside. The villages served only as points of exchange and collection of surpluses by the bureaucratic estates of the state. Thus, the singular individual did not achieve autonomy in relation to society, both due to the persistence of the hereditary social system (such as the Indian castes, for instance) and due to the constraints this system imposed on the emergence of private property as a legal attribute

[2] In fact, it is not a mistake to point out the formative character of the reality of mathematics; the point is that this is not a mathematical essence, but precisely an estrangement in everyday life. We will not dwell on this problem in this article.

of an autonomous individual. Thus, the *to-development* character of the categories of *genericity* and materialization is characterized by the high tenacity and elasticity of these societies in their social reproduction, unlike the classical modes of production of Ancient Greece and the Germanic societies.

In the classical mode of production of slavery, typical of ancient Greece and Rome, the countryside was an extension of the city, and the citizens of the city were landowners with the reverse being true as well, where to be a landowner it was necessary to be Greek or Roman. In short, there was a dialectical relationship of communal properties with private land ownership by the city dwellers (Marx, 2011). This model was based on the exploitation of slave labor, in which each landowner owned the enslaved, the means for work and the product thereof. It is in this model that the pillars of the concepts of Greek democracy emerge. Slaves were not granted rights, nor were they considered human by Greek citizens; however, among Greek citizens there were no classes, and everyone was equal before the assembly.

The classic slave model of production allowed idleness to the masters of enslaved people, so that they then had time to reflect and question the fundamentals of life. The great change from this system to others was to stop asking "how?" and start asking "why?": the same happened with the development of Mathematics (Struik, 1992). The Greeks' aversion to work allowed them to suspend daily life, since they were largely free from the problems associated with work. This model influenced the development of mathematics in very significant ways.

First, the Greek mode of production was based on territorial-military expansion for the acquisition of new enslaved people, while its model of war was that of plunder. This is how the Greeks took the riches of the ancient east and the peoples of the African continent. Territorial-military expansion and the method of plunder were the fundamental requirements for the reproduction of Greco-Roman society, in the same way that the endogeny of the ancient Orient was its peculiarity of social reproduction. The contact between the Greeks and the knowledge systems of ancient Mesopotamia and Egypt contributed to the development of the scientific and mathematical foundations later elaborated by the pre-Socratics, Plato, Aristotle, and Archimedes. Lukács (1978) elucidates two fundamental questions for the development of Greek scientific deanthropomorphization:

> In the first place, that a truly scientific grasp of objective reality is only possible through a radical break with the anthropomorphizing personifying mode of conception. The typical scientific mirroring of reality is a de-anthropoformation of both the object and the subject of knowledge: of the object, by cleansing its in-itself of all the additions of anthropomorphism (as far as possible); of the subject, by making his behavior in the face of reality consist of constantly criticizing his own intuitions, representations and conceptual formations to avoid the penetration of anthropomorphizing attitudes that defamed objectivity in the capture of reality. [...] And secondly, that its actual realization is bound up with the step towards the consciousness of philosophical materialism. (pp. 154-155, our translation)

The question of the true reflection/mirroring of objective reality was already raised by the pre-Socratics. For the development of Greek philosophy, this should be its motto. It is at this point that Plato's epistemological answers emerge, where the objective representation of reality was the result of an "illumination of sensitive intuition and representation" (Lukács, 1978, p. 160), and these took the form of concepts. For Aristotle (2002) the step to idealism, as a double reflex, would be like a return to anthropomorphism where

> […] we speak of man in himself, of the horse in himself, and of health in itself, without thereby having any other alteration of the object, just as when the existence of the gods is affirmed, but imagining them to be completely equal to men. For nothing else has been done in this way than to predicate of man the predicate of eternity, and in that other case nothing else has been done than to imagine ideas, equal to sensible objects, but with the predicate of eternity. (p. 97)

The criticism of Plato's double reflex moving towards the eternity of concepts and consequently a world of ideas is largely due to the philosopher's view that Mathematics was possessed of truth due to its character as a syllogism (Bicudo, 1998). Thus, the world of ideas endowed autonomous existence, from the generalization of geometry to all knowledge and as these could not be reconfigurations of the material world, but of another world, they would be nothing more than an extrapolation of the human being himself (Lukács, 1978). We thus realize that in Plato the conception of an *abstraction of abstraction* is already present; however, because he wanted to generalize it to all scientific knowledge, something peculiar to Mathematics, he sought to give the world of ideas the *status* of god, a new personification. Aristotle's criticism of Plato is due to his analysis of work. Plato, on the other hand, despised all forms of work and all knowledge that could be linked to it, such as mechanics itself.

However, Plato's emphasis on truth and its mathematical implication in conjunction with Greek philosophy's question of answering the "why" of things, mathematics that remained stagnant from the point of view of *genericity* took a leap. This leap is characterized by the efforts of Plato, and of the geometers, to consolidate the steps of a mathematical methodology that would correctly transfer the truth, the syllogistic practices of Greek logic. Due to their markedly material character, the periods of mathematics in the ancient East could be understood as periods of mathematical experimentation, while the Greeks experience a period of construction of mathematical rigor: the genesis of the *formal* category in mathematics. Mathematical formality was already present *in nuce* in the genesis of *mathematical genericity*; however, the Greek historical-social concreteness allowed the retreat of natural barriers in the field of Mathematics, consolidating its formal character. Even the formal character of the latter, though, as a form of formless content, necessitated movements contradictory to *genericity* itself, namely the moment of materialization. However, what allows Greek Mathematics to ascend to the formal is the same that made it stagnate: its mathematical thinkers despised work and contact with everyday life, the space of mathematical materialization, and in this way, the *value* of its production was only intrinsically consolidated, on itself, from which the idea that it is self-sufficient was born. Thus, as the intrinsic value inculcated a *must-be* in Greek mathematicians only in the face of what was already *genericity* in Mathematics, formalization focused only on itself. In Greek history, the few moments in which new mathematical *genericities* were produced were largely through the military problems of Archimedes or amusement machines.

Thus, we can conclude with two theses of the ontogenetic genesis of Greek mathematics. The first states that the idleness caused by classical slavery allowed the ascent to the more generic paths of Mathematics itself, and this could not have occurred without the military-territorial expeditions and Greco-Roman plunder, fundamental for the reproduction of their society. One of the great exponents, who often appears as a great traveler, is Thales of

Miletus, who maintained direct contact with the cultures of the ancient East, bringing his knowledge to the whole of Greek democracy (Struik, 1992).

Nevertheless, one should not neglect the great step of Greek mathematics in the development of the *mathematical formal*. From the formal character of mathematical *genericities*, it can enter into itself, producing problems about itself and solving them. The emergence of the *formal* in Mathematics makes up its autonomy, always relative, in the face of work. Formalizations are constituted as the means and mode of mathematical production (Badiou, 1972). Evidently, the formal was not constituted as we know it today; however, today the remnants of how we "demonstrate" mathematically always return to the works of Euclid, the first great systematizer of mathematical results in the ancient world. Along with the category of the *formal,* formal syllogism was born. To be able to advance in this category, we need to return to the *genericity* and the way of being that it inaugurated: *mathematical idealities*.

Mathematical idealities and the formal: the complex of the *model*

It is not new to understand that the objects of investigation of mathematics are ideal; however, little is discussed about the ontological character of these objectivities and their specific genesis. Before we move on to the specificity of mathematical idealities, let's understand the problem of the ideal in philosophy.

Despite being an ancient problem in Western philosophy that emerged with Plato, the *ideal* was exhaustively investigated by Kant. For him, the ideal is the rational overcoming of all contradictions, and given the impossibility of resolving all of them, it will always be unrealizable (Kant, 2001). Therefore, in Kantian philosophy, the ideal is not an image created at the end of a human activity, but indicates the direction and leads the human being to the constitution of a more correct image. From this character, Kant understands that the ideal can only be represented as art (beauty), while in the sciences (pure reason) it translates into the principle of non-contradiction. With Kant, there is an insurmountable barrier between ideal and material (sensory empirical reality).

In Hegel (2019a), the ideal becomes a moment of reality, an image of the spirit about itself in a continuous process of development that can therefore be realized in reality, even if partially. The ideal is an image of the end of the activity of the genre (spirit).

With the materialist inversion of Hegelian philosophy by the founders of Marxism, the ideal began to be developed as a reflection of the social contradictions in development, reflecting in the social consciousness historical-concrete, contradictory situations, their needs, tendencies of development, etc. The human being creates for himself an image of his reality, turning the ideal into an active force that organizes the consciousness of human beings and unifies them around socially concrete tasks Defined.

Despite this philosophical tradition on the problem of the ideal existing, there are recurrent definitions or elaborations that restrict it to some of its particular cases such as the phenomena of consciousness, thus making it impossible for ideal phenomena to exist outside consciousness. This conception implies a split in Mathematics, sometimes insurmountable, between the cognitive (seen as synonymous with the ideal) and the real. From this conception result the apologetic conceptions of the miracle of mathematical applications to physics and other sciences that we mentioned at the beginning of this work.

The problem of the ideal has always been linked to the problem of objectivity (or objective truth); that is, it has always been related to the forms of knowledge that cannot be explained by psychophysiology. Ideal things are beyond the individual psyche, but the latter imposes itself on the former. It is these phenomena, which have a specific kind of objectivity divergent from the singular objectivities perceived by the senses and independent of the individual body and mind, that constitute idealities, or the ideal *in general* (Ilyenkov, 2012).

Plato, despite his alienated interpretation of this complex of problems, is the first to highlight this category of phenomena as

> commonly held, necessarily universal, image patterns clearly opposed to a singular "soul" directing a human body, as an obligatory law for each "soul," with requirements which each subject must consider from childhood far more carefully than the exigencies of his own singular body with its fleeting and random states. (Ilyenkov, 2012, p. 154)[3]

They are normativities of culture that every individual is obliged to recognize in order to be able to develop his own vital activity. Thus, the individual is obliged to assimilate a certain specific reality, the ideal world, and such reality has nothing to do with the structure of the human body, much less and more specifically, with the human brain.

The ideal, or rather, idealities are the materially crystallized/objectified forms of social consciousness; they are representations historically established and legitimized by society (Ilyenkov, 2012). Such idealities make up a very peculiar world, but the question that arises is how this ideal in general, idealities in particular, relate to the "real."

When the Social Being makes its ontological leap with the emergence of the complex of work, it does not start to produce only material products, but ideal products. In this complex process of social intensification, the human being idealizes reality and then carries out its opposite process: *materialization* (objectification, reification, incarnation, etc.). Marx was the first to understand this, and he makes this relationship explicit in his analysis of the value form in capitalist society. Ilyenkov (2012, p. 163, our translation) explains and summarizes this analysis in a masterful way:

> The fact is that any sensorially perceptible object that satisfies a human need, any "use-value," can assume the "value-form." This is a purely universal form, completely indifferent to any sensuously perceptible material of its "incarnation [воплощения]," of its "materialization." The value-form is absolutely independent of the characteristics of the "natural body" of the commodity in which it "dwells," the form in which it is represented. Similarly, with money, which it also only expresses, it represents with its own specific body this mysterious reality, but it is not at all that reality itself. It is always something distinct from every material body, sensuously perceptible from its own "incarnation," from any corporeal reality. This mystical, mysterious reality does not possess its own material body, which is why it easily changes from one material form of its incarnation to another, persisting in all these "incarnations"

[3] Translation by Marcelo José de Souza e Silva is available on the marxists.org platform. For the purposes of the article, this translation was compared with the original, which is the main reference. It is taken here to be the most usual, despite significant differences when compared with the Spanish version. This debate is not our focus.

> and "metamorphoses," and even increasing thereby its own "incorporeal body," controlling the destiny and movement of all those singular bodies it inhabits, in which it temporarily "materializes." Including the human body.

The form of value in general is ideal, but this does not mean that it exists only in the consciousness or in the head, for its general form is possible only as a synthesis of the singularities of needs, just as the particularity of value which is embodied in a physical good can only do so in relation to value in universal form.

When an author writes a book, a painter paints a picture, or a student solves some mathematical problem, despite the physical existence of the book, the painting, or the writing/voice, such products are ideal because they require spiritual forces for their production. These ideal products exist to the extent that they embody physically, but their physical existence is established only from a relationship with the purely spiritual image of those who performed the activity. The written book is not the book thought, much less is the painted work the image created, just as the written reasoning is not the reasoning developed by the student. However, without their incarnation, beings would constitute non-beings. Only in their incarnation are they realized, but always in a contradictory relationship. In this sense, both the physical form and the ideal product are only understandable if taken in the contradictory and dialectical unity of identity and non-identity, of the singular, particular and universal. Mathematical idealities are objectivities, because they incarnate and can incarnate since they are crystallized reflections in the social consciousness. So, is every mathematical ideality a reflection of a material reality? And consequently, is every ideality capable of materialization?

According to Ilyenkov (2012), the ideal is a schema of objective human activity that *agrees* with the form existing outside the mind, and is precisely a schema and not properly the material reality. Agreeing does not mean mechanically reflecting the real in consciousness, but that idealities, the actively produced schema, holds the power to incarnate. In summary, ideality is

> the form of social human activity represented in the thing, reflecting objective reality; or, conversely, the form of human activity, which reflects objective reality, represented as a thing, as an object. "Ideality" is a kind of stamp imprinted on the substance of nature by social human vital activity, a way of functioning of the physical thing in the process of social human vital activity. Thus, all things involved in the social process acquire a new "form of existence," which is not included in their physical natures and differs from them altogether – their ideal form. (Ilyenkov, 2012, p. 181)

In other words, the ideal is a unity of body and mind, it is social consciousness as an incarnated human activity and a social body reflected in consciousness. Ideality is precisely this unity of contradictories, and only in this unity can its mystery be solved. Therefore, mathematical idealities *agree* with reality, and they agree precisely because they are only a quantitative abstraction from a qualitative abstraction of reality. That is, it is impossible for us to go looking for topological structures in reality as if they were things perceptible by the senses. On the other hand, these idealities incarnate in different dimensions of social structures that we sometimes do not even perceive, such as economic theories that define monetary policies, or a system of grades and academic credits of the education system, for example. However, we are skipping steps. Topological structures are idealities resulting from an active production of the formal already ontologically constituted in the complex of Mathematics, namely the model. The formal is placed, *in nuce,* in *genericity* precisely because this is the dialectical overcoming of the process of abstraction of abstraction. In this sense, we need to ontogenetically analyze this process in the constitution of the first mathematical idealities: number and space.

Humanity has not always been able to count, and the counting process is based on the immediate needs of the survival of the species. Gerdes (2014) analyzes the languages for numerals and exposes the thesis that in the beginning we knew how to count little, in Portuguese, for example, the speeches of one and two accept gender inflections (*um/uma*), (*dois/duas*) expressing qualities, while three does not present this inflection and is believed to have its origin in the French "très" which means "many." At first, we counted few things, and our words for the count also represented qualities. We can perceive the same situation in the relations of "three" and "four," etc. with their multiplicative inverses "one-third," "one-fourth" and so on. This relation does not exist for the "two," as it has "one half" as its multiplicative inverse, again expressing quantity and quality.

According to Gerdes (2014) and Struik (1992) the genesis of the concept of number is linked to the creation of words representing quantities for daily needs of hunting, feeding, fishing, etc. The first quantities, as we saw earlier, did not differentiate much between quantity and quality. A fat goat for example was worth more than two thin goats at feeding time. And over time and the continuous accumulation of this system of ideal productions, in a continuous dialectical process of idealization and materialization, mathematical idealities were conformed and began to represent quantitative relations between material beings as ideal relations between them: the joining of a group of three goats with a group of two goats constitutes a collection of five goats, and so on. The first relations of arithmetic operations are explicitly idealizations of material quantitative relations, and such material relations begin to require new linguistic representations for their idealization. It is this rich field of mediations that ontologically alters number to become pure quantity, and emerge as *genericity*. Lefebvre (1991) states that

> The notion of number proceeds from a momentary elimination of numbered objects: abstract number can designate all kinds of objects; moreover, it is expressly constituted by the understanding for the purpose of numbering objects, although it is later considered in itself by arithmetic. (p. 132)

To be "pure" means to be able to be understood in its universality, dispensing with the material singularity of which it is a reflection. To be a mathematical ideality is to agree with reality, without necessarily reflecting it as a mirror. The same process can be seen in the formation of mathematical idealities of space (geometric figures in general) as a result of the best adaptations of instruments, constructions, materials in general (Gerdes, 2014).

The point is that also in this process the geometric figures remain directly connected to their material representations to the point where new material needs allow the ontological alteration of these, constituting the emergence of a universality, the figure in general. In short, the ontogenetic process of the constitution of the number, as number in general, and of the figure as figure in general, is the constitution of the primary structures of idealities: the *genericities*. Genericity is precisely the ontogenetic moment of the model in which mathematical idealities already configure a double abstraction and are established as universals in relation to the material being, without, however, their relations having become autonomous from the material being: the operative relations between numbers are still restricted to the singular quantities of problems of the public administration of antiquity, to the relationship between physical or social quantities (taxes, for example). To overcome *genericity* and constitute oneself as formal is to autonomize these relations between idealities in relation to the material being, in such a way that these, which are universal in relation to the material being, are configured as singular in relation to the formal that emerges. I am talking about the process in which five – which as a number *in general* can assume the representation of any collection of things (particularization) or the representation of five goats (singularization) – is configured as a particular number of the collection of all numbers, and is the singularity of an autonomous ideal structure that is established in relation to other numbers. The ontogenetic constitution of the formal is this movement of denial of *genericity* that surpasses it by incorporation.

To expose the ontogenesis of number and space is to expose the material Being that they have idealized. A good ontogenetic indicator is Hegel's (2019b) exposition on the Being of magnitudes, the *quantum.* Hegel (2019b) dialectically extracts the *quantum* of being from quality and, in the same author, mathematical idealities can be understood as the ideal objectification of a double process of abstraction. But its importance lies in the specific way in which Hegel extracts the *quantum* of quality. It (the quantum) is precisely the "being assumed for itself [...] behaves identically towards the other and has thereby lost its determination [quality]" and more explicitly states that "quantity [quantum in general] is the determinateness that has become indifferent to being, a limit that is also no limit; it is the being for oneself that is purely and simply identical to the being for the other" (2019b, pp. 193, 197). In this sense, the decisive difference between quality and quantity consists in the fact that we change the quantity and the identity of what is changed remains the same. Therefore, we can increase from 5 to 10 apples, but the quality of being an apple does not change. Therefore, in Hegel (2019b), Mathematics is the science of *the quantum,* of the determinations of objects that do not describe them as such, that are indifferent to their content. In quantum science, content is irrelevant.

For Hegel (2019b), the *quantum,* as a being of greatness, has two moments: that of continuity and that of discretion. Discrete magnitude is the rupture of continuity, it is quantity that emerges from the count and is idealized in number. Continuous magnitude, as the magnitude of space, has only one limit in general, distinct from discrete magnitude, which is limited in specific, forming pluralities. What Hegel demonstrates is that one implies the other in a determination of reflection. That is to say, discrete magnitude is one in itself, and therefore continuous in its unity, while the science of the magnitude of space, geometry, has its one in the point, "but the point, in so far as it comes out of itself, becomes another, becomes line; it becomes, in *the relation,* a continuity, [...] in which the one is suppressed" (Hegel, 2019b, p. 2018). What the philosopher has shown is that the movement of negation between the discrete and the continuum in Mathematics, in which the number in its unity is the continuum, is an expression of a spatial quantity, and that space needs the point, its one, to constitute itself. The continuum to express its limit needs the discrete and the discrete to express its determinateness suppresses the continuum. The relations between number and space are configured by measurement (Hegel, 2019b), because space in its continuity can only compare itself and establish relations of equality and inequality, and in order to be able to operate with its quantities (act metrically) it needs number. On the other hand, number in its discretion is unity, but it is also numerical value as a result of operative relations with other numbers. Its determinacy as a numerical value requires the comparative possibility that is realized only in the continuum through the relations of equality and inequality. Therefore, the numerical value resulting from operation only finds its determination when it is configured in a comparable continuum, as a measure.

What Hegel (2019b) sometimes makes explicit in mystifying language is the ontogenesis of the first mathematical idealities (number and space) as double abstractions of magnitudes, material realities. In addition, he makes explicit the dialectical relationship of reflexive determination of the ontogenesis of these two *primary structures of idealities*, the *genericities* of arithmetic and geometry. With Hegel (2019b) we understand that with the process of looking at oneself, of idealities as qualitative subjectivities indifferent to being, as a peculiar phenomenon of the ideal world, the human being begins to realize that qualities indifferent to identities alien to the *quantum* are not indifferent to themselves. In other words, numbers and geometric figures may be indifferent to the material Being because they do not alter its determination, but they are not indifferent to themselves, because they are endowed with qualities: numbers and geometric figures are classified according to their characteristics, such as even, odd, prime, major, minor, etc.

When such idealities come to be classified and related according to their characteristics, the ontological leap from *genericity to formal* is gestated. The ontological change in the mathematical model is what allows it to emerge as a model of models, that is, as a social complex. This reconfigures the relations of determination between all the categories that existed until then in the Mathematics complex (idealities, *genericity*, materialization, *should-be*, value, alternative, etc.) and this alteration constitutes the fertile soil for the emergence of the Mathematics Education complex. The internal complex of the formal is configured with the determination of a secondary structure of idealities, as the ideal autonomization of the material relations between mathematical idealities; this secondary structure we will call *formal structure*, because it alters the primary structure of *genericity* and is related to many determinations of formal logic.

Lefebvre (1991) already elucidated the importance of logical formality as one of the ways of reducing material and concrete content to consciousness. From it one arrives at a kind of form without content, pure and rigorous, as if it had "nothing" substantial. This is the expression of the autonomization of the primary structures of idealities of the material world. They start to be configured as qualities indifferent to the material being, but not indifferent to themselves. However, this indifference to the material being,

> [t]his 'nothingness' is dialectical and not metaphysical: at the moment when it seems to vanish into this 'nothingness', and effectively vanishes if it is thus preserved, thought determines it precisely as the possibility of 'everything' apprehending. (p. 133)

For Lefebvre (1991), the *formal* is the momentary separation between form and content. Mathematical idealities are the form of human activity indifferent to content, and so they are to the detriment of the result of this momentary separation exerted by the movement of double abstraction. In this sense, the tautology of the principle of identity arises, which is infertile in materiality, but is the form of coherence of thought with itself. To say that A is A is purely a form in which in the material content it means nothing important. This is how the Frenchman draws attention to the fact that, despite a momentary separation, "the form is not separated from the content. The content, however, can remain as 'informed.' The thought form, therefore, is certainly the form of the content in thought" (Lefebvre, 1991, p. 136). Tautology is rigorous, but also sterile and inapplicable. Metaphysics takes tautology as a model of eternal truth and believes that it starts from the principle of the identity of thought and deduces it to objective identity. Thus, with metaphysics, identity ceases to be form and becomes the substance of beings, an internal particularity of them. The principle of identity becomes form *and* content, and comes into existence by itself, it is divinized.

Lefebvre (1991) very well perceives in this situation that the ontological priority of the formal lies in the dialectical relationship between *genericity* and materialization, and that although the formal is established as a possibility of apprehending everything, due to its apparently contentless form, the same form fades away if not materialized again. It is in this aspect that the intense and extensive interdependence between the *formal* and the contradictory complex of *genericity-materialization is demonstrated.* Despite its emptiness, the principle of non-contradiction has the function of keeping thought coherent with itself, rigorous, preserving the same definitions so that argumentation is possible and so that meanings are not lost. However, the principle of identity itself sets itself in motion and develops a content by inserting into identity the difference, the contradictory relation: If "A is A" we also have "A is not non-A." In short, the principle of identity, of rigorous thinking with oneself, inculcates contradiction and difference from a third element, the excluded third.

Moreover, the possibility of idealities indifferent to the material being not indifferent to themselves lies precisely in this characteristic of the necessity of materialization. The fact that secondary structures are no longer material configurations of relations between idealities does not alter the fact that their existence as an ideal necessitates subsequent incarnation in some material structure of society. In summary, the formal ontologically alters *genericity* when it emerges from it, but still needs to be reconverted into it in order to constitute itself as an ideal. This is so because the *ought-to-be* of the emergence of the formal is inculcated by the complex of materialization that poses itself as a reflexive determinant of *genericity*. The formal emerges from the need to solve social tasks that *genericity* was unable to grasp and, therefore, agrees with reality and embodies. But the objectification of the formal in reality is the reconfiguration of a certain primary ideal structure of *genericity*. In other words, the materialization of the formal as objectivity occurs not by what it has generated and denied, but in a *genericity* of a new type linked to new social needs.

With the genesis of the *formal,* all Mathematics became autonomous as a well-established science in relation to work and inaugurated a *new mathematical praxis*. This is the genesis of the mathematical model as the Being of Mathematics. In previous periods, materialization referred only to the materiality of the historical concreteness of the society in which it was present and, consequently, to the scientific manifestations of that same society. That is, mathematical knowledge – serving as an instrument for astronomy, administration, and other domains – enters the realm of the formal and the process of formalization. At this stage, grounded in strict rigor, mathematical genericity and materialization begin to reflect upon themselves, turning inward toward the very structure of the complex to which they belong. Now, generalization was no longer about problems of concreteness external to the complex of Mathematics, which began to create its own problems. This is the dynamic that the model exerts on the relationship between Mathematics and reality: from it, the latter relates to the other sciences, to daily life directly, and to itself. However, these relations are not autonomous from each other; they have degrees of relative dependence. It is enough to take the example of the formalization of non-Euclidean geometry, a new type of abstraction that produces *genericities different from the previous ones (Euclidean), but needs to have its importance estimated as a value,* in a process of materialization in the theoretical physics of Einstein's relativity. This may imply that it is physics that increasingly advances using mathematical results. However it is just the opposite; it is Mathematics in its processual dynamics of the mathematical model that advances on the problems of physics, chemistry and other sciences seeking *values* that inculcate new *should-be* to the model.

Present in the whole process of construction of the *formal* is the problem of truth. Truth, or correction, in the *praxis* of work is given by the correction of the causal links placed in the realization of the setting of the end. In Mathematics, as a formal field, truth is established as the exclusion of error, based on the principle of the excluded middle. However, as Hegel (2019b; 2018a; 2018b) already stated, in history this is pure empty tautology, because the material concreteness of history accepts error in the form of contradictions. It is in this sense that Lefebvre (1991) argues that in order for a mathematical truth, formalized as a structure of idealities, to also be true in material reality, it is necessary that it become a unity with its difference. This is the model-reality dynamic given the contradictory pair *genericity-materialization,* in which the model as a structure of idealities is objectified.

It is in this sense that the materialization of *mathematical genericities*, that is, the *mathematization of the world*, has three striking characteristics: that of transmission, construction and possession of truth. When Skovsmose (1999) and Gottschalk (2004) characterize mathematical materialization in a normative way in the face of reality, they crystallize the character of possession of the truth of the mathematical model. In fact, the possession of truth is one of the particularities of this model that is related to social objectivity; however, it is always linked to the other two in a relative interdependence.

What we mean by this is that in the Wittgensteinian view where Mathematics standardize reality and, therefore, must be relativized and criticized, since there is no single truth about the same reality, the crystallization of the contradiction between *genericity* and mathematical materialization to the point of autonomizing the two as different things is a metaphysical thought. Nevertheless, it is from Wittgenstein's considerations that the so-called philosophy of difference emerges, which, in short, rejects the ontodialectical character of reality and crystallizes differences as immanent and ahistorical. This turn seeks to oppose the formalist tendencies in the field of the philosophy of Mathematics, "which intend to take the content from the form, to study the empty and abstract possibility in order to obtain reality" (Lefebvre, 1991, p. 146). That is, while for the formalists reality is constructed referentially from the formalization of the sciences, Wittgenstein denies that there is anything referential to reality in Mathematics and constitutes it with a purely normative character. That said, the formalists and the Wittgensteinian perspectives represent two poles incapable of resolving the contradictory character between *genericity-materialization*, mainly because they do not understand the dialectic of the production of knowledge, reality and the subject-object interrelation.

The *formal* constitutes the predominant moment in the relationship with *genericity*, but it is subordinated to the moment of materialization. It is in *the formal* that the alternative chains of the consolidation of the model are structured; each choice, each question and each answer materialize in the *formal.* Nevertheless, Badiou (1972) says that the history of Mathematics is the history of mathematical formalizations. Here is presented the ontological alteration exerted by the formal over the internal complex of materialization. The dialectical pair *genericity-materialization*, the ontological character of materialization, was linked solely and exclusively to material reality. That is, the *shoulds-be* inculcated in *genericities* were always socially defined and valuable tasks for the immediate needs of daily life (ways of calculating the area of a certain type of soil, taxes with an increase in population, etc.). From the ontological alteration exerted by the formal that reconfigures and overcomes the previous determinacy of materialization by incorporation, it continues to inculcate the *ought-to-be* for the *genericity* of the immediate type of material daily life; however, it also now inculcates *ought-to-be* arising from the needs of the mathematical model itself, understood as *a model of models*. We can even speak of mediated needs through a characteristic example of these, which was the crisis of the foundations of Mathematics in the last century or the problem of the rigor of analysis with Cauchy and Weierstrass, internal needs of the model itself linked to its imperative of internal reproduction as a complex of complexes. In summary, the emergence of the formal complexified the determinations of the model and its relationship with the world.

The formal is the logical structure of the relations between mathematical ideal objects. Its objective existence is a result of a *duty-to-be* inculcated by materialization in *genericity*. However, the formal contradicts the complex of materialization because it is always "chasing," since in materialization all the contradictions that value the development of the formal are found. In this unfolds the process of rigorous crystallization of the creative mathematical experiments that occur from materialization and *genericity*. In this process of rigorous crystallization of the results of creative mathematical experimentation, the formal seeks to constitute itself as a universal language. However, because of its contradiction with the complex of materialization in which all the mediated needs of the formal reside, this universal language is only partially realized. The belief in the resolution of all these contradictions immanent to the formal and the consolidation of a universal language is what defines the formalism[4] capable of explaining any and all content; on the other hand, formalism not taken in a metaphysical interpretation has the task of manifesting the content in what is irreducible about it, such as contradiction and negation.

The materialization of the formal, as the production of a *genericity,* is always contradictory and insufficient for the social value that inculcated the *ought-to-be*; the return of *mathematical genericity* to the real, as a formal structure of mathematical idealities, is absolutely insufficient for human activity of which genericity materializes and seeks to participate. In short, the mathematical model is not enough to contribute decisively to the end of the activity. A chain of mediations with other non-mathematical idealities (other sciences, religion, morals, culture, etc.) is necessary for the correction of the chain of alternative acts that achieve the desired end.

As much as the *formal* can be characterized by its character of a syllogism, it is from it that the predominant characteristics of materialization emerge, namely, the transmission, construction and possession of truth. Let's look ahead.

When a thesis is established from a hypothesis, the truth present in it will be transmitted to the thesis, but it will first of all be transfigured into a new type of truth, namely the construction of another truth, and from this internal process in Mathematics itself the mathematical truth is always present. On the other hand, we can understand that the *new genericity* that is instituted as a thesis for mathematical demonstration originated in a problem of the natural sciences. Physics, for example, and its foundations, lent the *formal* its truth, and the *formal* as a form of "formless" content, of content *in nuce,* took possession of it. Together with the character of a syllogism intrinsic to itself, it transfigured and transmitted a truth of a new kind, a *mathematical genericity*, to the foundations of physics. However, this new type of truth of the foundations of physics has been criticized, since formal truth is not material truth, but part of it and contradictory. This is how the materialization of mathematical *genericity* is presented from an abstraction of the foundations of physics (these are also already an

[4] Belief destroyed by Gödel.

abstraction). The formal is always a *universalized genericity*, insofar as it retains within itself some content of the abstraction that generated it and of the materialization that it is incumbent on it to objectify.

At the same time, the act of the setting of the end of Mathematics, that of materializing itself as a model in historical concreteness, is directed and regulated by the *should-be* of materializing. And here, the *should-be* and the value that one wants to realize are mediated by the alternative character of the *formal.* Every process of formalizing a *genericity* for universality is endowed with an alternative character. For example, there are different proofs for the same theorem, at the same time that there are different ways of solving physical problems, and so on. Formalization is also a temporal sequence of alternatives, just as the *construction* of Mathematics, based on the complex of the model, is a constant set of formal acts, and thus, alternative acts. We can try to summarize this complex procedural and dialectical dynamics of the model, as a model of models, in the figure below.

The *value* that is objectified by the materialization of *mathematical genericity* always has a double character: intrinsic and extrinsic. Sometimes it seems that the intrinsic character, in which the importance of *genericity* for Mathematics is sufficient in itself, stands out; and at other times the extrinsic character of value stands out, when Mathematics is a social ideology. There is no materialization that is only important for Mathematics itself, or for social *praxis*; all materializations always have a value that is established as a contradictory form. The model has truth to the extent that it is *socially valuable.*

The medieval period was marked by the dominant dynamics of the dialectical pair of reflexive determination of *genericity-materialization*. The collapse of the Greco-Roman Empire, and the decay of its slave mode of production by the feudal formation of the Germanic barbarians, inaugurated a new model of social reproduction in Europe. The Germanic, or feudal, mode of production is characterized by the continuity of Greco-Roman tendencies within the scope of science. In this model, scientists, mathematicians, etc., were princes and sons of feudal lords, and consequently their leisure was the result of the servile exploitation of labor. With great emphasis on religious forces, an opening given by Plato according to Lukács (1978), the sciences went through great periods of difficulty that were broken on a large scale by the works of Copernicus and Galileo Galilei.

In the medieval period, a debate arose between theological ontology and scientific ontology, from which emerged the Bellarmine thesis, which states that ontological explanation belongs to theology and epistemological discussion to science The works of Newton and Euler were all marked in this respect. Unconcerned with carrying out their productions to a degree of universality given the productive stagnation of feudalism, their mathematical experiments such as mechanics and astronomy were established from the demands of everyday life (Struik, 1992). Their concepts were marked by the flexibility of rigor prominently present in *mathematical genericity*. Among the examples, the infinitesimals of Cavalieri so used in Newtonian productions.

The works of the two scientists are circumstantially important for the development of mathematics; however, their historical specificity prevented them from envisioning the possibility of moving from *genericity* to the universality of the formal; it was a period of unconcern with rigor, and with obscurantist practices to the point of leading Marx to write a history of differential and integral calculus trying to clear these obscurities.

The production of new *mathematical genericities*, based on the constitution of the formal, becomes the driving force of the history of Mathematics in a more evident way in the 1800s and 1900s. Struik (1992) explains that the 1800s were dominated by the character of mathematical experimentation without great emphasis on the formal character. The practice found in Euler, Newton, Leibniz, among others, and their most important references nourish a feeling at the beginning of the nineteenth century that Mathematics reached its limits and nothing more could be discovered. With little concern for the foundations of his work, that is, for the formal constitution of the model, the driving contradiction *genericity-materialization* was exhausted. This can be evidenced in the emphases of studies in calculus, mechanics, etc. The mathematicians cited were still linked to the Germanic model of production, namely the French and English feudalist monarchy or the Germanic and Italian feudal lords. In such models, scholars restricted themselves to thinking about the things that were given to them, while the work was relegated to the servants. Oblivious, in general, to the accumulation of capital by the bourgeoisie in the cities, the latter were little concerned at the height of their idleness with the linking of their theoretical production to the new demands of work, mainly due to the degenerating character of this period. This was marked by the opening of revolutionary periods, the destruction of feudalism by capitalism, the constitution of the first professional guilds, including that of mathematicians, and the inauguration of a new era in the division of labor.[5]

Cauchy's work in the foundation of the analysis, which is characterized as an ascent to the formal *genericity* of calculus, still closely linked to the problems of mechanics and astronomy in the period of degeneration of European feudalism,[6] allowed "new mathematical research to gradually emancipate itself from the previous tendency to see mechanics and astronomy as its final goal" (Struik, 1992, p. 201). It is the constitution of Mathematics as an autonomous complex no longer only in relation to work, but now in relation to all social reproduction.

The new demands of the emerging capitalist society meant that previous scientific knowledge was assimilated by technology and the social division of labor. However, due to its expansive character where everything "that was solid melts into the air, everything that was sacred is desecrated" (Marx; Engels, 1998, p. 42), these sciences were entrusted with a new level of knowledge due to the new problems that were posed by the sociality of the Social Being.

Now it is worth remembering that a large part of calculus was linked to mechanics and astronomy, even as *genericity* kept a material content related to the problems from which it was generated. In order for calculus to contribute intensively to the development of Mathematics – as a possessor, constructor and transmitter of truth in the materialization of the model – it had to pass to the universal, to the *formal*, so that its content became formless and could be configured as the possibility of "everything" to apprehend (Lefevbre, 1991). This step to the formal allowed the mathematical model to become emphatically powerful in the 1900s and to rise even more in its ideological power in the twentieth century, from the so-called revolutions of microelectronics and microinformatics. On the other hand, this step to the formal made it possible to materialize new fields of Mathematics, such as topology as a synthesis of

[5] It is possible to perceive in this historiographical analysis even the German formalist character and the opposition to the French innovative character, starting from the analysis of the Prussian and classical paths of the capitalist constitution of society. It would indeed be interesting to do historical research in this sense.
[6] The East still remained in the Asian mode of production, which is not surprising, given its tenacity and reproductive elasticity.

geometry, analysis and algebra. Algebra in its genesis, for example, is the result of historical-concrete processes that inculcated in the complex of Mathematics the need to find unknown values that sought to solve everyday problems of trade, profits, balance of public accounts, etc. The point is that, for its constitution of relations of equality, algebra needs the primary structure of geometry, and operative relations, of the primary structure of arithmetic. However, the interrelationship between the two mathematical *genericities* has always been very troubled (just remember the emphasis on geometries in Greek societies and the emphasis on arithmetic in Asian societies). In short, an ontological leap was needed that would allow an ontological alteration to the existing *genericities*.

Hegel (2019b) has already explained the relationship of the emergence of algebra with measure (unity of the discrete and the continuous). However, this is a mathematical ideality that is aimed at a *genericity* of a new type capable of emerging only after the ascent to the universal of geometry and arithmetic: the emergence of the formal. It is the formal that allows the interconnection between the structures of different *genericities*, producing a *genericity* of a new type. That is, it was the development of Greek abstract geometry (ascent to the formal) and the first Arabic formalizations of arithmetic (ascent to the formal) that constituted the fertile soil from which the *genericity* of algebra emerges,[7] as the relations of operation and comparisons necessary for the constitution of the new ideality, the unknown. The following is a small ontogenetic map of this process that lasted decades or centuries.

In addition, the emergence of technical and military colleges in Napoleonic France reinforced the character of continuous materialization as a value that inculcates new *duties-to-be* in the subjects who produce mathematics. The teaching model in these schools was copied for a long time, including in Brazilian schools.

I believe in this aspect that the model is characterized as a being of Mathematics, so that it is understood as a complex that constitutes, from its categories, the external dialectic between Mathematics and reality. The mathematical model, the contradictory and complex procedural dynamics of Mathematics in its totality is present as a substance in the different beings of the mathematical fields, be it geometry, algebra, etc. The mathematical model, as a unitary being that clarifies the mathematical interrelations with itself and with material reality, is a powerful step towards the understanding of Mathematics Education as a unified field seen as unity in diversity, synthesis of multiple determinations and generated by the dialectical dynamics of the various contradictions inherent to it.

On the other hand, this investigation does not intend to specify how one should research[8] Mathematics, although in our understanding, an ontological understanding of Mathematics brings powerful benefits in the epistemological specifications of the production of mathematical knowledge.

[7] Notwithstanding the frank development of algebra after the Arabic translations of Greek works and the Muslim domination over ancient Greek society.
[8] This would be precisely an epistemological investigation, which is not the focus of this article.

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